GTP 2012

Fourth Workshop on Game-Theoretic Probability and Related Topics

Imprecise multinomial processes

an overview of different approaches and how they are related to each other

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13 November 2012

What is an imprecise multinomial process?

A sequence of random variables

$$X_1, X_2, \dots, X_n, \dots$$

each assuming values in the same **finite** set

$$\mathscr{X} = \{H, T\}$$

RUNNING EXAMPLE

$$\{1, 2, 3, 4, 5, 6\}$$

What is an imprecise multinomial process?

A sequence of random variables

$$X_1, X_2, \ldots, X_n, \ldots$$

satisfying the **IID** property

INDEPENDENT

IDENTICALLY **D**ISTRIBUTED

What is an imprecise multinomial process?

A sequence of random variables

$$X_1 ?, X_2, \ldots, X_n, \ldots$$

satisfying the **IID** property

INDEPENDENT

IDENTICALLY **D**ISTRIBUTED ?

Modelling a single variable

How to model a single random variable?

The **precise** approach: **probability mass function** / **prevision**

probability mass function **p**

EQUIVALENT

$$\forall x \in \mathscr{X} \quad \mathbf{p}(x) \geq 0$$

$$\sum_{x \in \mathscr{X}} \mathbf{p}(x) = 1$$

prevision **P** (expectation operator)

$$\forall \, \mathbf{f} : \mathscr{X} \longrightarrow \mathbb{R}$$

$$\mathbf{P}(\mathbf{f}) = \sum_{x \in \mathscr{X}} \mathbf{p}(x)\mathbf{f}(x)$$

$$\mathbf{P}(\mathbf{f}) \geq \min \mathbf{f}$$

$$\mathbf{P}(\mathbf{f}_1 + \mathbf{f}_2) = \mathbf{P}(\mathbf{f}_1) + \mathbf{P}(\mathbf{f}_1)$$

$$\mathbf{P}(\lambda \mathbf{f}) = \lambda \mathbf{P}(\mathbf{f})$$

How to model a single random variable?

The **precise** approach: **probability mass function** / **prevision**

probability mass function **p**

$\forall x \in \mathcal{X} \quad \mathbf{p}(x) \geq 0$

$$\sum_{x \in \mathcal{X}} \mathbf{p}(x) = 1$$

EQUIVALENT

prevision **P** (expectation operator)

$\forall f : \mathcal{X} \longrightarrow \mathbb{R}$

$$\mathbf{P}(\mathbf{f}) = \sum_{x \in \mathcal{X}} \mathbf{p}(x)\mathbf{f}(x)$$

EXAMPLE: $\mathcal{X} = \{H,T\}$

$\mathbf{p}(H) = 4/10$

$\mathbf{p}(T) = 6/10$

$\mathbf{P}(\mathbf{f}) = 4/10\ \mathbf{f}(H) + 6/10\ \mathbf{f}(T)$

$I_H(H) = 1,\ I_H(T) = 0$

$\Rightarrow \mathbf{P}(I_H) = 4/10 = \mathbf{p}(H)$

$\mathbf{f}(H) = -1,\ \mathbf{f}(T) = 3$

$\Rightarrow \mathbf{P}(\mathbf{f}) = 1{,}4$

How to model a single random variable?

An **imprecise** approach: **credal set** / **coherent lower prevision** **[1]**

credal set $\mathcal{M}$

EQUIVALENT

closed and convex
set of probability
mass functions

coherent lower prevision $\underline{\mathbf{P}}$

$\forall\, \mathbf{f} : \mathscr{X} \longrightarrow \mathbb{R}$

$\underline{\mathbf{P}}(\mathbf{f}) = \min\{\mathbf{P}(\mathbf{f}) : \mathbf{p} \in \mathcal{M}\}$

COHERENCE:

$\underline{\mathbf{P}}(\mathbf{f}) \geq \min \mathbf{f}$

$\underline{\mathbf{P}}(\mathbf{f}_1+\mathbf{f}_2) \geq \underline{\mathbf{P}}(\mathbf{f}_1) + \underline{\mathbf{P}}(\mathbf{f}_1)$

$\underline{\mathbf{P}}(\lambda\mathbf{f}) = \lambda\underline{\mathbf{P}}(\mathbf{f})$

≥ 0

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EQUIVALENT

closed and convex **set** of probability mass functions

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EXAMPLE: $\mathscr{X} = \{H, T\}$

$$\mathbf{p}(H) = \theta \in [1/4, 1/2]$$

$$\mathbf{p}(T) = 1 - \theta$$

$$\underline{\mathbf{P}}(\mathbf{f}) = \min_{\theta \in [1/4,\, 1/2]} \{\theta \mathbf{f}(H) + (1-\theta)\mathbf{f}(T)\}$$

$$I_H(H) = 1,\ I_H(T) = 0$$

$$\Rightarrow \underline{\mathbf{P}}(I_H) = 1/4 = \underline{\mathbf{p}}(H)$$

$$\mathbf{f}(H) = -1,\ \mathbf{f}(T) = 3$$

$$\Rightarrow \underline{\mathbf{P}}(\mathbf{f}) = 1$$

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EQUIVALENT

coherent upper prevision $\overline{\mathbf{P}}$

EQUIVALENT

closed and convex **set** of probability mass functions

$$\forall\, \mathbf{f} : \mathscr{X} \longrightarrow \mathbb{R}$$

$$\overline{\mathbf{P}}(\mathbf{f}) = \max\{\mathbf{P}(\mathbf{f}) : \mathbf{p} \in \mathcal{M}\}$$

coherent lower prevision $\underline{\mathbf{P}}$

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EXAMPLE: $\mathcal{X} = \{H, T\}$

$\underline{\mathbf{P}}(\mathbf{I}_H) = 1/4$, $\overline{\mathbf{P}}(\mathbf{I}_H) = 1/2$

$\underline{\mathbf{P}}(\mathbf{f}) = 1$, $\overline{\mathbf{P}}(\mathbf{f}) = 2$

How to model a single random variable?

An **imprecise** approach: **credal set** / **coherent lower prevision** [1]

credal set $\mathcal{M}$ ⟷ EQUIVALENT ⟷ coherent upper prevision $\overline{\mathbf{P}}$

EQUIVALENT

closed and convex **set** of probability mass functions

$\forall\, \mathbf{f} : \mathscr{X} \longrightarrow \mathbb{R}$

$$\overline{\mathbf{P}}(\mathbf{f}) = \max\{\mathbf{P}(\mathbf{f}) : \mathbf{p} \in \mathcal{M}\}$$
$$= \max\{-\mathbf{P}(-\mathbf{f}) : \mathbf{p} \in \mathcal{M}\}$$
$$= -\min\{\mathbf{P}(-\mathbf{f}) : \mathbf{p} \in \mathcal{M}\}$$
$$= -\underline{\mathbf{P}}(-\mathbf{f})$$

coherent lower prevision $\underline{\mathbf{P}}$

EQUIVALENT

$\forall\, \mathbf{f} : \mathscr{X} \longrightarrow \mathbb{R}$

$$\underline{\mathbf{P}}(\mathbf{f}) = \min\{\mathbf{P}(\mathbf{f}) : \mathbf{p} \in \mathcal{M}\}$$

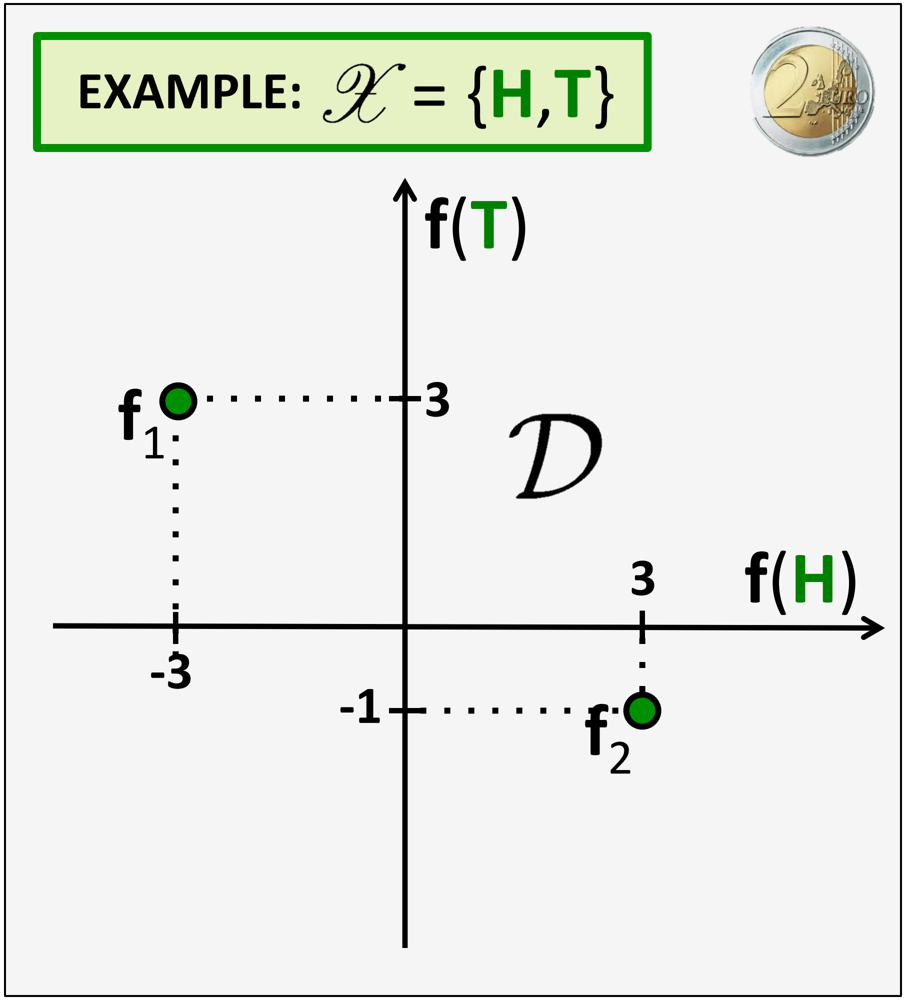
We will focus on **lower** previsions!

How to model a single random variable?

An **imprecise** approach: **coherent set of desirable gambles** [1]

A coherent set of desirable gambles $\mathcal{D}$

We model a **subject's beliefs** about a variable by looking at the **gambles he is willing to accept** on its value



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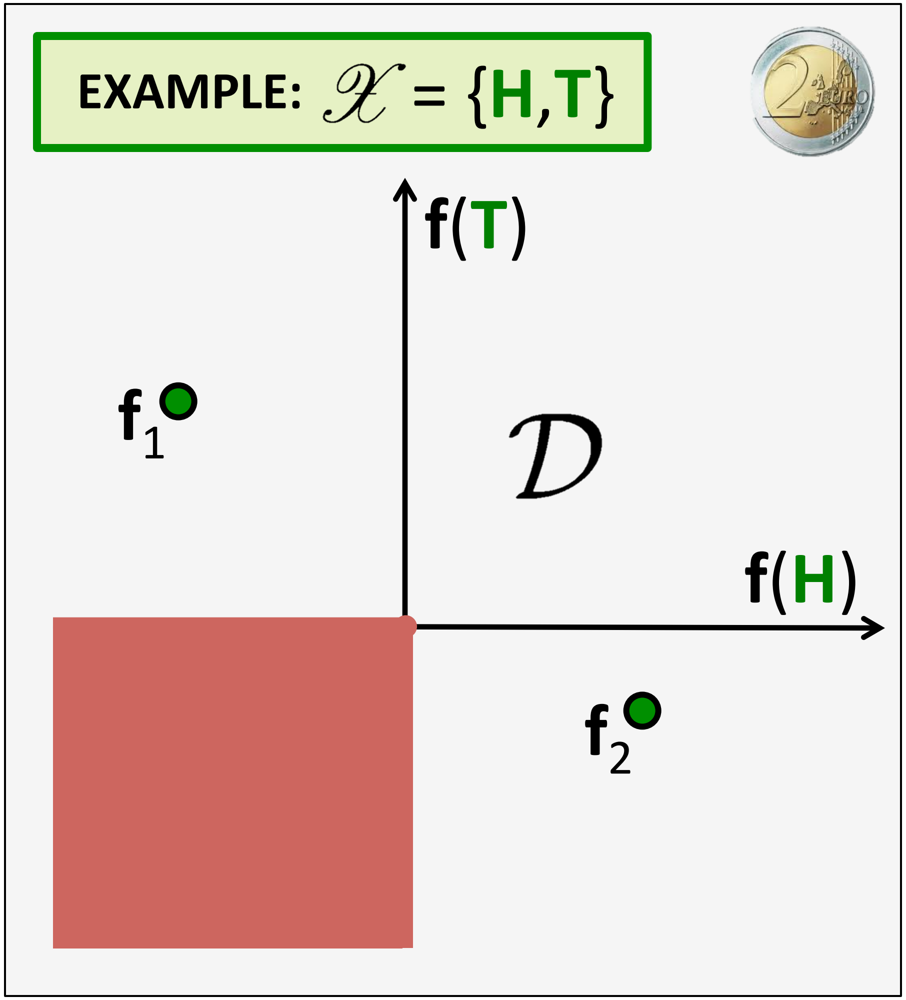
COHERENCE:

$$\mathbf{f} \leq 0 \quad \Rightarrow \quad \mathbf{f} \notin \mathcal{D}$$



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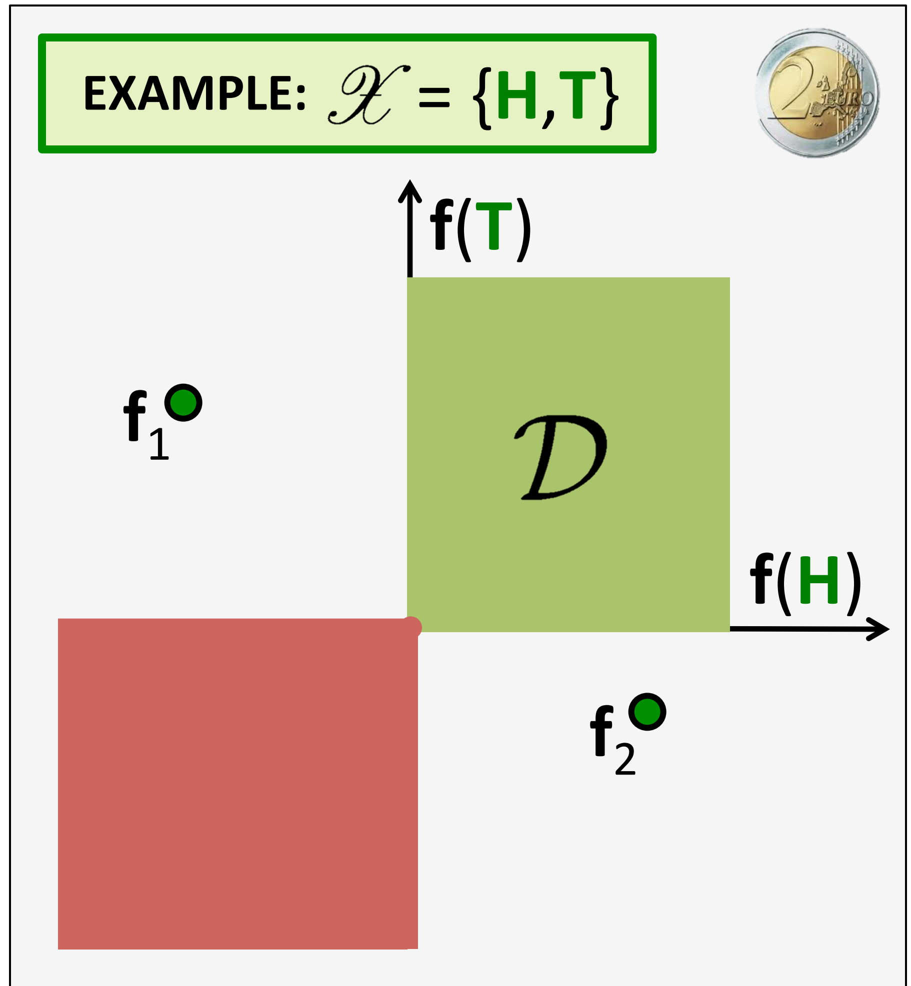
$$\mathbf{f} > 0 \quad \Rightarrow \quad \mathbf{f} \in \mathcal{D}$$



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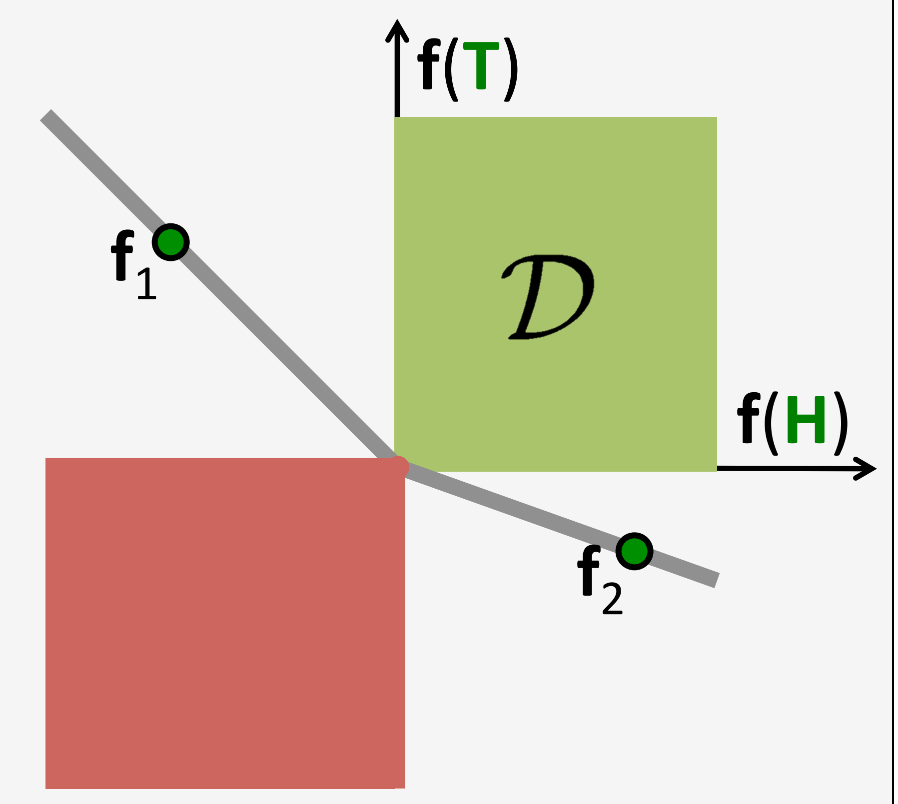

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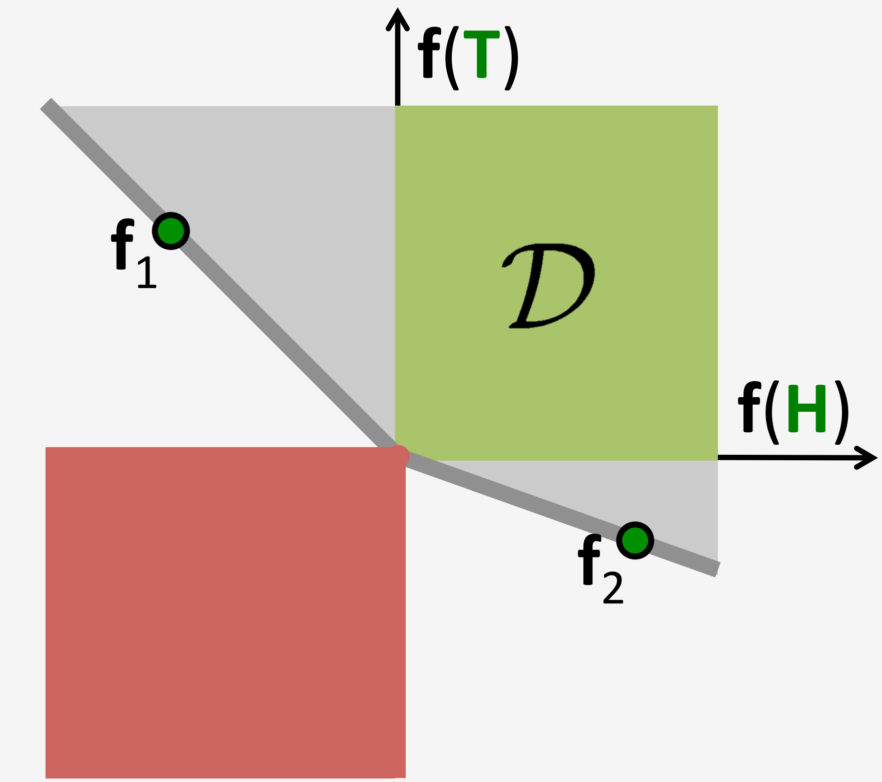
$$\mathbf{f} > 0 \Rightarrow \mathbf{f} \in \mathcal{D}$$
$$\mathbf{f} \in \mathcal{D} \Rightarrow \lambda\mathbf{f} \in \mathcal{D} \quad (\lambda > 0)$$
$$\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{D} \Rightarrow \mathbf{f}_1 + \mathbf{f}_2 \in \mathcal{D}$$

EXAMPLE: $\mathscr{X} = \{H, T\}$



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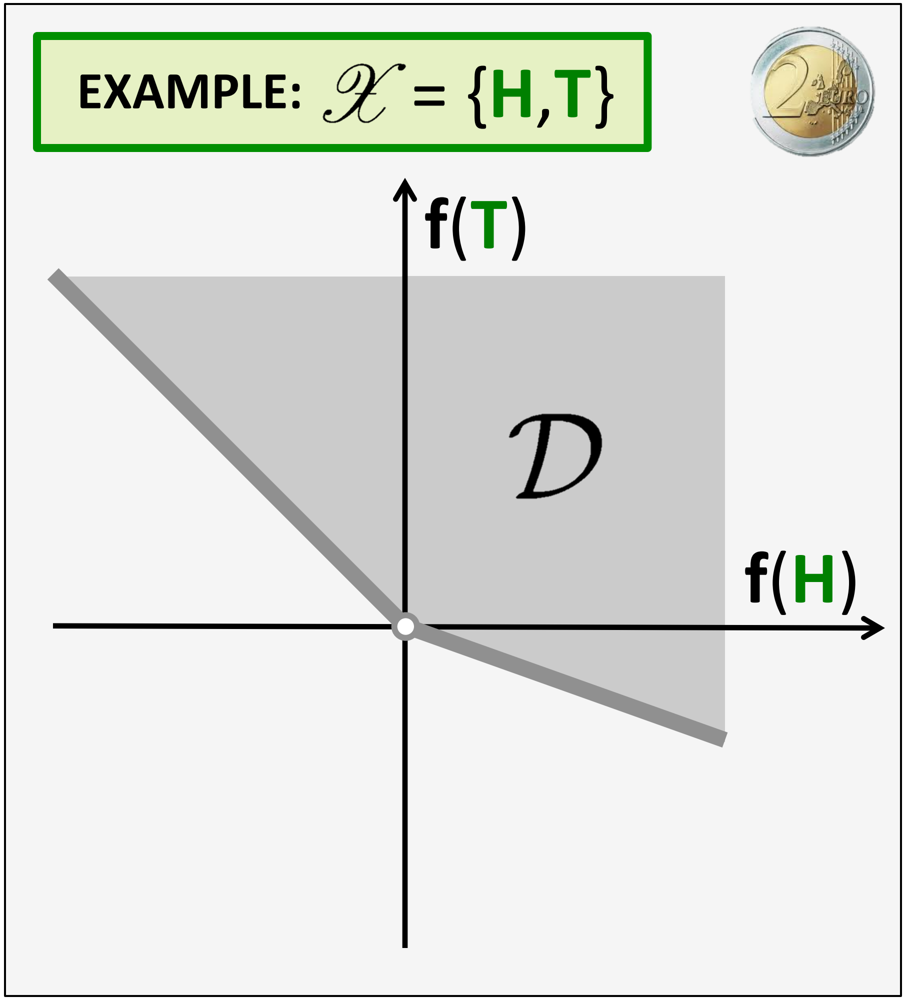

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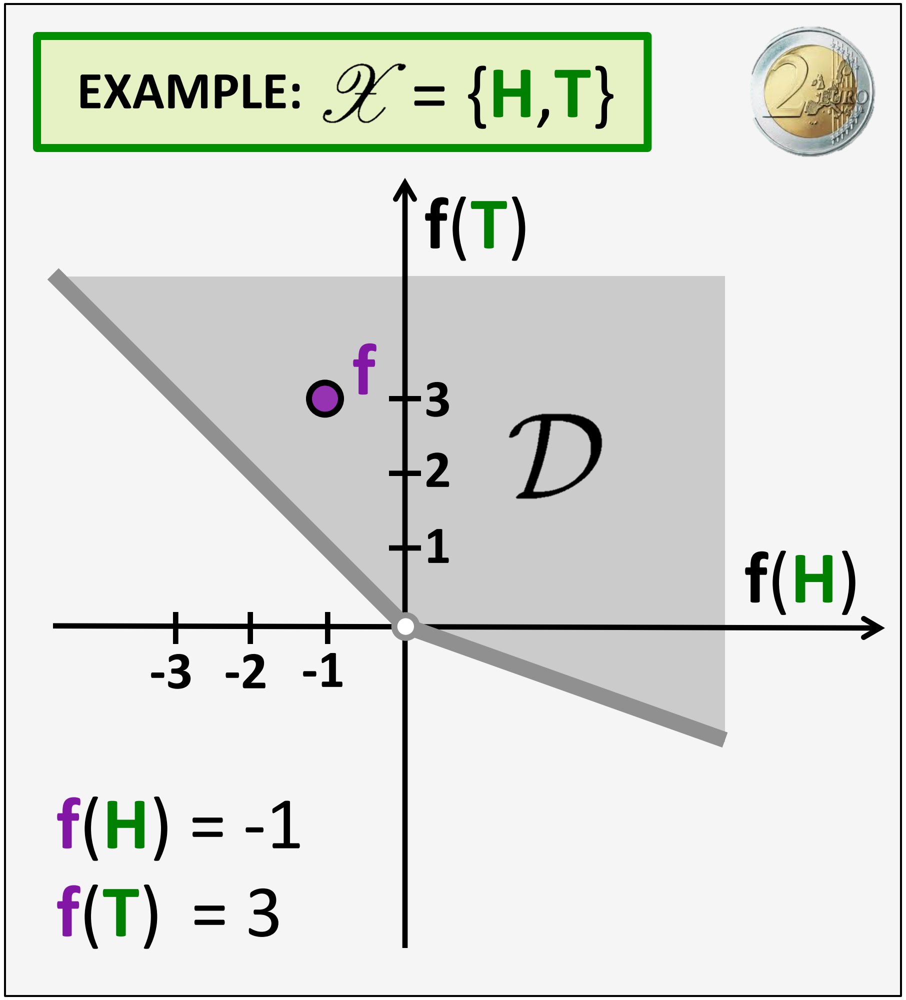
$\mathbf{f} \in \mathcal{D}$



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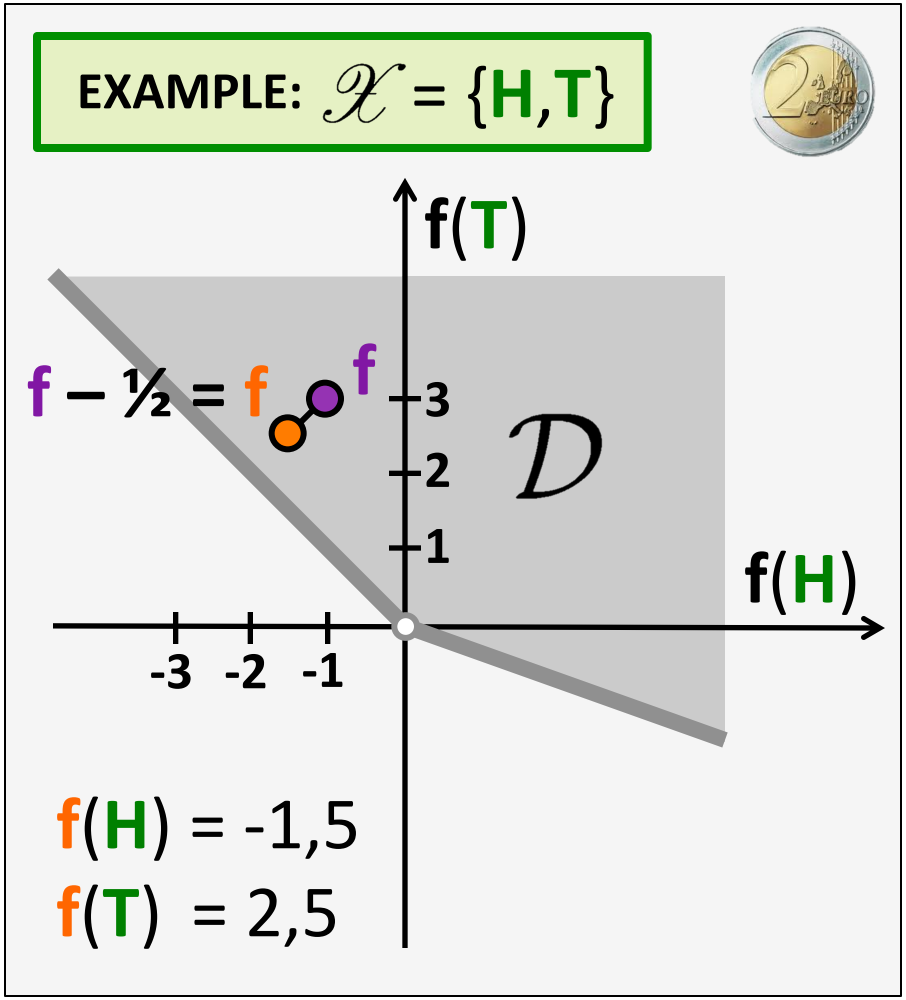
We model a **subject's beliefs** about a variable by looking at the **gambles he is willing to accept** on its value

buying price $\boldsymbol{\mu}$

$$\mathbf{f} - \boldsymbol{\mu} \in \mathcal{D}$$



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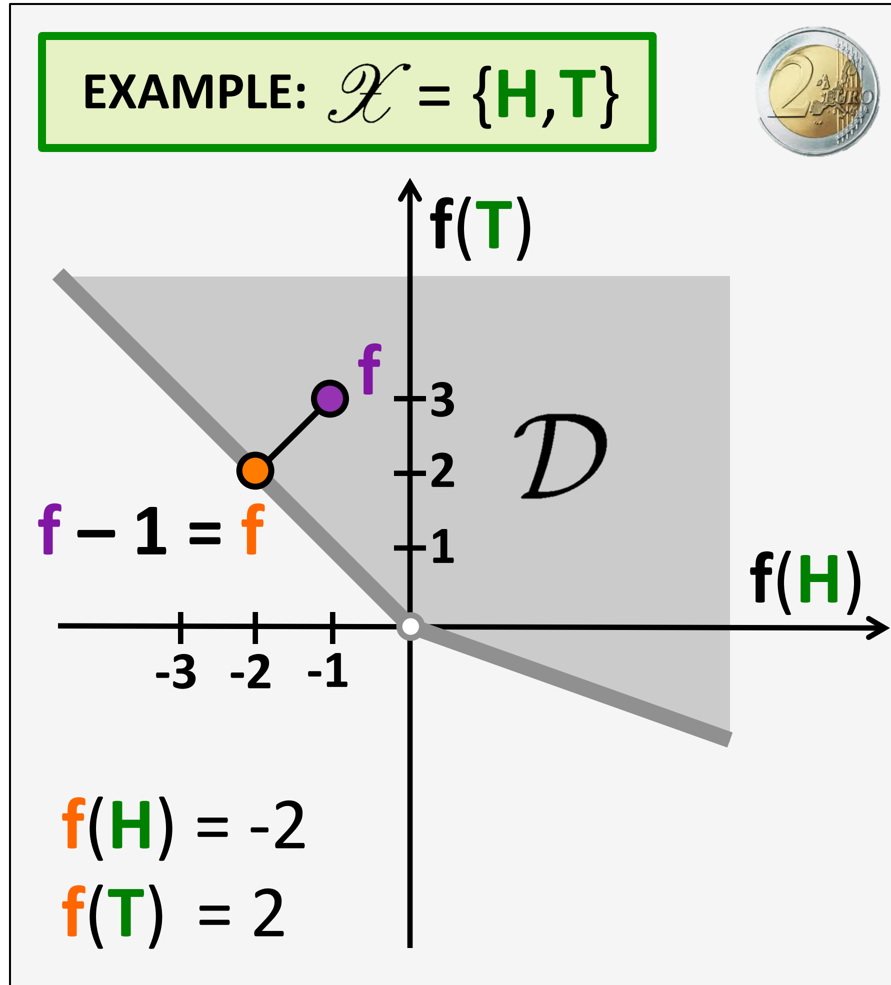
supremum buying price

$$\sup\{\boldsymbol{\mu} : \mathbf{f} - \boldsymbol{\mu} \in \mathcal{D}\}$$



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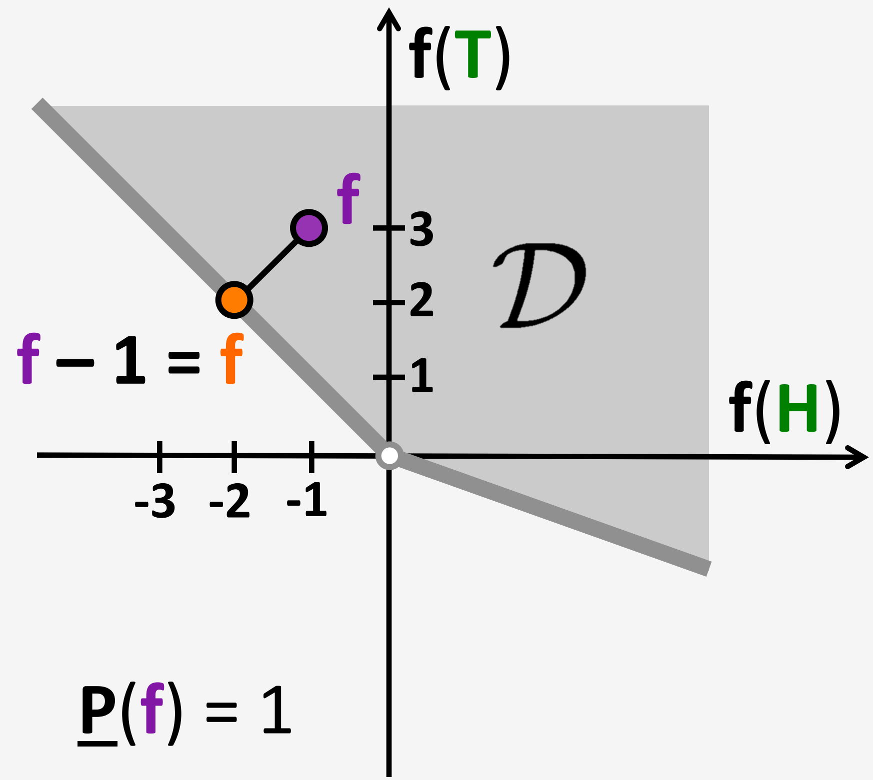
We model a **subject's beliefs** about a variable by looking at the **gambles he is willing to accept** on its value

coherent lower prevision $\underline{\mathbf{P}}$

supremum buying price

$$\underline{\mathbf{P}}(\mathbf{f}) = \sup\{\boldsymbol{\mu} : \mathbf{f} - \boldsymbol{\mu} \in \mathcal{D}\}$$

EXAMPLE: $\mathscr{X} = \{\mathbf{H},\mathbf{T}\}$



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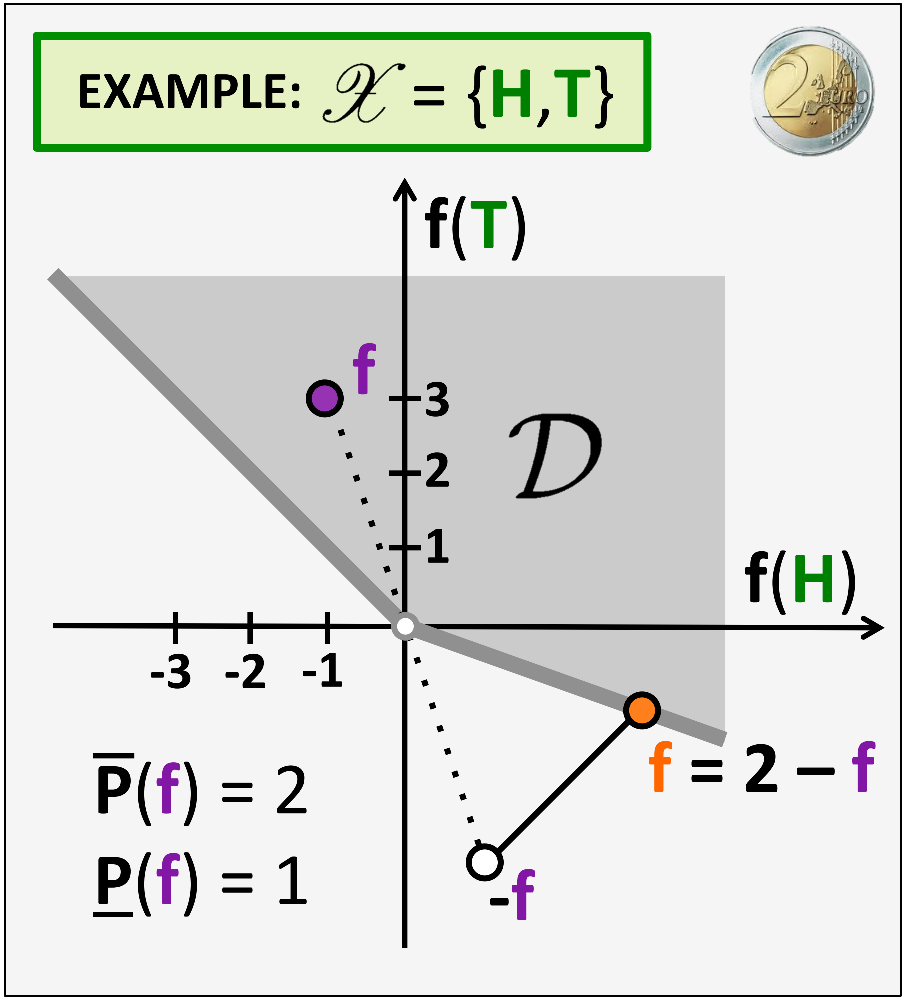
coherent upper prevision $\overline{\mathbf{P}}$

infimum selling price

$$\overline{\mathbf{P}}(\mathbf{f}) = \inf\{\boldsymbol{\mu} : \boldsymbol{\mu} - \mathbf{f} \in \mathcal{D}\}$$



Precise multinomial process

The precise multinomial process

A sequence of random variables

$$X_1, X_2, \ldots, X_n, \ldots$$

satisfying the **IID** property

INDEPENDENT

IDENTICALLY **D**ISTRIBUTED

The precise multinomial process

A sequence of random variables

$$X_1, X_2, \dots, X_n, \dots$$

The precise multinomial process

A sequence of random variables

$$X_1, X_2, \dots, X_n, \dots$$

|| TIME CONSISTENCY

$$X_1, X_2, \dots, X_n, \dots, X_m, \dots$$

MARGINALISATION

The precise multinomial process

A sequence of random variables

$$X_1, X_2, \ldots, X_n$$

The precise multinomial process

A sequence of random variables

$$X_1, X_2, X_3$$

The precise multinomial process

$$X_1, X_2, X_3$$

$p_1 \cdot p_2 \cdot p_3 = p_{1,2,3}$ **INDEPENDENT**

$\parallel \quad \parallel \quad \parallel$

$p \quad p \quad p$ **IDENTICALLY DISTRIBUTED**

The precise multinomial process

$$\begin{matrix} x_1 & x_2 & x_3 \\ \shortparallel & \shortparallel & \shortparallel \\ X_1, & X_2, & X_3 \end{matrix}$$

$$p_1(x_1) \cdot p_2(x_2) \cdot p_3(x_3) = p_{1,2,3}(x_1,x_2,x_3)$$

$$\begin{matrix} \shortparallel & \shortparallel & \shortparallel \\ p(x_1) & p(x_2) & p(x_3) \end{matrix}$$

$$\forall\, f : \mathscr{X}^3 \longrightarrow \mathbb{R}$$

$$P_{1,2,3}(f) = \sum_{x_1 \in \mathscr{X}} \sum_{x_2 \in \mathscr{X}} \sum_{x_3 \in \mathscr{X}} p_{1,2,3}(x_1,x_2,x_3) f(x_1,x_2,x_3)$$

The precise multinomial process

$$\begin{array}{ccc} \text{H} & \text{T} & \text{T} \\ \| & \| & \| \\ X_1, & X_2, & X_3 \end{array}$$

EXAMPLE: $\mathscr{X} = \{H,T\}$

$\mathbf{p}(H) = 4/10$, $\mathbf{p}(T) = 6/10$

$$\mathbf{p}_1(H) \cdot \mathbf{p}_2(T) \cdot \mathbf{p}_3(T) = \mathbf{p}_{1,2,3}(H,T,T)$$

$$\begin{array}{ccc} \| & \| & \| \\ p(H) & p(T) & p(T) \end{array}$$

$= 0{,}144$

$A = \{(H,H,H),(H,T,T)\}$

$$\mathbf{P}_{1,2,3}(I_A) = \mathbf{p}_{1,2,3}(H,H,H) + \mathbf{p}_{1,2,3}(H,T,T) = 0{,}208$$

Forward irrelevant multinomial process

The forward irrelevant multinomial process

An **interpretation** for the precise multinomial process

$$X_1, \quad X_2, \quad X_3$$

IDENTICALLY DISTRIBUTED

$$\mathbf{p}(X_1) \qquad \mathbf{p}(X_2) \qquad \mathbf{p}(X_3)$$

$$\parallel \qquad \parallel \qquad \parallel$$

INDEPENDENT

$$\mathbf{p}_1(X_1) \cdot \mathbf{p}_2(X_2) \cdot \mathbf{p}_3(X_3) = \mathbf{p}_{1,2,3}(X_1,X_2,X_3)$$

The forward irrelevant multinomial process

An **interpretation** for the precise multinomial process

$$X_1, \quad X_2, \quad X_3$$

IDENTICALLY DISTRIBUTED

$$\mathbf{p}(X_1) \qquad \mathbf{p}(X_2) \qquad \mathbf{p}(X_3)$$

$$\| \qquad \| \qquad \|$$

INDEPENDENT

$$\mathbf{p}_1(X_1) \cdot \mathbf{p}_2(X_2) \cdot \mathbf{p}_3(X_3) = \mathbf{p}_{1,2,3}(X_1,X_2,X_3)$$

$$\mathbf{p}_1(X_1) \cdot \mathbf{p}_2(X_2|X_1) \cdot \mathbf{p}_3(X_3|X_1,X_2) = \mathbf{p}_{1,2,3}(X_1,X_2,X_3)$$

The forward irrelevant multinomial process

An **interpretation** for the precise multinomial process

$$X_1, \quad X_2, \quad X_3$$

IDENTICALLY DISTRIBUTED

INDEPENDENT

$\mathbf{p}(X_1)$	$\mathbf{p}(X_2)$	$\mathbf{p}(X_3)$
$=$	$=$	$=$
$\mathbf{p}_1(X_1)$	$\mathbf{p}_2(X_2)$	$\mathbf{p}_3(X_3)$
$=$	$=$	$=$

$$\mathbf{p}_1(X_1) \cdot \mathbf{p}_2(X_2|X_1) \cdot \mathbf{p}_3(X_3|X_1,X_2) = \mathbf{p}_{1,2,3}(X_1,X_2,X_3)$$

The value of previous variables is **irrelevant** for our beliefs about the current one !

The forward irrelevant multinomial process

An **interpretation** for the precise multinomial process

$X_1, \quad X_2, \quad X_3$

IDENTICALLY DISTRIBUTED

INDEPENDENT

X_1	X_2	X_3
$P(\ )$	$P(\ )$	$P(\ )$
=	=	=
$P_1(\ )$	$P_2(\ )$	$P_3(\ )$
=	=	=
$P_1(\ )$	$P_2(\ \mid X_1)$	$P_3(\ \mid X_1, X_2)$

$P_1(\ )$? $P_2(\ \mid X_1)$? $P_3(\ \mid X_1, X_2)$? $P_{1,2,3}(\ )$

The value of previous variables is **irrelevant** for our beliefs about the current one !

The forward irrelevant multinomial process

An **interpretation** for the precise multinomial process

$$X_1, \quad X_2, \quad X_3$$

IDENTICALLY DISTRIBUTED

INDEPENDENT

$P(\ )$	$P(\ )$	$P(\ )$
$=$	$=$	$=$
$\mathbf{P}_1(\ )$	$\mathbf{P}_2(\ )$	$\mathbf{P}_3(\ )$
$=$	$=$	$=$
$\mathbf{P}_1(\ )$	$\mathbf{P}_2(\ \vert X_1)$	$\mathbf{P}_3(\ \vert X_1,X_2)$

$$\mathbf{P}_{1,2,3}(\,\mathbf{f}(X_1,X_2,X_3)\,) = \mathbf{P}_1(\,\mathbf{P}_2(\,\mathbf{P}_3(\,\mathbf{f}(X_1,X_2,X_3)\,|\,X_1,X_2)\,|\,X_1)\,)$$

$$= P\,(\,P\,(\,P\,(\,\mathbf{f}(X_1,X_2,X_3)\qquad)\qquad)\,)$$

The forward irrelevant multinomial process

Described using **coherent lower previsions**

$$X_1, \quad X_2, \quad X_3$$

IDENTICALLY DISTRIBUTED

FORWARD IRRELEVANCE

$$\underline{P}(\ ) \qquad \underline{P}(\ ) \qquad \underline{P}(\ )$$

$$\parallel \qquad \parallel \qquad \parallel$$

$$\underline{P}_1(\ ) \qquad \underline{P}_2(\ ) \qquad \underline{P}_3(\ )$$

$$\parallel \qquad \parallel \qquad \parallel$$

$$\underline{P}_1(\ ) \quad ? \quad \underline{P}_2(\ |X_1) \quad ? \quad \underline{P}_3(\ |X_1,X_2) \quad ? \quad \underline{P}_{1,2,3}(\ )$$

The value of previous variables is **irrelevant** for our beliefs about the current one !

The forward irrelevant multinomial process

Described using **coherent lower previsions** **[2]**

$X_1, \quad X_2, \quad X_3$

IDENTICALLY DISTRIBUTED

FORWARD IRRELEVANCE

$\underline{P}(\ ) \quad \underline{P}(\ ) \quad \underline{P}(\ )$

$\| \quad \| \quad \|$

$\underline{P}_1(\ ) \quad \underline{P}_2(\ ) \quad \underline{P}_3(\ )$

$\| \quad \| \quad \|$

$\underline{P}_1(\ ) \quad \underline{P}_2(\ |X_1) \quad \underline{P}_3(\ |X_1,X_2)$

$$\underline{P}_{1,2,3}(\, \mathbf{f}(X_1,X_2,X_3)\,) = \underline{P}_1(\, \underline{P}_2(\, \underline{P}_3(\, \mathbf{f}(X_1,X_2,X_3) | X_1,X_2) \,|X_1)\,)$$

$$= \underline{P}\,(\, \underline{P}\,(\, \underline{P}\,(\, \mathbf{f}(X_1,X_2,X_3) \qquad) \qquad)\,)$$

The forward irrelevant multinomial process

Described using **coherent lower previsions**

EXAMPLE: $\mathscr{X} = \{H,T\}$

$$\underline{P}(f) = \min_{\theta \in [1/4,\, 1/2]} \{\theta f(H) + (1-\theta) f(T)\}$$

$$A = \{(H,H,H),(H,T,T)\}$$

$$\underline{P}_{1,2,3}(\, I_A(X_1,X_2,X_3)\,) = ?$$

$$\underline{P}_3(\, I_A(H,H,X_3)\,) = 1/4$$

$$I_A(H,H,H) = 1$$

$$I_A(H,H,T) = 0$$

$$\underline{P}_{1,2,3}(\, f(X_1,X_2,X_3)\,) = \underline{P}_1(\, \underline{P}_2(\, \underline{P}_3(\, f(X_1,X_2,X_3) \,|\, X_1,X_2) \,|\, X_1)\,)$$
$$= \underline{P}(\, \underline{P}(\, \underline{P}(\, f(X_1,X_2,X_3)\,)\,)\,)$$

The forward irrelevant multinomial process

Described using **coherent lower previsions**

EXAMPLE: $\mathscr{X} = \{H,T\}$

$$\underline{P}(f) = \min_{\theta \in [1/4,\, 1/2]} \{\theta f(H) + (1-\theta) f(T)\}$$

$$A = \{(H,H,H),(H,T,T)\}$$

$$\underline{P}_{1,2,3}(\, I_A(X_1,X_2,X_3)\,) = ?$$

$$\underline{P}_3(\, I_A(H,H,X_3)\,) = 1/4$$

$$\underline{P}_3(\, I_A(H,T,X_3)\,) = 1/2$$

$$\underline{P}_3(\, I_A(T,H,X_3)\,) = 0$$

$$\underline{P}_3(\, I_A(T,T,X_3)\,) = 0$$

$$\underline{P}_{1,2,3}(\, f(X_1,X_2,X_3)\,) = \underline{P}_1(\, \underline{P}_2(\, \underline{P}_3(\, f(X_1,X_2,X_3) \,|\, X_1,X_2) \,|\, X_1)\,)$$

$$= \underline{P}(\, \underline{P}(\, \underline{P}(\, f(X_1,X_2,X_3) \,)\,)\,)$$

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Described using **coherent lower previsions**

EXAMPLE: $\mathscr{X} = \{H,T\}$

$$\underline{P}(f) = \min_{\theta \in [1/4,\, 1/2]} \{\theta f(H) + (1-\theta) f(T)\}$$

$$A = \{(H,H,H),(H,T,T)\}$$

$$\underline{P}_{1,2,3}(\, I_A(X_1,X_2,X_3)\,) = ?$$

$$\underline{P}_3(\, I_A(H,H,X_3)\,) = 1/4$$

$$\underline{P}_3(\, I_A(H,T,X_3)\,) = 1/2$$

$$\underline{P}_2(\, \underline{P}_3(\, I_A(H,X_2,X_3)\,)\,) = 3/8$$

$$\underline{P}_{1,2,3}(\, f(X_1,X_2,X_3)\,) = \underline{P}_1(\, \underline{P}_2(\, \underline{P}_3(\, f(X_1,X_2,X_3) \,|\, X_1,X_2) \,|\, X_1)\,)$$

$$= \underline{P}(\, \underline{P}(\, \underline{P}(\, f(X_1,X_2,X_3)\,)\,)\,)$$

The forward irrelevant multinomial process

Described using **coherent lower previsions**

EXAMPLE: $\mathscr{X} = \{H,T\}$

$$\underline{P}(\mathbf{f}) = \min_{\theta \in [1/4,\, 1/2]} \{\theta \mathbf{f}(H) + (1-\theta)\mathbf{f}(T)\}$$

$$A = \{(H,H,H),(H,T,T)\}$$

$$\underline{P}_{1,2,3}(\, I_A(X_1,X_2,X_3)\,) = ?$$

$$\underline{P}_2(\, \underline{P}_3(\, I_A(H,X_2,X_3)\,)\,) = 3/8$$

$$\underline{P}_2(\, \underline{P}_3(\, I_A(T,X_2,X_3)\,)\,) = 0$$

$$\underline{P}_{1,2,3}(\, \mathbf{f}(X_1,X_2,X_3)\,) = \underline{P}_1(\, \underline{P}_2(\, \underline{P}_3(\, \mathbf{f}(X_1,X_2,X_3) \,|\, X_1,X_2) \,|\, X_1)\,)$$

$$= \underline{P}(\, \underline{P}(\, \underline{P}(\, \mathbf{f}(X_1,X_2,X_3)\,)\,)\,)$$

The forward irrelevant multinomial process

Described using **coherent lower previsions**

EXAMPLE: $\mathscr{X} = \{H,T\}$

$$\underline{P}(\mathbf{f}) = \min_{\theta \in [1/4,\, 1/2]} \{\theta \mathbf{f}(H) + (1-\theta)\mathbf{f}(T)\}$$

$$A = \{(H,H,H),(H,T,T)\}$$

$$\underline{P}_2(\, \underline{P}_3(\, I_A(H,X_2,X_3))\,) = 3/8$$

$$\underline{P}_2(\, \underline{P}_3(\, I_A(T,X_2,X_3))\,) = 0$$

$$\underline{P}_{1,2,3}(\, I_A(X_1,X_2,X_3)\,) = 3/32 = \underline{P}_1(\, \underline{P}_2(\, \underline{P}_3(\, I_A(X_1,X_2,X_3))))$$

$$\underline{P}_{1,2,3}(\, \mathbf{f}(X_1,X_2,X_3)\,) = \underline{P}_1(\, \underline{P}_2(\, \underline{P}_3(\, \mathbf{f}(X_1,X_2,X_3) \,|\, X_1,X_2) \,|\, X_1)\,)$$

$$= \underline{P}(\, \underline{P}(\, \underline{P}(\, \mathbf{f}(X_1,X_2,X_3) \quad) \quad)\,)$$

The forward irrelevant multinomial process

Described using **coherent lower previsions**

EXAMPLE: $\mathscr{X} = \{H,T\}$

$$\underline{P}(\mathbf{f}) = \min_{\theta \in [1/4,\, 1/2]} \{\theta \mathbf{f}(H) + (1-\theta)\mathbf{f}(T)\}$$

$$A = \{(H,H,H),(H,T,T)\}$$

$$\underline{\mathbf{P}}_{1,2,3}(\ \mathbf{I}_A(X_1,X_2,X_3)\ ) = 3/32$$

$$\overline{\mathbf{P}}_{1,2,3}(\ \mathbf{I}_A(X_1,X_2,X_3)\ ) = -\ \underline{\mathbf{P}}_{1,2,3}(\ -\ \mathbf{I}_A(X_1,X_2,X_3)\ ) = 11/32$$

Independent multinomial process

The independent multinomial process

An **interpretation** for the precise multinomial process

$$X_1, \quad X_2, \quad X_3$$

IDENTICALLY DISTRIBUTED

$\mathbf{p}(X_1)$ $\quad$ $\mathbf{p}(X_2)$ $\quad$ $\mathbf{p}(X_3)$

$\parallel$ $\quad$ $\parallel$ $\quad$ $\parallel$

INDEPENDENT

$$\mathbf{p}_1(X_1) \cdot \mathbf{p}_2(X_2) \cdot \mathbf{p}_3(X_3) = \mathbf{p}_{1,2,3}(X_1,X_2,X_3)$$

The independent multinomial process

An **interpretation** for the precise multinomial process

$$X_1, \quad X_2, \quad X_3$$

IDENTICALLY DISTRIBUTED

INDEPENDENT

$$\mathbf{p}(X_1) = \mathbf{p}_1(X_1) = \mathbf{p}_1(X_1)$$

$$\mathbf{p}(X_2) = \mathbf{p}_2(X_2) = \mathbf{p}_2(X_2 | X_1)$$

$$\mathbf{p}(X_3) = \mathbf{p}_3(X_3) = \mathbf{p}_3(X_3 | X_1, X_2)$$

The value of previous variables is **irrelevant** for our beliefs about the current one !

The independent multinomial process

An **interpretation** for the precise multinomial process

$$X_1, \quad X_2, \quad X_3$$

IDENTICALLY DISTRIBUTED

INDEPENDENT

$p(X_1)$	$p(X_2)$	$p(X_3)$
$=$	$=$	$=$
$\mathbf{p}_1(X_1)$	$\mathbf{p}_2(X_2)$	$\mathbf{p}_3(X_3)$
$=$	$=$	$=$
$\mathbf{p}_1(X_1 \mid X_2, X_3)$	$\mathbf{p}_2(X_2 \mid X_1, X_3)$	$\mathbf{p}_3(X_3 \mid X_1, X_2)$

The value of previous **and future** variables is **irrelevant** for our beliefs about the current one !

The independent multinomial process

An **interpretation** for the precise multinomial process

$X_1, \quad X_2, \quad X_3$

IDENTICALLY DISTRIBUTED

INDEPENDENT

$P(X_1)$	$P(X_2)$	$P(X_3)$
=	=	=
$\mathbf{P}_1(X_1)$	$\mathbf{P}_2(X_2)$	$\mathbf{P}_3(X_3)$
=	=	=
$\mathbf{P}_1(X_1 \mid X_2, X_3)$	$\mathbf{P}_2(X_2 \mid X_1, X_3)$	$\mathbf{P}_3(X_3 \mid X_1, X_2)$

The value of previous **and future** variables is **irrelevant** for our beliefs about the current one !

The independent multinomial process

Described using **coherent lower previsions** **[3]**

X_1,	X_2,	X_3
$\underline{P}(X_1)$	$\underline{P}(X_2)$	$\underline{P}(X_3)$
=	=	=
$\underline{P}_1(X_1)$	$\underline{P}_2(X_2)$	$\underline{P}_3(X_3)$
=	=	=
$\underline{P}_1(X_1 \mid X_2, X_3)$	$\underline{P}_2(X_2 \mid X_1, X_3)$	$\underline{P}_3(X_3 \mid X_1, X_2)$

IDENTICALLY DISTRIBUTED

EPISTEMICALLY INDEPENDENT

? $\underline{P}_{1,2,3}(\quad)$

The value of previous **and future** variables is **irrelevant** for our beliefs about the current one !

The independent multinomial process

Described using **coherent sets of desirable gambles** **[4]**

$X_1, \quad X_2, \quad X_3$

$\mathcal{D} \quad \mathcal{D} \quad \mathcal{D}$

$\shortparallel \quad \shortparallel \quad \shortparallel$

$\mathcal{D}_1 \quad \mathcal{D}_2 \quad \mathcal{D}_3$

$\shortparallel \quad \shortparallel \quad \shortparallel$

$\mathcal{D}_{1|X_2,X_3} \quad \mathcal{D}_{2|X_1,X_3} \quad \mathcal{D}_{3|X_1,X_2}$

IDENTICALLY DISTRIBUTED

EPISTEMICALLY INDEPENDENT

? $\mathcal{D}_{1,2,3}$

The value of previous **and future** variables is **irrelevant** for our beliefs about the current one !

The independent multinomial process

Described using **coherent sets of desirable gambles** **[4]**

$$X_1, \quad X_2, \quad X_3$$

IDENTICALLY DISTRIBUTED

$$\mathcal{D} \qquad \mathcal{D} \qquad \mathcal{D}$$

$$= \qquad = \qquad =$$

$$\mathcal{D}_1 \qquad \mathcal{D}_2 \qquad \mathcal{D}_3$$

EPISTEMICALLY INDEPENDENT

$$= \qquad = \qquad =$$

$$\mathcal{D}_{1|X_2,X_3} \qquad \mathcal{D}_{2|X_1,X_3} \qquad \mathcal{D}_{3|H,H}$$

? $\mathcal{D}_{1,2,3}$

$$\mathcal{D}_{3|H,H} \ni \quad \Leftrightarrow \quad \mathcal{D}_{1,2,3} \ni$$

$$\mathbf{I}_{\{X_1=H\}} \cdot \mathbf{I}_{\{X_2=H\}} \cdot \mathbf{f}_{3|H,H} \longrightarrow \mathcal{D}_{1,2,3}$$

The independent multinomial process

Described using **coherent sets of desirable gambles** **[4]**

$$\mathbf{f} \in \mathcal{D}_{1,2,3}$$

$$\Leftrightarrow \mathbf{f} = \sum_{x_2 \in \mathcal{X}} \sum_{x_3 \in \mathcal{X}} \boxed{\mathbf{f}_{1|x_2,x_3}} \cdot \mathbf{I}_{\{\mathbf{X}_2 = x_2\}} \cdot \mathbf{I}_{\{\mathbf{X}_3 = x_3\}}$$

$$+ \sum_{x_1 \in \mathcal{X}} \sum_{x_3 \in \mathcal{X}} \mathbf{I}_{\{\mathbf{X}_1 = x_1\}} \cdot \boxed{\mathbf{f}_{2|x_1,x_3}} \cdot \mathbf{I}_{\{\mathbf{X}_3 = x_3\}}$$

$$+ \sum_{x_1 \in \mathcal{X}} \sum_{x_2 \in \mathcal{X}} \mathbf{I}_{\{\mathbf{X}_1 = x_1\}} \cdot \mathbf{I}_{\{\mathbf{X}_2 = x_2\}} \cdot \boxed{\mathbf{f}_{3|x_1,x_2}}$$

IDENTICALLY DISTRIBUTED

EPISTEMICALLY INDEPENDENT

$\boxed{\in \mathcal{D}}$

The independent multinomial process

Described using **coherent sets of desirable gambles** [4]

$$\mathbf{f} \in \mathcal{D}_{1,2,3}$$

$$\Leftrightarrow \mathbf{f} = \sum_{x_2 \in \mathcal{X}} \sum_{x_3 \in \mathcal{X}} \boxed{\mathbf{f}_{1|x_2,x_3}} \cdot \mathbf{I}_{\{X_2=x_2\}} \cdot \mathbf{I}_{\{X_3=x_3\}}$$

$$+ \sum_{x_1 \in \mathcal{X}} \sum_{x_3 \in \mathcal{X}} \mathbf{I}_{\{X_1=x_1\}} \cdot \boxed{\mathbf{f}_{2|x_1,x_3}} \cdot \mathbf{I}_{\{X_3=x_3\}}$$

$$+ \sum_{x_1 \in \mathcal{X}} \sum_{x_2 \in \mathcal{X}} \mathbf{I}_{\{X_1=x_1\}} \cdot \mathbf{I}_{\{X_2=x_2\}} \cdot \boxed{\mathbf{f}_{3|x_1,x_2}}$$

IDENTICALLY DISTRIBUTED

EPISTEMICALLY INDEPENDENT

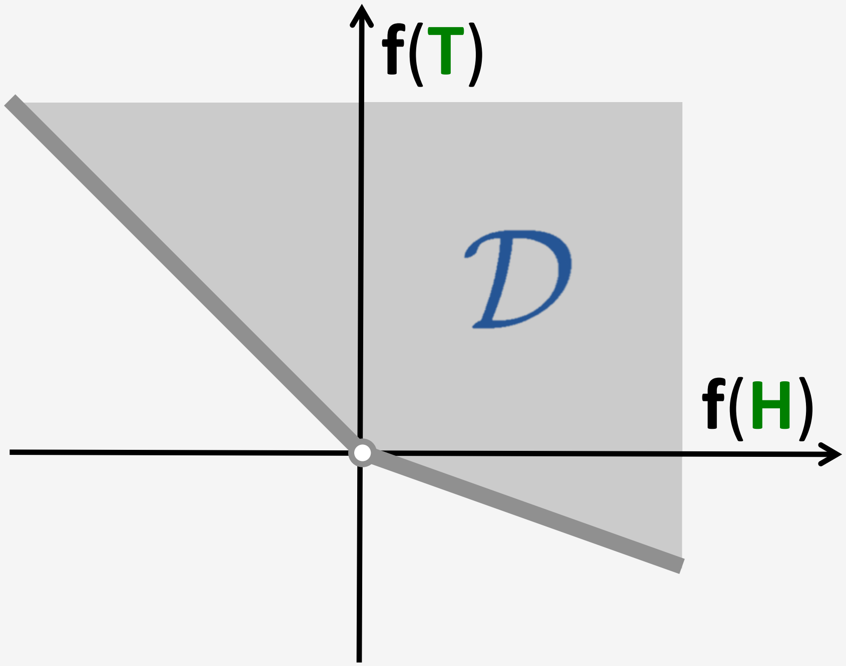
$\boxed{\in \mathcal{D}}$

$$\underline{\mathbf{P}}_{1,2,3}(\mathbf{f}) = \sup\{\boldsymbol{\mu} : \mathbf{f} - \boldsymbol{\mu} \in \mathcal{D}_{1,2,3}\}$$

The independent multinomial process

Described using **coherent sets of desirable gambles**

EXAMPLE: $\mathscr{X} = \{H, T\}$



$A = \{(H,H,H),(H,T,T)\}$

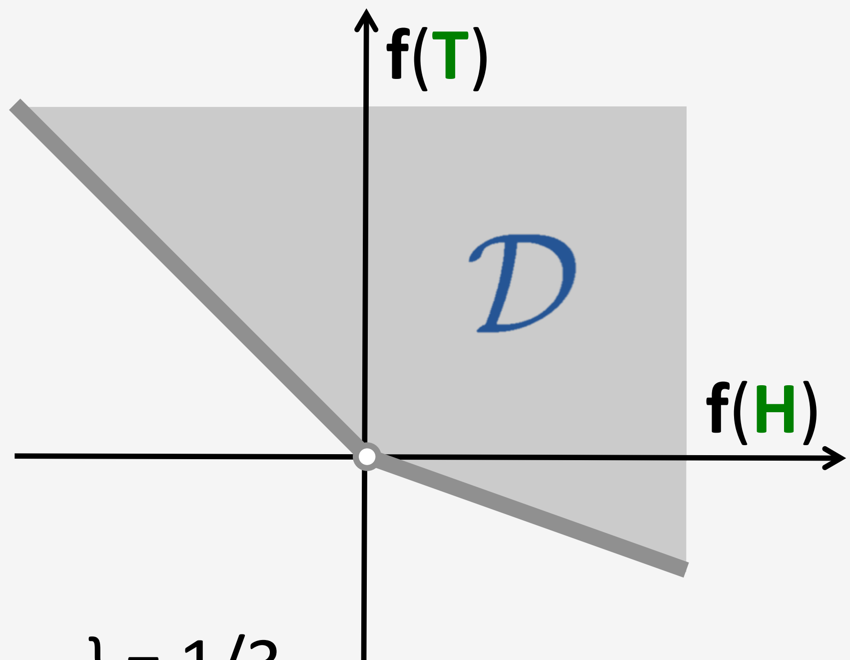
$\underline{P}_{1,2,3}(I_A) = \sup\{\mu : I_A - \mu \in \mathcal{D}_{1,2,3}\} = 1/10$

The independent multinomial process

Described using **coherent sets of desirable gambles**

EXAMPLE: $\mathscr{X} = \{H, T\}$

$A = \{(H,H,H),(H,T,T)\}$



$$\overline{P}_{1,2,3}(I_A) = \inf\{\mu : I_A - \mu \in \mathcal{D}_{1,2,3}\} = 1/3$$

$$\underline{P}_{1,2,3}(I_A) = \sup\{\mu : I_A - \mu \in \mathcal{D}_{1,2,3}\} = 1/10$$

Permutability

Permutability

Consider any permutation $\boldsymbol{\pi}$ of the set of indices **{1, 2, 3}**

Symmetry of the precise multinomial process

$$\mathbf{p}_{1,2,3}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3) = \mathbf{p}_{1,2,3}(\mathbf{X}_{\pi(1)}, \mathbf{X}_{\pi(2)}, \mathbf{X}_{\pi(3)})$$

$$\mathbf{P}_{1,2,3}(\ \mathbf{f}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3)\ ) = \mathbf{P}_{1,2,3}(\ \mathbf{f}(\mathbf{X}_{\pi(1)}, \mathbf{X}_{\pi(2)}, \mathbf{X}_{\pi(3)})\ )$$

Permutability

Permutability of the imprecise multinomial process

Consider any permutation π of the set of indices **{1, 2, 3}**

$$\underline{P}_{1,2,3}(\, f(X_1,X_2,X_3)\,) = \underline{P}_{1,2,3}(\, f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)})\,)$$

$$f(X_1,X_2,X_3) \in \mathcal{D}_{1,2,3} \Leftrightarrow f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)}) \in \mathcal{D}_{1,2,3}$$

Symmetry of the precise multinomial process

$$p_{1,2,3}(X_1,X_2,X_3) = p_{1,2,3}(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)})$$

$$P_{1,2,3}(\, f(X_1,X_2,X_3)\,) = P_{1,2,3}(\, f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)})\,)$$

Permutability

Permutability of the imprecise multinomial process

Consider any permutation π of the set of indices **{1, 2, 3}**

$$\underline{\mathbf{P}}_{1,2,3}(\, \mathbf{f}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3)\,) = \underline{\mathbf{P}}_{1,2,3}(\, \mathbf{f}(\mathbf{X}_{\pi(1)}, \mathbf{X}_{\pi(2)}, \mathbf{X}_{\pi(3)})\,)$$

$$\mathbf{f}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3) \in \mathcal{D}_{1,2,3} \Leftrightarrow \mathbf{f}(\mathbf{X}_{\pi(1)}, \mathbf{X}_{\pi(2)}, \mathbf{X}_{\pi(3)}) \in \mathcal{D}_{1,2,3}$$

The **forward irrelevant** multinomial process becomes **equivalent** with the **independent** multinomial process if we additionally impose **permutability** as a required property!

Strong multinomial process

The strong multinomial process

An **interpretation** for the precise multinomial process

$$X_1, X_2, X_3$$

IDENTICALLY DISTRIBUTED

$$\mathbf{p}_1 \cdot \mathbf{p}_2 \cdot \mathbf{p}_3 = \mathbf{p}_{1,2,3}$$

INDEPENDENT

$$\begin{matrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \\ \shortparallel & \shortparallel & \shortparallel \\ \mathbf{p} & \mathbf{p} & \mathbf{p} \end{matrix}$$

The strong multinomial process

An **interpretation** for the precise multinomial process

$$X_1 , X_2 , X_3$$ **IDENTICALLY DISTRIBUTED**

$$\mathbf{p}_1 \cdot \mathbf{p}_2 \cdot \mathbf{p}_3 = \mathbf{p}_{1,2,3}$$ **INDEPENDENT**

$$\begin{matrix} \Cap & \Cap & \Cap \\ \{\mathbf{p}\} & \{\mathbf{p}\} & \{\mathbf{p}\} \end{matrix}$$

The strong multinomial process

Described using **credal sets**

$$X_1, X_2, X_3$$

$$\mathbf{p}_1 \cdot \mathbf{p}_2 \cdot \mathbf{p}_3 = \mathbf{p}_{1,2,3}$$

$$\Cap \quad \Cap \quad \Cap \quad \Cap$$

$$\mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M}_{1,2,3}$$

IDENTICALLY DISTRIBUTED

(STRONGLY) INDEPENDENT

TAKE CONVEX CLOSURE!

The strong multinomial process

Described using **credal sets** / **coherent lower previsions**

$$X_1 \, , \, X_2 \, , \, X_3$$

IDENTICALLY DISTRIBUTED

(STRONGLY) INDEPENDENT

$$\mathbf{p}_1 \cdot \mathbf{p}_2 \cdot \mathbf{p}_3 = \mathbf{p}_{1,2,3}$$

$$\Cap \quad \Cap \quad \Cap \quad \Cap$$

$$\mathcal{M} \quad \mathcal{M} \quad \mathcal{M} \quad \mathcal{M}_{1,2,3}$$

TAKE CONVEX CLOSURE!

$$\underline{\mathbf{P}}_{1,2,3}(\, \mathbf{f}(X_1,X_2,X_3)\,) = \min\{\mathbf{P}_{1,2,3}(\, \mathbf{f}(X_1,X_2,X_3)\,) : \mathbf{p}_{1,2,3} \in \mathcal{M}_{1,2,3}\}$$

The strong multinomial process

Described using **credal sets** / **coherent lower previsions**

EXAMPLE: $\mathscr{X} = \{H,T\}$

$$\mathcal{M} = \{\, \mathbf{p} : \mathbf{p}(H) = \theta \in [1/4,\ 1/2],\ \mathbf{p}(T) = 1-\theta \,\}$$

$$A = \{(H,H,H),(H,T,T)\}$$

$$\underline{\mathbf{P}}_{1,2,3}(\ \mathbf{I}_A(X_1,X_2,X_3)\ ) = \min_{\substack{\theta_1 \in [1/4,\ 1/2] \\ \theta_2 \in [1/4,\ 1/2] \\ \theta_3 \in [1/4,\ 1/2]}} \{\theta_1(\theta_2\theta_3+(1-\theta_2)(1-\theta_3)) = 1/8$$

$$\underline{\mathbf{P}}_{1,2,3}(\ \mathbf{f}(X_1,X_2,X_3)\ ) = \min\{\mathbf{P}_{1,2,3}(\ \mathbf{f}(X_1,X_2,X_3)\ ) : \mathbf{p}_{1,2,3} \in \mathcal{M}_{1,2,3}\ \}$$

The strong multinomial process

Described using **credal sets** / **coherent lower previsions**

EXAMPLE: $\mathscr{X} = \{H,T\}$

$$\mathcal{M} = \{\ \mathbf{p} : \mathbf{p}(H) = \theta \in [1/4,\ 1/2],\ \mathbf{p}(T) = 1-\theta\ \}$$

$$A = \{(H,H,H),(H,T,T)\}$$

$$\underline{\mathbf{P}}_{1,2,3}(\ \mathbf{I}_A(X_1,X_2,X_3)\ ) = \min_{\substack{\theta_1 \in [1/4,\ 1/2] \\ \theta_2 \in [1/4,\ 1/2] \\ \theta_3 \in [1/4,\ 1/2]}} \{\theta_1(\theta_2\theta_3+(1-\theta_2)(1-\theta_3)) = 1/8$$

$$\overline{\mathbf{P}}_{1,2,3}(\ \mathbf{I}_A(X_1,X_2,X_3)\ ) = \max_{\substack{\theta_1 \in [1/4,\ 1/2] \\ \theta_2 \in [1/4,\ 1/2] \\ \theta_3 \in [1/4,\ 1/2]}} \{\theta_1(\theta_2\theta_3+(1-\theta_2)(1-\theta_3)) = 5/16$$

Exchangeable multinomial process

The exchangeable multinomial process

An **interpretation** for the precise multinomial process

$$X_1 , X_2 , X_3$$ **IDENTICALLY DISTRIBUTED**

$$\mathbf{p}_1 \cdot \mathbf{p}_2 \cdot \mathbf{p}_3 = \mathbf{p}_{1,2,3}$$ **INDEPENDENT**

$$\begin{matrix} \parallel & \parallel & \parallel \\ \mathbf{p} & \mathbf{p} & \mathbf{p} \end{matrix}$$

The exchangeable multinomial process

An **interpretation** for the precise multinomial process

$$X_1 , X_2 , X_3$$

IDENTICALLY DISTRIBUTED

$$\mathbf{p}_1 \cdot \mathbf{p}_2 \cdot \mathbf{p}_3 = \mathbf{p}_{1,2,3}$$

INDEPENDENT

$$\begin{array}{ccccc} \mathbf{p}_1 & & \mathbf{p}_2 & & \mathbf{p}_3 \\ \shortparallel & & \shortparallel & & \shortparallel \\ \mathbf{p} & = & \mathbf{p} & = & \mathbf{p} \\ & & \in & & \\ & & \{\mathbf{p}\} & & \end{array}$$

The exchangeable multinomial process

Described using **credal sets**

$$X_1, X_2, X_3$$

IDENTICALLY DISTRIBUTED

$$\mathbf{p}_1 \cdot \mathbf{p}_2 \cdot \mathbf{p}_3 = \mathbf{p}_{1,2,3}$$

$$\mathbf{p}_1 = \mathbf{p},\quad \mathbf{p}_2 = \mathbf{p},\quad \mathbf{p}_3 = \mathbf{p};\qquad \mathbf{p} = \mathbf{p} = \mathbf{p}$$

$$\mathbf{p}_{1,2,3} \in \mathcal{M}_{1,2,3}$$

$$\mathbf{p} \in \mathcal{M}$$

INDEPENDENT (Sensitivity analysis)

TAKE CONVEX CLOSURE!

The exchangeable multinomial process

Described using **credal sets** / **coherent lower previsions**

$$X_1 , X_2 , X_3$$ **IDENTICALLY DISTRIBUTED**

$$\begin{array}{ccccccc} \mathbf{p}_1 & \cdot & \mathbf{p}_2 & \cdot & \mathbf{p}_3 & = & \mathbf{p}_{1,2,3} \\ \shortparallel & & \shortparallel & & \shortparallel & & \in \\ \mathbf{p} & = & \mathbf{p} & = & \mathbf{p} & & \mathcal{M}_{1,2,3} \\ & & \in & & & & \\ & & \mathcal{M} & & & & \end{array}$$

INDEPENDENT (Sensitivity analysis)

TAKE CONVEX CLOSURE!

$$\underline{\mathbf{P}}_{1,2,3}(\ \mathbf{f}(X_1,X_2,X_3)\ ) = \min\{\mathbf{P}_{1,2,3}(\ \mathbf{f}(X_1,X_2,X_3)\ ) : \mathbf{p}_{1,2,3} \in \mathcal{M}_{1,2,3}\}$$

The exchangeable multinomial process

Described using **credal sets** / **coherent lower previsions**

EXAMPLE: $\mathscr{X} = \{H,T\}$

$$\mathcal{M} = \{\, \mathbf{p} : \mathbf{p}(H) = \theta \in [1/4,\, 1/2],\ \mathbf{p}(T) = 1-\theta \,\}$$

$$A = \{(H,H,H),(H,T,T)\}$$

$$\underline{\mathbf{P}}_{1,2,3}(\, \mathbf{I}_A(X_1,X_2,X_3)\,) = \min_{\theta\in[1/4,\,1/2]} \{\theta(\theta^2+(1-\theta)^2) = 5/32$$

$$\underline{\mathbf{P}}_{1,2,3}(\, \mathbf{f}(X_1,X_2,X_3)\,) = \min\{\mathbf{P}_{1,2,3}(\, \mathbf{f}(X_1,X_2,X_3)\,) : \mathbf{p}_{1,2,3} \in \mathcal{M}_{1,2,3}\,\}$$

The exchangeable multinomial process

Described using **credal sets** / **coherent lower previsions**

EXAMPLE: $\mathscr{X} = \{H, T\}$

$\mathcal{M} = \{\, \mathbf{p} : \mathbf{p}(H) = \theta \in [1/4, 1/2], \mathbf{p}(T) = 1-\theta \,\}$

$A = \{(H,H,H),(H,T,T)\}$

$\underline{P}_{1,2,3}(\, I_A(X_1,X_2,X_3)\,) = \min_{\theta \in [1/4, 1/2]} \{\theta(\theta^2+(1-\theta)^2) = 5/32$

$\overline{P}_{1,2,3}(\, I_A(X_1,X_2,X_3)\,) = \max_{\theta \in [1/4, 1/2]} \{\theta(\theta^2+(1-\theta)^2) = 1/4$

An overview

An overview of the different approaches

EXAMPLE: $\mathscr{X} = \{H,T\}$

$A = \{(H,H,H),(H,T,T)\}$

Local models

Precise: $\mathbf{p}(H) = 4/10$, $\mathbf{p}(T) = 6/10$

Imprecise: $\mathcal{M} = \{\, \mathbf{p} : \mathbf{p}(H) = \theta \in [1/4, 1/2], \mathbf{p}(T) = 1-\theta \,\}$

Multinomial processes

	$\underline{P}_{1,2,3}(\, I_A(X_1,X_2,X_3)\,)$	$\overline{P}_{1,2,3}(\, I_A(X_1,X_2,X_3)\,)$
Precise:	2496/12000	2496/12000
Forward irrelevant:	1125/12000	4125/12000
Independent:	1200/12000	4000/12000
Strong:	1500/12000	3750/12000
Exchangeable:	1875/12000	3000/12000

Exchangeability

Exchangeability

Consider any permutation $\boldsymbol{\pi}$ of the set of indices **{1, 2, 3}**

Symmetry of the precise multinomial process

$$\mathbf{p}_{1,2,3}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3) = \mathbf{p}_{1,2,3}(\mathbf{X}_{\pi(1)}, \mathbf{X}_{\pi(2)}, \mathbf{X}_{\pi(3)})$$

$$\mathbf{P}_{1,2,3}(\ \mathbf{f}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3)\ ) = \mathbf{P}_{1,2,3}(\ \mathbf{f}(\mathbf{X}_{\pi(1)}, \mathbf{X}_{\pi(2)}, \mathbf{X}_{\pi(3)})\ )$$

$$\mathbf{P}_{1,2,3}(\ \mathbf{f}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3) - \mathbf{f}(\mathbf{X}_{\pi(1)}, \mathbf{X}_{\pi(2)}, \mathbf{X}_{\pi(3)})\ ) = 0$$

Exchangeability

Exchangeability of the imprecise multinomial process

Consider any permutation $\boldsymbol{\pi}$ of the set of indices **{1, 2, 3}**

$$\underline{\mathbf{P}}_{1,2,3}(\ \mathbf{f}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3) - \mathbf{f}(\mathbf{X}_{\pi(1)},\ \mathbf{X}_{\pi(2)},\ \mathbf{X}_{\pi(3)})\ ) \geq 0$$

Symmetry of the precise multinomial process

$$\mathbf{p}_{1,2,3}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3) = \mathbf{p}_{1,2,3}(\mathbf{X}_{\pi(1)},\ \mathbf{X}_{\pi(2)},\ \mathbf{X}_{\pi(3)})$$

$$\mathbf{P}_{1,2,3}(\ \mathbf{f}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3)\ ) = \mathbf{P}_{1,2,3}(\ \mathbf{f}(\mathbf{X}_{\pi(1)},\ \mathbf{X}_{\pi(2)},\ \mathbf{X}_{\pi(3)})\ )$$

$$\mathbf{P}_{1,2,3}(\ \mathbf{f}(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3) - \mathbf{f}(\mathbf{X}_{\pi(1)},\ \mathbf{X}_{\pi(2)},\ \mathbf{X}_{\pi(3)})\ ) = 0$$

Exchangeability

Exchangeability of the imprecise multinomial process

Consider any permutation π of the set of indices **{1, 2, 3}**

$$\underline{P}_{1,2,3}(f(X_1,X_2,X_3) - f(X_{\pi(1)}, X_{\pi(2)}, X_{\pi(3)})) \geq 0$$

MAIN RESULT:

All four imprecise multinomial processes become **equivalent** with the **exchangeable** multinomial process if we additionally impose **exchangeability** (for all finite sequences) and **time consistency** as required properties!

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