On the intermediate structure of arithmetical transfinite recursion and Π^1_1 comprehension

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(This is a joint work with Keita Yokoyama.)

Reverse mathematics is a research program which investigate the complexity of mathematical theorems. More precisely, we consider the complexity of a mathematical theorem as the strength of an axiom of second order arithmetic which is sufficient and needed to prove the theorem.

The structure of complexity in reverse mathematics has five big milestones, namely RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0, and some recent studies focus on problems having intermediate complexity between ATR_0 and Π^1_1-CA_0. Theorems belonging to this area are roughly separated into two parts by the complexity of formulas : the Π^1_2 part and the Π^1_3 part. In this talk, we will give a fine separation of the Π^1_2 part.

For this purpose, we use a characterization of Π^1_1-CA_0 based on hyperjump. Let X be a set. We denote $\mathrm{HJ}(X)$ the set of indices of X-computable well-oderings, and call it the hyperjump of X. For each $k > 0$, Π^1_1-CA_0 is equivalent to $\mathsf{ACA}_0 + \forall X \exists Y (Y = \mathrm{HJ}^k(X))$.

According to this fact, the complexity of Π^1_1-CA_0 should be separated into the hierarchy of $\{\forall X \exists Y (Y = \mathrm{HJ}^k(X)) : k \in \omega\}$. To apply this idea to the Π^1_2 part, we introduce a Π^1_2 weakening of them.

Definition. *Let $k > 0$. Define $\beta\mathsf{RFN}(k)$ by $\forall X \exists \mathcal{M}$: coded ω-model($\mathcal{M} \models \mathsf{ACA}_0 + \exists Y (Y = \mathrm{HJ}^k(X))$.*

We will show that the hierarchy of $\beta\mathsf{RFN}$ gives a nice separation of the Π^1_2 part of Π^1_1-CA_0.

Theorem. *The following holds.*

- *Each $\beta\mathsf{RFN}(k)$ is a Π^1_2 statement provable from Π^1_1-CA_0*
- *Over ACA_0, the hierarchy of $\beta\mathsf{RFN}(n)$ is strictly increasing.*
- $|\Pi^1_1\text{-}\mathsf{CA}_0|_{\Pi^1_2} = |\mathsf{ACA}_0 + \{\beta\mathsf{RFN}(k) : k \in \omega\}|_{\Pi^1_2}$. *Here, $|T|_{\Pi^1_2}$ denotes $\{\sigma \in \Pi^1_2 : T \vdash \sigma\}$.*

In addition, some Π^1_2 consequences of Π^1_1-CA_0 can be characterized as follows.

Theorem. *Over ACA_0, the following assertions are equivalent.*

- $\{\beta\mathsf{RFN}(k) : k \in \omega\}$
- $\{\mathrm{rel}(\Sigma^0_1)_k\mathsf{Det} : k \in \omega\}$ *where $\mathrm{rel}(\Sigma^0_1)_k\mathsf{Det}$ says that for any $(\Sigma^0_1)_k$ games in the Baire space, there is a pseudo-winning strategy S for one player in the sense that any arithmetically definable strategy S' from S never win against S.*
- $\{\mathrm{rel}\Sigma^0_k\mathsf{Ram} : k \in \omega\}$ *where $\mathrm{rel}\Sigma^0_k\mathsf{Ram}$ says the existence of psuedo-homogeneous set for each 2-coloring for a Σ^0_k subset of $[\mathbb{N}]^{\mathbb{N}}$.*