Effective Strong Measure Zero

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Joint work with Takayuki Kihara+

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Abstract

- E. Borel introduced the concept, strong measure zero in 1919.
- We give some characterization of the concept through techniques and results obtained in Algorithmic Randomness Theory.

First, we introduce the following three concepts:

- Strong measure zero
- Effective strong measure zero
- Strong Martin-Löf measure zero

Strong Measure Zero

- Definition (E. Borel, 1919)
  $X \subset \mathbb{R}$ is a strong measure zero set $\iff$
  $\forall \{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+ \ \exists \{I_n\}_{n\in\mathbb{N}}$: open intervals with $|I_n| < \varepsilon_n$
  $X \subset \bigcup_{n\in\mathbb{N}} I_n$.
- Borel conjectured BC:
  $\{$Strong measure zero sets$\} = \{$Countable sets$\}$.
- BC is independent from ZFC.
  - (Sierpiński, 1928) Continuum Hypothesis implies $\neg$BC.
  - (Laver, 1976) ZFC+BC $\nvdash \perp$ (if ZFC $\nvdash \perp$).

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Besicovitch's Theorem

Definitions For $\mu : 2^{<\mathbb{N}} \to [0,1]$,

- The induced outer measure μ^* of μ is defined by $\mu^*(X) = \inf\{\sum_{\sigma\in A}\mu(\sigma) : A \subset 2^{<\mathbb{N}}, X \subset \bigcup_{\sigma\in A}[\![\sigma]\!]\}$ for all $X \subset 2^{\mathbb{N}}$.
- X is of μ-zero $\iff \mu^*(X) = 0$.
- μ is atomless $\iff \forall f \in 2^{\mathbb{N}},\ \mu^*(f) = 0$.
- μ is a premeasure $\iff \forall\sigma \in 2^{<\mathbb{N}}$, $\mu(\sigma 0), \mu(\sigma 1) \le \mu(\sigma) \le \mu(\sigma 0) + \mu(\sigma 1)$.
  - In this case, we have $\mu^*([\![\sigma]\!]) = \mu(\sigma)$.

Theorem (due to A.S.Besicovitch, 1933)
$X \subset 2^{\mathbb{N}}$ is a strong measure zero set $\iff$
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Effectivizations of Strong Measure Zero

Definitions For $X \subset 2^{\mathbb{N}}$,

- X is an effective strong measure zero set $\iff$ $\forall$ comp. atomless premeasure $\mu : 2^{<\mathbb{N}} \to [0, 1]$, $\mu^*(X) = 0$.
- (Kihara/Miyabe) X is a strong Martin-Löf measure zero set $\iff$ $\forall$ comp. atomless premeasure $\mu : 2^{<\mathbb{N}} \to [0, 1]$, X is of Martin-Löf μ-zero,

  Here, X is Martin-Löf μ-zero for a premeasure $\mu : 2^{<\mathbb{N}} \to [0, 1]$ $\iff$ $\exists$ comp. descending seq. $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of c.e. open sets $\forall n \in \mathbb{N}$ $[\mu^*(\mathcal{U}_n) \leq 2^{-n}$ and $X \subset \mathcal{U}_n]$.

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Here, in terms of semimeasures and complexities, we give some characterizations of the concepts we have defined.

Semimeasures and The A Priori Complexity

Definitions

- $\nu : 2^{<\mathbb{N}} \to [0,1]$ is a semimeasure $\iff$ $\forall \sigma \in 2^{<\mathbb{N}}$, $\nu(\sigma) \geq \nu(\sigma 0) + \nu(\sigma 1)$.
- A left-computable semimeasure ν_0 is optimal $\iff$ $\forall$ l.-c. semimeasure $\nu_1 \exists c \in \mathbb{R} \forall \sigma \in 2^{<\mathbb{N}}$, $\nu_1(\sigma) \leq c\nu_0(\sigma)$.
    - (Levin) There is such a l.-c. semimeasure.
- Fix an optimal l.-c. semimeasure $\nu_{\mathrm{opt}} : 2^{<\mathbb{N}} \to [0,1]$. A priori complexity KA of $\sigma \in 2^{<\mathbb{N}}$ is defined by $\mathrm{KA}(\sigma) = -\log_2 \nu_{\mathrm{opt}}(\sigma)$.

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Characterizations via KA and Semimeasures

Theorem (due to Hudelson/H./Simpson/Yokoyama)

For a comp. premeasure $\mu : 2^{<\mathbb{N}} \to [0,1]$ and $X \subset 2^{\mathbb{N}}$, TFAE:

- X is a Martin-Löf μ-zero set.
- X contains no μ-complex element w.r.t. KA, i.e., $\neg\exists f \in X, c \in \mathbb{N} \forall \sigma \subset f, \mathrm{KA}(\sigma) \geq -\log_2(\mu(\sigma)) - c$.
- $\exists$ l.-c. semimeasure $\nu \forall f \in X$, $\sup_{\sigma \subset f} \nu(\sigma)/\mu(\sigma) = \infty$.

Corollary

For a comp. premeasure $\mu : 2^{<\mathbb{N}} \to [0,1]$ and $X \subset 2^{\mathbb{N}}$, TFAE:

- X is a μ-zero set.
- X contains no μ-complex element w.r.t. KA relative to some real, i.e., $\neg\exists f \in X, g \in 2^{\mathbb{N}}, c \in \mathbb{N} \forall \sigma \subset f$, $\mathrm{KA}^g(\sigma) \geq -\log_2(\mu(\sigma)) - c$.
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Theorem (Kihara/Miyabe) For $X \subset 2^{\mathbb{N}}$, TFAE:

- X is a strong Martin-Löf measure zero set.
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- $\forall$ comp. atomless premeasure $\mu \exists$ l.-c. semimeasure ν $\forall f \in X$, $\sup_{\sigma \subset f} \nu(\sigma)/\mu(\sigma) = \infty$.

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Characterizations via KA and Semimeasures

By relativizing to all reals, we have:

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Corollary Assume Borel Conjecture is true. Then, every uncountable subset of $2^{\mathbb{N}}$ has an element which is complex relative to some real.

- $\because$ Let X be uncountable and let μ be an atomless premeasure in the above condition.
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Here, in terms of martingales, we give characterizations of our main concepts. It is obtained easily using the characterizations by semimeasures.

Martingales

Definitions (related to Schnorr, Lutz)

- Any $O : 2^{<\mathbb{N}} \to [1, \infty)$ is called odds.
- $M : 2^{<\mathbb{N}} \to [0, \infty)$ is O-gale $\iff$ $M(\sigma) = M(\sigma 0)/O(\sigma 0) + M(\sigma 1)/O(\sigma 1)$.
  - Intuitively, an O-gale M is a betting strategy of a gambler:
    - at a stage σ, she/he has her/his capital $M(\sigma)$,
    - she/he divides $M(\sigma)$ into two $M(\sigma 0)/O(\sigma 0)$ and $M(\sigma 1)/O(\sigma 1)$ to bet 0 and 1 respectively as her/his conjecture of the next value, and
    - she/he gets $M(\sigma 0)$ if the next value is 0, $M(\sigma 1)$ if it is 1.
  - Note martingale = 2-martingale.
- $M : 2^{<\mathbb{N}} \to [0, \infty)$ is O-supergale $\iff$ $M(\sigma) \geq M(\sigma 0)/O(\sigma 0) + M(\sigma 1)/O(\sigma 1)$.

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Semimesures vs Martingales

Definitions

- The induced odds O_μ and the induced (O_μ-)supergale M_μ^ν by a premeasure $\mu : 2^{<\mathbb{N}} \to (0,1]$ and a semimeasure ν are defined as $O_\mu(\emptyset) = 1/\mu(\emptyset)$ and $O_\mu(\sigma i) = \mu(\sigma)/\mu(\sigma i)$; and $M_\mu^\nu(\sigma) = \nu(\sigma)/\mu(\sigma)$.
- The induced premeasure μ_O and the induced semimeasure ν_O^M by odds $O : 2^{<\mathbb{N}} \to [1,\infty)$ with $O(\sigma 0)^{-1} + O(\sigma 1)^{-1} \geq 1$ and an O-supergale M are defined as $\mu_O(\sigma) = (\prod_{\tau \subset \sigma} O(\tau))^{-1}$ and $\nu_O^M(\sigma) = \mu_O(\sigma) M(\sigma)$.

Proposition These maps are a bijection and its inverse b/w;

- the set of all pairs of premeasures $\mu : 2^{<\mathbb{N}} \to (0,1]$ and semimeasures; and
- the set of all pairs of odds O with $O(\sigma 0)^{-1} + O(\sigma 1)^{-1} \geq 1$ and O-supergales.

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Characterizations via Martingales

Proposition $\forall f \in 2^{\mathbb{N}}, \sup_{\sigma \subset f} M^{\nu}_{\mu}(\sigma) = \sup_{\sigma \subset f} \nu(\sigma)/\mu(\sigma)$.

- Since $M^{\nu}_{\mu}(\sigma) = \nu(\sigma)/\mu(\sigma)$.

Theorem For $X \subset 2^{\mathbb{N}}$, TFAE

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  $\exists$ semimeasure $\nu \forall f \in X$, $\sup_{\sigma \subset f} \nu(\sigma)/\mu(\sigma) = \infty$.
- $\forall$ comp. acceptable odds $O : 2^{<\mathbb{N}} \to [1, \infty)$
  $\exists O$-supergale $M \forall f \in X$, $\sup_{\sigma \subset f} M(\sigma) = \infty$,
  - where O is acceptable $\iff$
    $\forall g \in 2^{\mathbb{N}}$, $\prod_{n \in \mathbb{N}} O(g \upharpoonright n) = \infty$.

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Again, by relativization, we have

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Summary

TFAE:

- X is a strong measure zero set.
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This talk is based on:

K.Higuchi and T.Kihara, On effectively closed sets of effective strong measure zero, preprint.

Thank you for your attention!

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- $\neg\exists$ atomless premeasure μ $\forall g \in 2^{\mathbb{N}}$ $\exists f \in X$ f is μ-complex relative to g.
- $\forall$ atomless premeasure $\mu \exists$ semimeasure $\nu \forall f \in X$, $\sup_{\sigma \subset f} \nu(\sigma)/\mu(\sigma) = \infty$.
- $\forall$ acceptable odds $O \exists O$-supergale $M \forall f \in X$, $\sup_{\sigma \subset f} M(\sigma) = \infty$.

This talk is based on:

K.Higuchi and T.Kihara, On effectively closed sets of effective strong measure zero, preprint.

Thank you for your attention!