How powerful are integer-valued martingales?

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Based on work of L. Bienvenu, A. Chalcraft, R. Dougherty, C. Freiling, F. Stephan & J. Teutsch

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Bibliography

The talk is based on the two following papers:

[BST] L. Bienvenu, F. Stephan, J. Teutsch. ***How powerful are integer-valued martingales?*** Theory of Computing Systems, 51(3):330-351, 2012.

[CDFT] A. Chalcraft, R. Dougherty, C. Freiling, J. Teutsch. ***How to build a probability-free casino***. Information and Computation, 211:160-164, 2012.

1. Randomness = unpredictability

Randomness for infinite sequences

In the previous talks, randomness for infinite binary sequences was discussed, in particular the notion of Martin-Löf randomness.

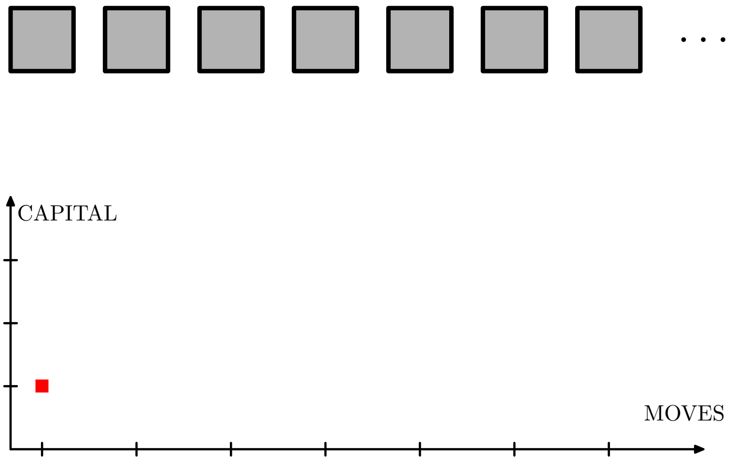
A sequence $X \in 2^\omega$ is Martin-Löf random
iff it avoids every effectively null Π^0_2 class
iff all its initial segments are incompressible (= have high Kolmogorov complexity).

The unpredictability paradigm

Schnorr proposed another approach: an infinite binary sequence X is random if no computable gambling strategy makes (a lot of) money by betting on the values of the bits of X.

Strategy:

- starts with some finite capital
- at each turn, bets some amount of money – between 0 and current capital – on the value of the next bit (e.g. bets $\$0.3$ on the value 1 for next bit)
- next bit is revealed, the strategy doubles its stake if correct, loses it otherwise
- strategy succeeds if the capital reaches arbitrarily high values throughout the game (i.e. the $\limsup$ of the capital is $+\infty$).



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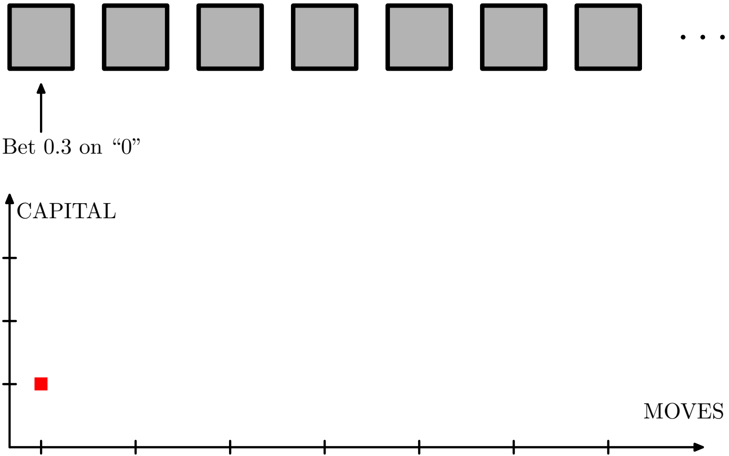

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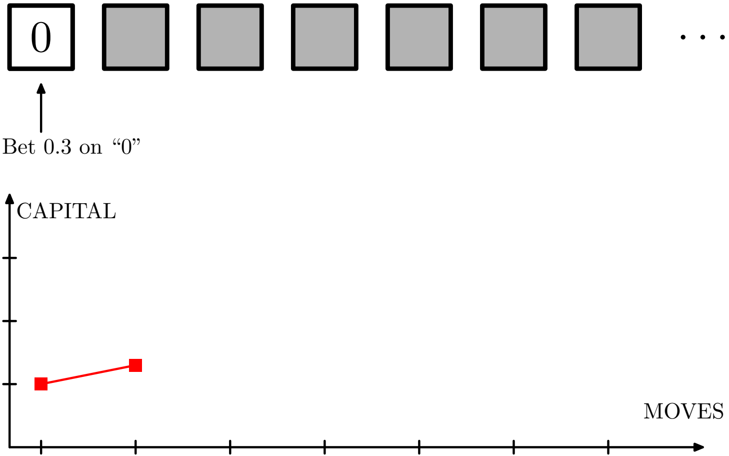
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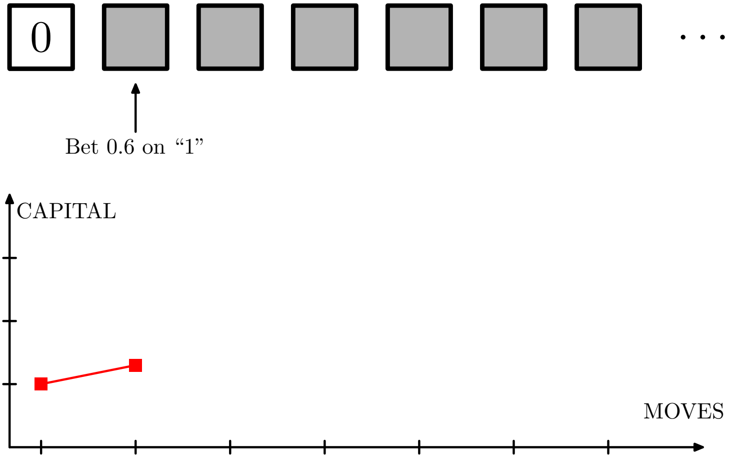
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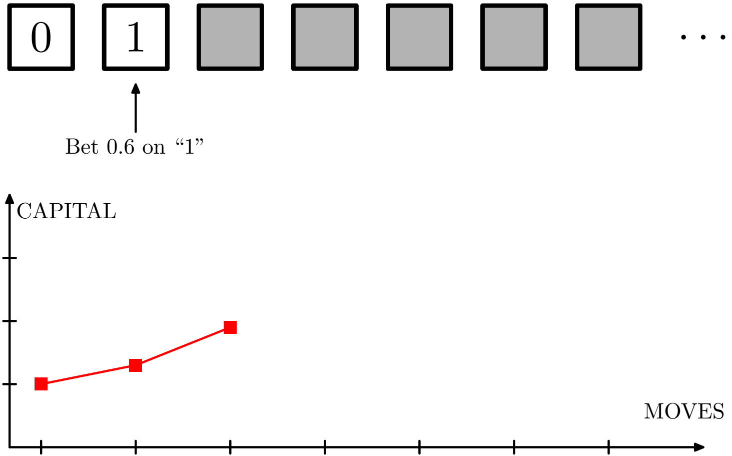
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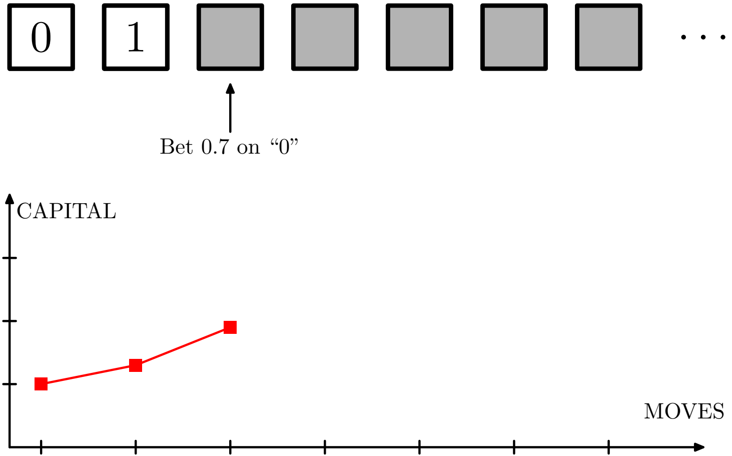
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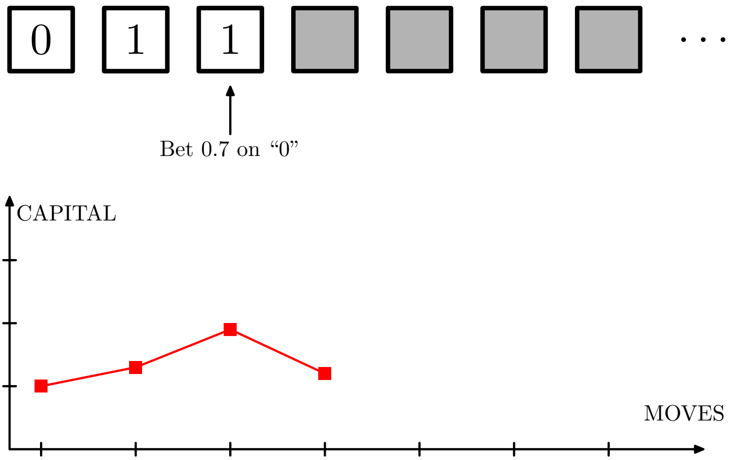
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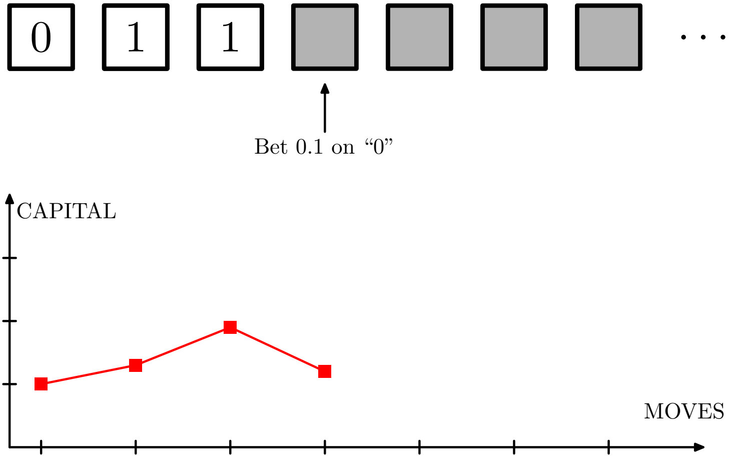
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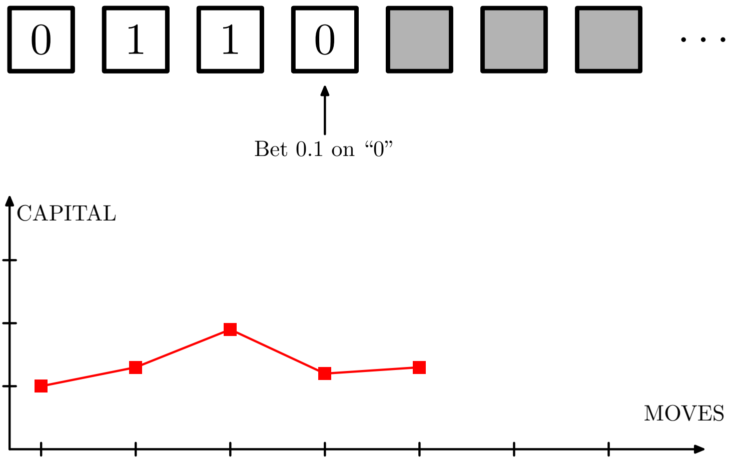
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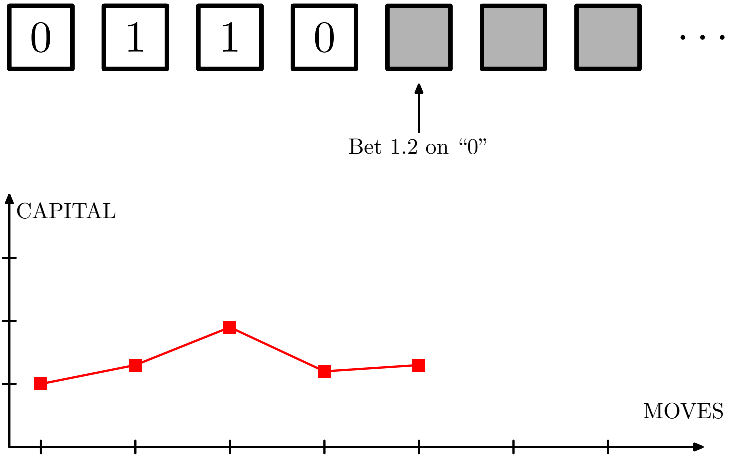
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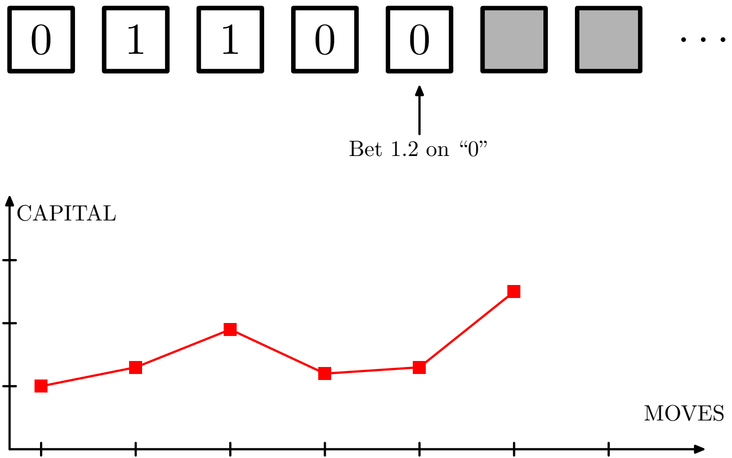
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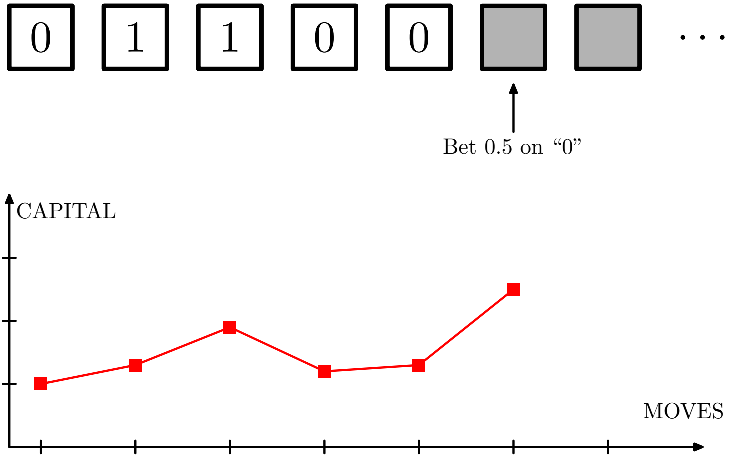
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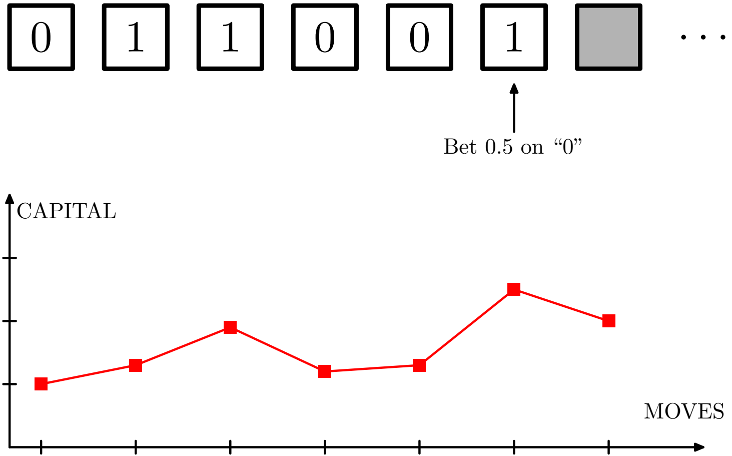
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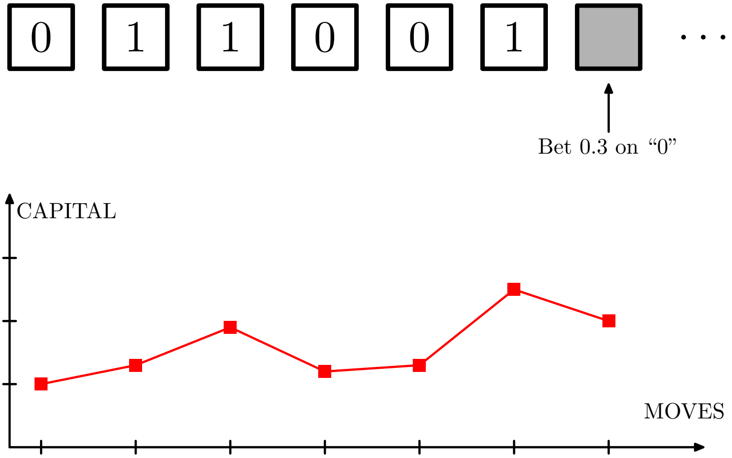
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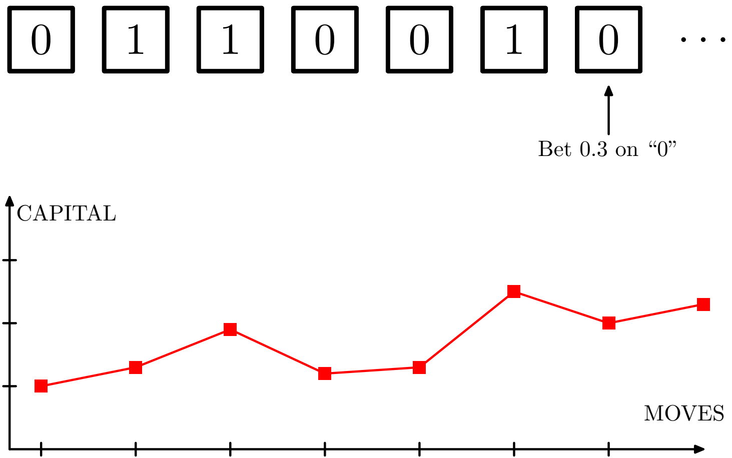
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Computable randomness (1)

A strategy can be thought as a function which maps a finite sequence of bits (the sequence of bits already revealed) to a bet, i.e. a 0/1 guess and a stake (any real number between 0 and current capital). This allows us to talk about computable strategies.

A sequence $X \in 2^\omega$ is computably random if there is no computable strategy which succeeds on X (with succeeds = reaches arbitrarily high values of capital).

Computable randomness (2)

Examples of non-computably random sequences:

- computable sequences
- sequences which do not satisfy the law of large numbers
- sequences such that any prefix contains more 0's than 1's
- sequences X such that $X(i) = 0$ for all i power of 2

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- sequences X such that $X(i) = 0$ for all i power of 2

Computable randomness is significantly weaker than Martin-Löf randomness, but is sufficient to capture most classical probability laws.

Recently, it was shown by Brattka, Miller, Nies that it has strong connections with computable analysis (differentiability).

2. Integer-valued strategies

A more realistic gambling model

In any “real-world” situation involving gambling, the money is discrete, i.e. the values of possible bets are all multiples of an “atomic” value (one cent, one token, etc.).

Therefore, it makes sense to look at integer-valued strategies. We call $\mathbb{N}$-random a sequence X such that no computable integer-valued strategy succeeds on X.

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Does this give a weaker notion?

$\mathbb{N}$-valued strategies are weak (1)

Computable $\mathbb{N}$-valued strategies are much weaker than real-valued ones, in the sense that they can be defeated more easily.

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indeed *n* strategies together can be thought as a single player.

One typically builds computably random reals by diagonalization, adding more and more strategies.

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Suppose we want to defeat an $\mathbb{N}$-valued strategy S which has already seen some finite sequence σ of bits. There are two cases:

- **either** S does not make any bet (= stake 0) on any extension of σ. In that case, S is dormant and is defeated no matter what extension of σ we choose.
- **or** S bets on some extension τ of σ, we pick the shortest such τ, and chose the extension $\tau 0$ if the strategy guesses 1 and the extension $\tau 1$ if the strategy guesses 0. This causes the strategy to lose at least 1.

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We repeat this process. The second case cannot happen infinitely often, so eventually the strategy is either broke or "gives up".

$\mathbb{N}$-randomness and category (1)

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This is very large. For example the set of sequences that satisfy the Law of Large Numbers is meager.

$\mathbb{N}$-randomness and category (2)

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For all n, weakly-$(n+1)$-generic $\Rightarrow$ n-generic $\Rightarrow$ weakly-n-generic.

$\mathbb{N}$-randomness and category (3)

How much genericity is needed to guarantee $\mathbb{N}$-randomness?

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Theorem (BST)

Any weakly-2-generic sequence X is $\mathbb{N}$-random. This does not hold anymore with 1-genericity in place of weak-2-genericity.

$\mathbb{N}$-randomness vs Kurtz randomness (1)

There is another well-known "randomness" notion whose corresponding class of random reals is co-meager: Kurtz randomness. A sequence X is Kurtz random if it belongs to every Σ^0_1 set of measure 1. Since a Σ^0_1 of measure 1 must be dense, any 1-generic sequence is Kurtz random.

So Kurtz random does not imply $\mathbb{N}$-randomness, but does the reverse implication hold?

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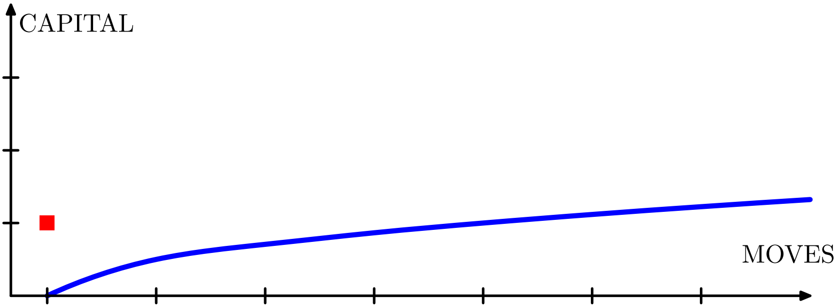
We show that it is not the case.

$\mathbb{N}$-randomness vs Kurtz randomness (2)

To prove this, we use a characterization of Kurtz randomness in terms of strategies.

Theorem (Wang)

A sequence X is ***not*** *Kurtz random if and only if there is a computable real-valued strategy S and a computable non-decreasing unbounded function h such that the capital of S after n bets on X is at least $h(n)$.*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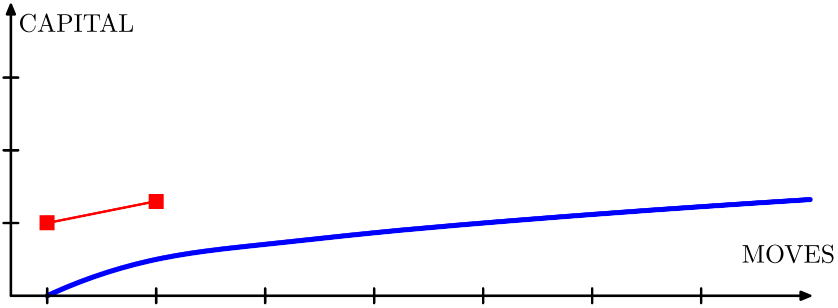
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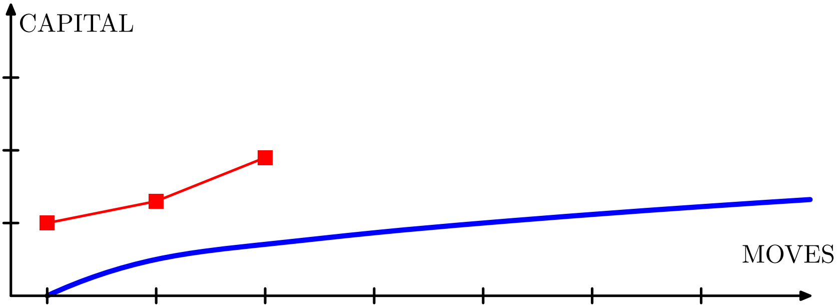
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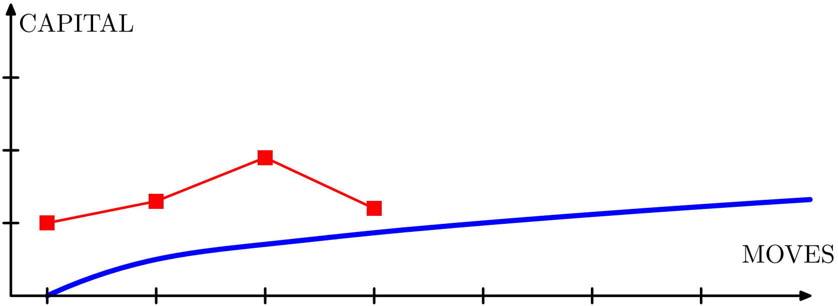
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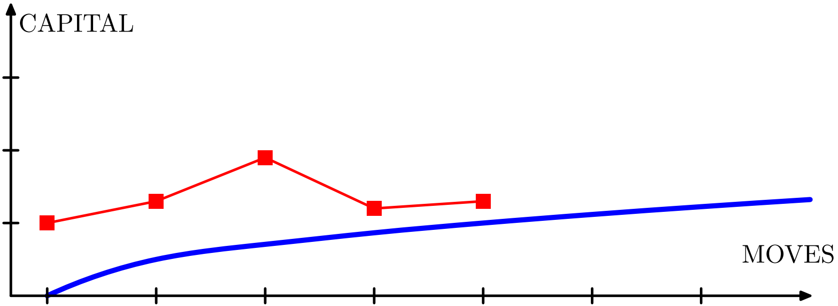
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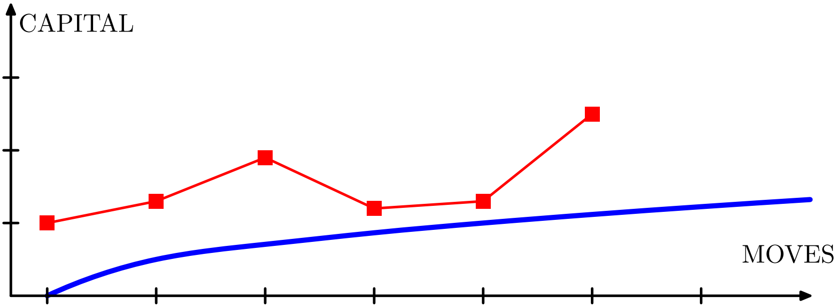
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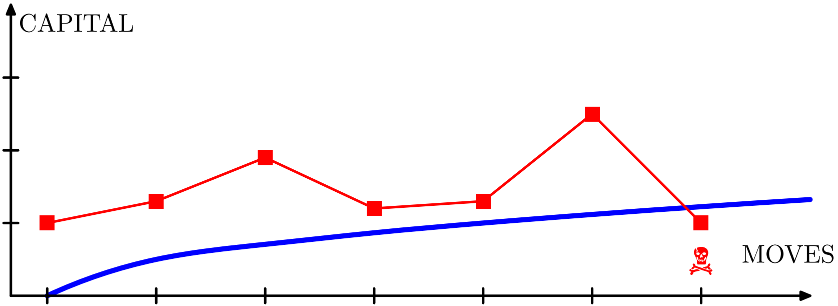
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$\mathbb{N}$-randomness vs Kurtz randomness (4)

It is sufficient to build a computable probability measure μ such that when choosing X at random w.r.t. μ:

- With μ-probability 1, X is $\mathbb{N}$-random
- With μ-probability 1, X is not Kurtz random

Idea: we use generalized Bernoulli measures. These are the measures we get when for all i the i-th bit is chosen at random independently of the others, but with probablity $1/2 + \delta_i$ to be 0.

$\mathbb{N}$-randomness vs Kurtz randomness (3)

A result of Bienvenu and Merkle (after Muchnik): If $\sum_i \delta_i^2 = +\infty$, then with μ-probability 1, X is not Kurtz random. Indeed, the **optimal strategy** which for all i guesses that the i-th bit will be 0 and bets a fraction $2\delta_i$ of its capital will make money steadily, ensuring that X not Kurtz random.

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Also, if a strategy S' **bets too much** on the i-th bit (stake $\gg$ fraction $2\delta_i$ of the capital), **then it loses quickly**.

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Also, if a strategy S' **bets too much** on the i-th bit (stake $\gg$ fraction $2\delta_i$ of the capital), **then it loses quickly**.

Idea: take the δ_i such that $\sum_i \delta_i^2 = +\infty$ but sufficiently small in such a way that any strategy wins very slowly, and hence, for an $\mathbb{N}$-valued strategy, betting 1 or more at stage i is too costly. One can take

$$\delta_i = \left(i \log i\right)^{-1/2}$$

3. Bounding the bets

An even weaker model

In many situations, on top of being discrete, the set of possible bets is bounded (hence finite). If $\mathcal{V}$ is a finite set of integers, we call $\mathcal{V}$-random a sequence X which defeats all computable martingales which are only allowed to bet amounts of money belonging to $\mathcal{V}$.

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Does this too make a difference?

Yes:

Theorem (BST)

For any finite set $\mathcal{V}$, $\mathcal{V}$-randomness is weaker than $\mathbb{N}$-randomness.

$\mathbb{N}$-randomness vs $\mathcal{V}$-randomness (1)

Idea of the proof: we split the set of integers (positions of bits) into consecutive blocks. Start with 1 block of length 1, then 2 blocks of lenght 2, 4 blocks of length 3 and so on, i.e. 2^n intervals of length $n + 1$ for all n in order.

We make sure that $X \in 2^\omega$ satisfies the condition

For any block B, there is an $i \in B$ such that $X(i) = 0$ (*)

$\mathbb{N}$-randomness vs $\mathcal{V}$-randomness (2)

Any X satisfying the condition $(*)$ can be defeated by an $\mathbb{N}$-valued strategy. Use the D'Alembert martingale on each block: first bet $\$1$ on value 0, and every time one loses, double the stake on the next bit. This guarantees a gain of $\$1$ per block, and thus a gain of 2^n on the set of blocks of length $n+1$. This is enough to ensure that the strategy has enough money to play the D'Alembert martingales on blocks of length $n+2$.

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However, a strategy which is only allowed bounded bets cannot use the D'Alembert martingale, and in fact it is not too hard to diagonalize against all $\mathcal{V}$-valued strategies while enforcing condition $(*)$.

The importance of $\mathcal{V}$ (1)

Ok, putting a bound on the bets makes a difference, but does the set of allowed values matter itself?

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Example: three players are betting on a sequence X.

- Asuka can make bets in $\mathcal{A} = \{0, 1, 2, 3, 4, 5\}$
- Bachiko can make bets in $\mathcal{B} = \{1, 2, 3, 4, 10, 20\}$
- Chihiro can make bets in $\mathcal{C} = \{0, 2, 4, 6, 8, 10, 15\}$

Does any of the players have an advantage?

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Pairs ($\mathcal{A}$,$\mathcal{B}$), ($\mathcal{B}$,$\mathcal{C}$) are not clear. However, one can see immediately that Chihiro has an advantage over Asuka. The reason is that $2\mathcal{A} \subseteq \mathcal{B}$, so Chihiro can just copy Asuka's strategy, just making bets twice bigger (and even has the opportunity to use the 15 option; if we remove this option then the rules of Asuka and Chihiro are "equivalent").

The importance of $\mathcal{V}$ (2)

This observation gives us directly:

Lemma

If for some $k \in \mathbb{Q}^{>0}$ we have $(k \cdot \mathcal{A}) \subseteq \mathcal{B}$, then $\mathcal{B}$-randomness implies $\mathcal{A}$-randomness.

The importance of $\mathcal{V}$ (3)

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Theorem (CDFT)

If $\mathcal{B}$-randomness implies $\mathcal{A}$-randomness, then there is some $k \in \mathbb{Q}^{>0}$ such that $(k \cdot \mathcal{A}) \subseteq \mathcal{B}$.

The importance of $\mathcal{V}$ (4)

Proof sketch. Suppose that $(k \cdot \mathcal{A}) \not\subseteq \mathcal{B}$ for any k. We want to show that there is an X which can be defeated by a computable $\mathcal{A}$-strategy S but no computable $\mathcal{B}$-strategy (in fact, we can diagonalize against any countable set of $\mathcal{B}$-strategies).

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Proof sketch. Suppose that $(k \cdot \mathcal{A}) \not\subseteq \mathcal{B}$ for any k. We want to show that there is an X which can be defeated by a computable $\mathcal{A}$-strategy S but no computable $\mathcal{B}$-strategy (in fact, we can diagonalize against any countable set of $\mathcal{B}$-strategies).

Think of a situation where S is the strategy of a stooge player, who works in collusion with the casino which provides the sequence X.

Let $\mathcal{A} = \{a_1, a_2, ..., a_n\}$. The strategy S will be simple: bet $\$a_1$ on 0, then $\$a_2$, $\$a_3$, ..., $\$a_n$, and repeat. This is clearly computable!!

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  - if $k' \geq k$, the casino output a 0 (both S and M win some money, but S **wins more proportionally to its capital**)
  - if $k' < k$, the casino output a 1 (both S and M lose some money, but M **loses more proportionally to its capital**)

The importance of $\mathcal{V}$ (5)

During phase 1, the ratio $(\text{capital of } S)/(\text{capital of } M)$ is non-decreasing over time. One can show by contradiction that in fact it has to tend to $+\infty$. If not, it tends to some k^*. But then, the best strategy for M to prevent the increase of $(\text{capital of } S)/(\text{capital of } M)$ would be to multiply each bet of S by $1/k^*$. By assumption, this is not possible.

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When this ratio becomes large enough, then the casino enters **phase 2**, where it will try to make *M* lose money no matter how much it costs *S*. Since *S* is sufficiently wealthy at the beginning of this phase, it will “survive” until the end, i.e., when *M* is defeated.

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To complete the proof, a bit more care is needed: how to play against different players, how to handle the case where players appear to be dormant (stops betting) but could wake up later, etc.

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Does $\mathbb{N}$-randomness has interesting interactions with computability theory? (e.g. what is lowness for $\mathbb{N}$-randomness?)

Can the separation of $\mathcal{A}$-randomness and $\mathcal{B}$-randomness always be witnessed by a 1-generic?

Thank you

どうもありがとうございました