Part 1: Brownian motion and Kolmogorov complexity
Part 2: Teaching asset pricing using GTP

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PART 1: Brownian motion and Kolmogorov complexity

Kolmogorov Complexity and Schnorr's Theorem

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Thus, the randomness of the infinite object A is reduced to randomness of finite approximations. Does something similar hold in other settings?

Random Walk

Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of independent random variables each satisfying $\mathbb{P}\{X_i = 1\} = \mathbb{P}\{X_i = -1\} = \frac{1}{2}$.

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$$S_n = \sum_{i=1}^{n} X_i.$$

The sequence of random variables $\{S_n\}_{n\in\mathbb{N}}$ is a *random walk on* $\mathbb{Z}$.

Random Walk

The expected value of each X_i is

$$\mathbb{E}(X_i) = \sum_a a \cdot \mathbb{P}\{X_i = a\} = \frac{1}{2} - \frac{1}{2} = 0.$$

and so

$$\mathbb{E}(S_n) = \mathbb{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{E}(X_i) = 0.$$

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$$\mathrm{Var}(X_i) = \mathbb{E}\left[(X_i - \mathbb{E}[X_i])^2\right] = \mathbb{E}[X_i^2] = 1.$$

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Thus,

$$\mathrm{Var}(S_n) = \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = \sum_{i=1}^{n} 1 = n.$$

Random Walk

For $0 \leq k \leq n$, we have

$$\mathrm{Var}\left(\frac{S_k}{\sqrt{n}}\right) = \left(\frac{1}{\sqrt{n}}\right)^2 \mathrm{Var}(S_k) = \frac{k}{n}.$$

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The *piecewise linearization* $\ell_X(t)$ of $X = (X_1, \ldots, X_n)$ is a piecewise linear function with

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Thus whenever t is of the form $\frac{k}{n}$, we have

$$\mathrm{Var}(\ell_X(t)) = t.$$

Brownian Motion

Brownian motion is a random function $W : [0, 1] \to \mathbb{R}$ with $\mathrm{Var}(W(t)) = t$ for all $t \in [0, 1]$; a "random walk with infinitely small steps".

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It satisfies

$$\mathbb{P}\left(W(t) \leq y\right) = \lim_{n \to \infty} \mathbb{P}\left(\ell(t) \leq y\right).$$

Approximating Brownian Motion

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ℓ_X is an approximation of W, similarly to how a finite binary sequence $A \upharpoonright n = (A(0), \ldots, A(n-1))$ is an approximation of $A = (A(0), A(1), \ldots)$.

It is also true that almost surely, Brownian motion is close to a piecewise linearization of a random walk; we may choose $X = (X_1, \ldots, X_n)$ so that

$$\sup_{0 \leq t \leq 1} |\ell_X(t) - W(t)| \approx 0.$$

Can we find such an X which has high Kolmogorov complexity?

Asarin's *Schnorr's Theorem* for Brownian Motion, and a strange constant

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Theorem (Asarin, *Automation and Remote Control* (1986))

$W \in C[0,1]$ is a Martin-Löf random Brownian motion if and only if there is a constant c such that for all but finitely many $n \in \mathbb{N}$ there is a string $x = (x_1, \ldots, x_n) \in \{0,1\}^n$ such that

$$\sup_{0 \leq t \leq 1} |\ell_x(t) - W(t)| \leq n^{-1/10} \text{ and } K(x_1, \ldots, x_n) \geq n - c.$$

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The constant $1/10$ is also used in Fouché, *Adv. Math.* (2000).

Reducing Wiener measure to Lebesgue measure

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Theorem (Kjos-Hanssen and Nerode)

Let $\pi : [0,1] \to C[0,1]$ be a certain partial mapping from the proof of Carathéodory's measure algebra isomorphism theorem. The following are equivalent:

- *w is a Martin-Löf random real.*
- *$\pi(w)$ is defined and is a Martin-Löf random Brownian motion;*

Moreover, if W is a Martin-Löf random Brownian motion then there is a real w with $\pi(w) = W$.

Approximating Brownian Motion

Theorem (Szabados (2001))

There exists a probability space on which is defined a random variable W and a double sequence $X_{i,n}$, $1 \leq i \leq n$, $n = 4^m$, $m \geq 1$, such that $\{X_{1,n}, \ldots, X_{n,n}\}$ are independent for each n with

$$\mathbb{P}\{X_{i,n} = 1\} = \mathbb{P}\{X_{i,n} = -1\} = \frac{1}{2},$$

such that the marginal distribution of W is Brownian motion, and there are constants c_1, c_2, and $\alpha \geq 2$, such that if $X^{(n)} = (X_{1,n}, \ldots, X_{n,n})$,

$$\mathbb{P}\left\{ \sup_{0 \leq t \leq 1} |W(t) - \ell_{X^{(n)}}(t)| \geq c_1 \frac{\log n}{\sqrt{n}} \right\} \leq c_2/n^{\alpha}.$$

Note: the sequences $X_{1,n}, \ldots, X_{n,n}$ for different n are necessarily dependent.

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Szabados' proof relies on work of Komlós, Major and Tusnády (1975). As Szabados proves a more general result, the theorem quoted here may have been known earlier.

Using Szabados' result

Thus

$$\mathbb{P} \bigcup_{n=4^m>N} \left\{ \sup_{0\le t\le 1} |W(t)-\ell_n(t)| \ge c_1 \frac{\log n}{\sqrt{n}} \right\}$$

$$\le \sum_{n>N} c_2/n^{\alpha} \le c_2/N.$$

If n is large enough, say $n \ge N_\varepsilon$, then

$$c_1 \frac{\log n}{\sqrt{n}} < \frac{n^{\varepsilon}}{\sqrt{n}}.$$

So for $N \ge N_\varepsilon$,

$$\mathbb{P} \bigcup_{n=4^m>N} \left\{ \sup_{0\le t\le 1} |W(t)-\ell_n(t)| \ge n^{\varepsilon-\frac{1}{2}} \right\} \le c_2/N.$$

Lemma

If $\mathbb{P}$ is a distribution on sequences

$$\{x_n\}_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \{0,1\}^n$$

such that the marginal distribution of x_n is uniform on $\{0,1\}^n$, then $\mathbb{P}(\exists b \forall n\ K(x_n) \geq n - b) = 1$.

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Proof.

Using $\sum_{\text{all } \sigma} 2^{-K(\sigma)} \leq \sum_{p \text{ halts}} 2^{-|p|} \leq 1$,

$$\mathbb{P}(\exists n\ K(x_n) < n - b) \leq \sum_{\{\sigma : K(\sigma) < |\sigma| - b\}} \mathbb{P}(x_{|\sigma|} = \sigma)$$

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$$= \sum_{\{\sigma : K(\sigma) < |\sigma| - b\}} 2^{-|\sigma|} \leq \sum_{\text{all } \sigma} 2^{-(K(\sigma)+b)} \leq 2^{-b}$$

$\square$

Definition

$f : [0,1] \to \mathbb{R}$ is Hölder continuous of order γ if $\exists C\ \forall x, y \in [0,1]$ $|f(x) - f(y)| \le C|x-y|^\gamma$.

Explaining the constant $1/10$

Theorem (Kjos-Hanssen)

For all $\varepsilon > 0$, Asarin's Theorem $_{\textit{with } \mathbb{N} \textit{ replaced by } \{4^m \,:\, m \in \mathbb{N}\}}$ *holds with the constant $1/10$ replaced by $1/2 - \varepsilon$, but not by $1/2$.*

Proof sketch.

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Proof sketch. Replacing $1/10$ by $1/2$ is impossible, as it would imply that Brownian motion is Hölder continuous of order $1/2$. Let π be as in Carathéodory's measure algebra isomorphism theorem; so a Brownian motion W is $\pi(w)$ for a real w. For $b \geq N_\varepsilon$, the set $U_b =$

$$\bigcup_{n=4^m>b} \bigcap_{x\in\{0,1\}^n} \left\{ w : \sup_{0\leq t\leq 1} |\ell_x(t) - \pi(w)(t)| \leq n^{\varepsilon-\frac{1}{2}} \to K(x) < n - b \right\} \tag{1}$$

is Σ^0_1.

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is Σ^0_1.
Let $X^{(n)} = (X_{1,n}, \ldots, X_{n,n})$ be Szabados' string, and let $V_b =$

$$\bigcup_{n=4^m>b} \left\{ w : \sup_{0\leq t\leq 1} |\ell_{X^{(n)}}(t) - \pi(w)(t)| \leq n^{\varepsilon-\frac{1}{2}} \to K(X^{(n)}) < n - b \right\}.$$

Explaining the constant 1/10

We have $U_b \subseteq V_b$.
By Szabados' Theorem,

$$\mathbb{P} \bigcup_{n=4^m>b} \left\{ w : \sup_{0\leq t\leq 1} |\ell_{X^{(n)}}(t) - \pi(w)(t)| > n^{\varepsilon - \frac{1}{2}} \right\} \leq c_2/b.$$

By the Lemma,

$$\mathbb{P}(\exists n\ K(X_{1,n}, \ldots, X_{n,n}) < n - b) \leq 2^{-b}.$$

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By the Lemma,

$$\mathbb{P}(\exists n \; K(X_{1,n}, \ldots, X_{n,n}) < n - b) \le 2^{-b}.$$

So

$$\mathbb{P}(U_b) \le \mathbb{P}(V_b) \le \frac{c_2}{b} + 2^{-b}.$$

Thus if $W = \pi(w)$ is Martin-Löf random then there is some b such that for all $n = 4^m > b$, there is some $x = (x_1, \ldots, x_n)$ with $\ell_x(t)$ within $n^{\varepsilon - \frac{1}{2}}$ of $W(t)$ and $K(x) \ge n - b$.
End of Proof Sketch.

Proof Summary

Let $X^{(n)}$ be the nth Szabados' random walks.
With high probability,

(1) $X^{(n)}$ has high complexity and $\ell_{X^{(n)}}$ is near the Brownian motion.

Therefore, with high probability,

(2) there is some $x \in \{0,1\}^n$ of high complexity such that ℓ_x is near the Brownian motion.

Property (2) is simple enough that it will be possessed by every Martin-Löf random Brownian motion.

We do not need to show in any sense that (1) will hold for Martin-Löf random Brownian motion.

PART 2: Teaching the binomial asset pricing model using GTP

$$
\begin{tikzcd}
 & & \boxed{S_2(HH) = 12} \\
 & \boxed{S_1(H) = 8, r_1(H) = 1/4} \arrow[ur, "H"'] \arrow[dr, "T"] & \\
\boxed{S_0 = 4, r_0 = 1/4} \arrow[ur, "H"'] \arrow[dr, "T"] & & \boxed{S_2(TH) = 8} \\
 & \boxed{S_1(T) = 2, r_1(T) = 1/2} \arrow[ur, "H"'] \arrow[dr, "T"] & \\
 & & \boxed{S_2(TT) = 2}
\end{tikzcd}
$$

Figure: A stochastic volatility, random interest rate model.

Game theoretic probability for asset pricing

$$\overline{\mathbb{P}}(E) = \inf\{a \mid \forall F\, \exists S\, \forall R\, (\mathcal{K}_0 = a \,\&\, E \Rightarrow \sup_n \mathcal{K}_n \geq 1)\}$$

Here

- F is "forecaster" which we ignore in this simple model.
- S is "skeptic" which means a hedging strategy.
- R is "reality" which is one of the 4 paths.
- $\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2$ is the initial capital and the next two capitals

Defining GTP when there is an interest rate

$\overline{\mathbb{P}}(E) =$ the minimum amount of capital needed to set up a strategy/portfolio guaranteeing that we will receive

$$(1 + R_0)(1 + R_1(\omega))1_E(\omega)$$

(equivalent to $1_E(\omega)$ using discounting) at time n.

Example

The sample space Ω has probability 1 since we may put \$1 in the bank and receive $(1 + R_0)(1 + R_1(\omega))$ at time $n = 2$.

Defining GTP when there is an interest rate: more natural way

$\overline{\mathbb{P}}(E) =$ the minimum amount of capital needed to set up a strategy/portfolio guaranteeing that we will receive *the contents of a bank account initially holding \$1*, if E occurs, and 0 otherwise, at time n.

Upper and lower probabilities

In general $\overline{\mathbb{P}}(A) = 1 - \underline{\mathbb{P}}(A)$ and

$$\overline{\mathbb{P}}(A) \geq \underline{\mathbb{P}}(A)$$

although in this simple model they are equal. (When are they not?)

Risk-neutral measure

$\tilde{\mathbb{P}}(E)$ defined by:

$$\tilde{\mathbb{E}}_n(S_{n+1}) = (1+R)S_n$$

where R is the interest rate. Let $\tilde{p}$ be the probability of head (H), and

$$S_{n+1}(H) = uS_n$$

$$S_{n+1}(T) = dS_n,$$

then it follows that

$$\tilde{p} = \frac{1+R-d}{u-d}$$

$\tilde{\mathbb{P}} = \overline{\mathbb{P}}$ under the No-arbitrage assumption

For the proof that $\tilde{\mathbb{P}} = \overline{\mathbb{P}}$ we need to use the assumption of *no arbitrage* which corresponds to a game-theoretic assumption that one cannot get arbitrarily rich.

The real world measure

$\mathbb{P}(E)$ defined by:

$$\mathbb{E}_n(S_{n+1}) = (1+\varepsilon)S_n$$

where $\varepsilon \geq R$ is constant. Let p be the probability of head (H), and $S_{n+1}(H) = uS_n$, $S_{n+1}(T) = dS_n$, then it follows that

$$p = \frac{1+\varepsilon-d}{u-d}$$

The risk-neutral/game-theoretic probability gives no reward for risk-taking

If it costs 50 cent to set up a portfolio that will pay the proceeds from a \$1 bank account if A occurs and 0 otherwise, then in what sense is $\mathbb{P}(A) = 1/2$? If there is no reward for taking a risk, then $\mathbb{P}(A) = 1/2$, because then the expected reward for our 50 cent investment is the same as if we put the 50 cent in the bank.

Convenience argument for risk-neutral measure

In addition to being the game-theoretic or “hedge” measure, for option pricing it is more convenient to use $\tilde{\mathbb{P}}$ than $\mathbb{P}$.

Suppose we adopt the axiom that $\mathbb{E}(S_1) = (1+\epsilon)S_0$ where $\epsilon \neq r$. This is similar to a risk-neutral outlook except that, well, it is no longer risk-neutral.

Since V_1 and S_1 depend on head vs. tail, we have $V_1 = \alpha S_1 + \beta$ for some constants α and β that can be calculated.

The relationship between V_1 and V_0 has the property that

$$E(V_1/V_0) = E(V_1)/V_0 = \frac{\alpha E(S_1) + \beta}{\alpha S_0 + \frac{\beta}{1+r}}$$

(since a bond is still risk-free hence risk-neutral when the interest rate is known)

$$= \frac{\alpha(1+\epsilon)S_0 + \beta}{\alpha(1+r)S_0 + \beta} \cdot (1+r)$$

When $\epsilon = r$ this simplifies to $1 + r$; in particular then $E(V_1/V_0)$ is the same for all securities V when $\epsilon = r$. For any value of $\epsilon \neq r$, $E(V_1/V_0)$ depends on α and β.

Can we still find V_0 if $\epsilon \neq r$? Yes, the value of $V_0 = \alpha S_0 + \frac{\beta}{1+r}$ does not depend on ϵ. But using $\epsilon \neq r$, we cannot relate V_0 to $E(V_1)$ in a simple way.