# The rate of convergence of strong law of large numbers and convergence of series of moderate and small deviations in the unbounded forecasting game

Shin-ichiro Takazawa

Department of Mathematics, Graduate School of Science, Kobe University

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# Outline

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3. Convergence of series of moderate and small deviation probabilities in game-theoretic probability
4. Related results for a sequence of martingale differences

# Convergence of series of moderate and small deviation probabilities in measure-theoretic probability

# Convergence of series of moderate and small deviation probabilities in measure-theoretic probability

**Theorem (Davis, 1968)**

Let $X_1, X_2, \dots$ be i.i.d. random variables, and set $S_n = X_1 + \cdots + X_n$.

(1) $E[X_1] = 0,\ E[X_1^2] < \infty$

$$\iff \sum_{n=2}^{\infty} \frac{\log n}{n} P\left(|S_n| > \varepsilon\sqrt{n \log n}\right) < \infty,\ \forall \varepsilon > 0.$$

(2) $E[X_1] = 0,\ E[X_1^2] < \infty$

$$\implies \sum_{n=3}^{\infty} \frac{1}{n} P\left(|S_n| > \varepsilon\sqrt{n \log\log n}\right) < \infty,\ \forall \varepsilon > \sigma\sqrt{2}.$$

# Convergence of series of moderate and small deviation probabilities in measure-theoretic probability

**Theorem (Stoica, 2005)**

Set $S_n = X_1 + \cdots + X_n$.

(1) For any $p > 2$ and $L^p$-bounded martingale differences $(X_n)_{n \geq 1}$,

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P\left(|S_n| > \varepsilon \sqrt{n \log n}\right) < \infty, \; 0 \leq \delta < \frac{p}{2} - 1, \; \varepsilon > 0.$$

(2) For any $p > 2$ and $L^p$-bounded martingale differences $(X_n)_{n \geq 1}$,

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} P\left(|S_n| > \varepsilon \sqrt{n \log \log n}\right) < \infty, \; \varepsilon > 0.$$

## Convergence of series of moderate and small deviation probabilities in measure-theoretic probability

* To derive these type results in the framework of game-theoretic probability, we focus on the series

$$\sum_{n=2}^{\infty} c_n \bar{P}\left(|S_n| \geq \varepsilon\sqrt{n \log n}\right),\ \varepsilon > 0,$$

and the series

$$\sum_{n=3}^{\infty} c_n \bar{P}\left(|S_n| \geq \varepsilon\sqrt{n \log\log n}\right),\ \varepsilon > 0,$$

where $c_n > 0$ and $\sum_n c_n = \infty$.

The rate of convergence of strong law of large numbers in the unbounded forecasting game

# The rate of convergence of strong law of large numbers in the unbounded forecasting game

- In the game-theoretic approach, we only assume the protocol of the game.
- In the protocol of the game, there exists no probabilistic assumption on the behavior of Reality.
- A probabilistic behavior of Reality follows from the protocol of the game.

# The rate of convergence of strong law of large numbers in the unbounded forecasting game

THE UNBOUNDED FORECASTING GAME

**Protocol:**

$\mathcal{K}_0 := 1.$
FOR $n = 1, 2, \ldots$ :
  Skeptic announces $M_n \in \mathbb{R}$ and $V_n \geq 0$.
  Reality announces $x_n \in \mathbb{R}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(x_n^2 - v)$.
END FOR

- Skeptic must keep $\mathcal{K}_n \geq 0$.
- Skeptic tries to become rich (i.e. $\mathcal{K}_n \to \infty$) and Reality tries to prevent it.

# The rate of convergence of strong law of large numbersin the unbounded forecasting game

- In order to prevent Skeptic from becoming infinitely rich, Reality is forced to behave probabilistically.

**Theorem (Shafer and Vovk, 2001)**

Let $S_n = \sum_{i=1}^n x_i$.

(1) In the unbounded forecasting game, Skeptic can force $\frac{S_n}{n} \to 0$.

(2) In the unbounded forecasting game, Skeptic can force

$$|x_n| = o\left(\sqrt{\frac{n}{\log\log n}}\right) \Rightarrow \limsup_{n\to\infty} \frac{S_n}{\sqrt{n \log\log n}} \le \sqrt{2v}.$$

# The rate of convergence of strong law of large numbers in the unbounded forecasting game

- If Reality does not behave probabilistically, then Skeptic's capital process $\mathcal{K}_n$ diverges ($\mathcal{K}_n \to \infty$).

∗ We investigate the growth rate of Skeptic's capital process when Reality does not behave probabilistically in the unbounded forecasting game.

# Convergence of series of moderate and small deviation probabilities in game-theoretic probability

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THE UNBOUNDED FORECASTING GAME WITH DOUBLE HEDGES

**Protocol:**

$\mathcal{K}_0 := 1$.

FOR $n = 1, 2, \dots$:

Skeptic announces $M_n \in \mathbb{R}$, $V_n \in \mathbb{R}$ and $W_n \geq 0$.

Reality announces $x_n \in \mathbb{R}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(x_n^2 - v) + W_n(x_n^4 - w)$,

$v, w > 0$.

END FOR

# Convergence of series of moderate and small deviation probabilities in game-theoretic probability

**Theorem 1**

*Let $S_n = \sum_{i=1}^{n} x_i$. In the unbounded forecasting game with double hedges, Skeptic can force*

$$\limsup_{n\to\infty} \frac{|S_n|}{\sqrt{n \log \log n}} \le 2\sqrt{v}.$$

**Remark**

In the unbounded forecasting game with double hedges, Skeptic can force the upper bound of the law of the iterated logarithm without conditions on Reality's moves.

# Convergence of series of moderate and small deviation probabilities in game-theoretic probability

* We investigate the growth rate of Skeptic's capital process when Reality violates the law of the single logarithm, that is,

$$|S_n| \geq \varepsilon\sqrt{n \log n},\ \varepsilon > 0,$$

or when Reality violates the law of the iterated logarithm, that is,

$$|S_n| \geq \varepsilon\sqrt{n \log\log n},\ \varepsilon > 0,$$

in the unbounded forecasting game with double hedges.

# Convergence of series of moderate and small deviation probabilities in game-theoretic probability

## Theorem 2

*In the unbounded forecasting game with double hedges, we have the following results.*

(1) *For all $\delta \geq 0$ and $\varepsilon > 0$,*

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} \bar{P}\left(|S_n| \geq \varepsilon\sqrt{n \log n}\right) < \infty.$$

(2) *For all $0 \leq \delta \leq 1$ and $\varepsilon > 2\sqrt{v(1-\delta)}$,*

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)^{\delta}} \bar{P}\left(|S_n| \geq \varepsilon\sqrt{n \log \log n}\right) < \infty.$$

# Convergence of series of moderate and small deviation probabilities in game-theoretic probability

## Lemma 3

*Let $T_n = \sum_{i=1}^{n}(x_i^2 - v)$. For any $m$, we consider the following strategy $\mathcal{P}(m)$:*

$$\begin{cases} M_i = 0,\ V_i = \dfrac{2}{(v^2+w)m} T_{i-1},\ W_i = \dfrac{1}{(v^2+w)m}, & 1 \le i \le m, \\ M_i = 0,\ V_i = 0,\ W_i = 0, & i > m. \end{cases}$$

*Then, in the unbounded forecasting game with double hedges, for $n \ge m$, Skeptic's capital process $\mathcal{K}_n^{\mathcal{P}(m)}$ is bounded as*

$$\mathcal{K}_n^{\mathcal{P}(m)} \ge \frac{T_m^2}{(v^2+w)m}.$$

# Convergence of series of moderate and small deviation probabilities in game-theoretic probability

**Lemma 4**

*For any $t \in \mathbb{R}$, we consider the following strategy $\mathcal{P}(t)$:*

$$M_n = \mathcal{K}_{n-1} \frac{t \exp\left(-\frac{t^2}{2} v\right)}{1 + \frac{t^2}{2} v}, \ V_n = \mathcal{K}_{n-1} \frac{\frac{t^2}{2} \exp\left(-\frac{t^2}{2} v\right)}{1 + \frac{t^2}{2} v}, \ W_n = 0.$$

*Then, in the unbounded forecasting game with double hedges, Skeptic's capital process $\mathcal{K}_n^{\mathcal{P}(t)}$ is bounded as*

$$\mathcal{K}_n^{\mathcal{P}(t)} \geq \exp\left\{ t \sum_{i=1}^{n} x_i - \frac{t^2}{2} \sum_{i=1}^{n} \left(x_i^2 + v\right) \right\}.$$

# Convergence of series of moderate and small deviation probabilities in game-theoretic probability

Proof of Theorem 2.

(1) By Lemma 3 and Lemma 4, in the unbounded forecasting game with double hedges, there exists a strategy for Skeptic $\mathcal{P}(t)$ such that

$$\mathcal{K}_n^{\mathcal{P}(t)} \geq \frac{T_n^2}{2(v^2+w)n} + \frac{1}{2}\exp\left\{tS_n - \frac{t^2}{2}\left(T_n + 2vn\right)\right\}.$$

For any $\varepsilon > 0$ and any $\theta > 0$, let

$$E_n^1 = \{\, \omega \mid |T_n| \geq \theta n \,\},$$
$$E_n^2 = \left\{\, \omega \mid S_n \geq \varepsilon\sqrt{n \log n},\ |T_n| \leq \theta n \,\right\}.$$

# Convergence of series of moderate and small deviation probabilities in game-theoretic probability

By Lemma 3, on $E_n^1$, for any $t \in \mathbb{R}$,

$$\mathcal{K}_n^{\mathcal{P}(t)} \geq \frac{\theta^2}{2(v^2 + w)} n.$$

Set

$$t_\theta = \frac{\varepsilon}{2v + \theta} \sqrt{\frac{\log n}{n}}.$$

Then, by Lemma 4, on $E_n^2$,

$$\mathcal{K}_n^{\mathcal{P}(t_\theta)} \geq \frac{1}{2} n^{\varepsilon^2/(4v+2\theta)}.$$

# Convergence of series of moderate and small deviation probabilities in game-theoretic probability

Let

$$E_n = \left\{ \omega \mid S_n \geq \varepsilon\sqrt{n \log n} \right\} \left( \subseteq E_n^1 \cup E_n^2 \right).$$

For some $\beta > 0$, there exists a strategy for Skeptic $\mathcal{P}(\beta)$ such that on $E_n$,

$$\mathcal{K}_n^{\mathcal{P}(\beta)} \geq Cn^{\beta}, \quad C\text{: constant.}$$

It follows that

$$\bar{P}(E_n) \leq C^{-1} n^{-\beta}.$$

Therefore, for any $\delta \geq 0$ and $\varepsilon > 0$,

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} \bar{P}\left( |S_n| \geq \varepsilon\sqrt{n \log n} \right) < \infty.$$

# Convergence of series of moderate and small deviation probabilities in game-theoretic probability

(2) For any $\varepsilon > 0$, let

$$E_n' = \left\{ \omega \mid S_n \geq \varepsilon\sqrt{n \log\log n} \right\}.$$

For any $\theta > 0$ and for sufficiently large $n$, there exists a strategy for Skeptic $\mathcal{P}'(\theta)$ such that on $E_n'$,

$$\mathcal{K}_n^{\mathcal{P}'(\theta)} \geq \frac{1}{2}(\log n)^{\varepsilon^2/(4v+2\theta)}.$$

Therefore, for any $0 \leq \delta \leq 1$ and $\varepsilon > 2\sqrt{v(1-\delta)}$,

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)^{\delta}} \bar{P}\left(|S_n| \geq \varepsilon\sqrt{n \log\log n}\right) < \infty.$$

# Related results for a sequence of martingale differences

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## Proposition 5

*Let $(X_i)_{i\geq 1}$ be a sequence of martingale differences and $S_n = \sum_{i=1}^{n} X_i$. For some constants $C$ and $D$, we assume that $E[|X_i|^4] \leq C$ and $E[X_i^2 \mid \mathcal{F}_{i-1}] \leq D$ a.s. Then, we have the following results.*

(1) *For all $\delta \geq 0$ and $\varepsilon > 0$,*

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P\left(|S_n| \geq \varepsilon\sqrt{n \log n}\right) < \infty.$$

(2) *For all $0 \leq \delta \leq 1$ and $\varepsilon > 2\sqrt{D(1-\delta)}$,*

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)^{\delta}} P\left(|S_n| \geq \varepsilon\sqrt{n \log\log n}\right) < \infty.$$

## Remark

- For $\delta = 1$ the second part of Proposition 5 is contained in Stoica's second result.
- By assuming that $E[X_i^2 \mid \mathcal{F}_{i-1}] \leq D$ a.s., we can drop the condition on $\delta$ in the first part of Proposition 5 and obtain the second part of Proposition 5 for $0 \leq \delta < 1$.

- By measure-theoretic proof, we can derive stronger results than Proposition 5.

# Related results for a sequence of martingale differences

### Theorem 6

*Let $(X_i)_{i\geq 1}$ be a sequence of martingale differences and $S_n = \sum_{i=1}^{n} X_i$. For $p > 2$ and some constants $C$ and $D$, we assume that $E[|X_i|^p] \leq C$ and $E[X_i^2 \mid \mathcal{F}_{i-1}] \leq D$ a.s. Then, we have the following results.*

(1) *For all $\delta \geq 0$ and $\varepsilon > 0$,*

$$\sum_{n=2}^{\infty} \frac{(\log n)^{\delta}}{n} P\left(|S_n| \geq \varepsilon\sqrt{n \log n}\right) < \infty.$$

(2) *For all $0 \leq \delta \leq 1$ and $\varepsilon > 2\sqrt{D(1-\delta)}$,*

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)^{\delta}} P\left(|S_n| \geq \varepsilon\sqrt{n \log\log n}\right) < \infty.$$