A coding theoretic approach to Szilárd's engine game

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Game theoretic probability theory [1], proposed by Shafer and Vovk in 2001, is a theory that deals with stochastic behavior without assuming a probability space. Let us recall a typical result of game theoretic probability theory. Consider the following game.

Predictive game by Shafer and Vovk

Let $\Omega := \{1, 2, \ldots, A\}$ and $p : \Omega \to (0, 1)$ such that $\sum_{\omega \in \Omega} p(\omega) = 1$.

Players : Skeptic and Reality.

Protocol : $W_0 = 1$.

FOR $n \in \mathbb{Z}_{>0}$:

Skeptic announces $\alpha_n \in \mathbb{R}^\Omega$.

Reality announces $\omega_n \in \Omega$.

$W_n := W_{n-1} \left\{ 1 + \sum_{a \in \Omega} \alpha_n(a)(\delta_{\omega_n}(a) - p(a)) \right\}$.

END FOR.

Apparently, this game is in favor of Reality because Reality announces ω_n *after* hearing Skeptic's bet α_n, preventing Skeptic from becoming rich. However, Shafer and Vovk showed the following surprising result:

Theorem 1 (Game-theoretic law of large numbers [1])**.** *Skeptic has a prudent strategy* $\alpha : \Omega^* \to \mathbb{R}^\Omega$ *that forces* $\lim_{n\to\infty} W_n = \infty$ *unless*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_a(\omega_i) = p(a) \quad (\forall a \in \Omega).$$

Here, a strategy is called prudent if $W_n > 0$ *for all* $n \in \mathbb{Z}_{>0}$*.*

After Shafer and Vovk's work, some alternative proofs have been obtained [2, 3]. In this talk, we present yet another proof of Theorem 1, putting emphasis on a coding theoretic aspect of the game, and clarify a close connection to the following renowned result in algorithmic randomness theory.

Theorem 2. *An infinite sequence* $\omega^\infty \in \Omega^\infty$ *is Martin-Löf* μ*-random if and only if*

$$\limsup_{\omega^n \sqsubset \omega^\infty} (-\log \mu(\omega^n) - K(\omega^n)) < \infty,$$

where $K(\omega^n)$ *is the prefix complexity of* $\omega^n \in \Omega^n$*.*

While Theorem 2 involves the prefix complexity $K(\omega^n)$ that is uncomputable, our argument is based on a computable universal coding scheme, leading to a game theoretic analogue of Theorem 2. We also elucidate a game theoretic formulation of Szilárd's engine and discuss its interpretation in terms of the second law of thermodynamics.

References

[1] G. Shafer and V. Vovk: *Probability and finance: It's only a game!* (Wiley, 2001).

[2] M. Kumon, A. Takemura and K. Takeuchi, "Capital process and optimality properties of a Bayesian Skeptic in coin-tossing games," Stochastic Analysis and Applications", **26** Issue 6, 1161-1180 (2008).

[3] K. Takeuchi, M. Kumon and A. Takemura, "Multistep Bayesian strategy in coin-tossing games and its application to asset trading games in continuous time," Stochastic Analysis and Applications, **26** Issue 5, 842-861 (2009).

[1] Joint work with A. Fujiwara