Finitely Presented expansions of semigroups, algebras, and groups

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Plan

- Finitely presented universal algebras
- The Bergstra-Tucker problem
- The NFP theorem
- Applications of the NFP theorem

Universal algebras

Definition

A **universal algebra** $\mathcal{A}$ is a tuple $(A; f_1, \ldots, f_n, c_1, \ldots, c_m)$, where

- $A \neq \emptyset$ is the *domain* of $\mathcal{A}$,
- Each f_i is a total function $A^{k_i} \to A$, a *basic operation* of $\mathcal{A}$.
- Each c_j is a distinguished element of $\mathcal{A}$.

The **signature** of $\mathcal{A}$ is the sequence $f_1, \ldots, f_n, c_1, \ldots, c_m$.

Terms

Definition

Each constant symbol and a variable x is a **term.** If $t_1, \ldots, t_k$ are terms and f is function symbol of arity k then the expression $f_i(t_1, \ldots, t_k)$ is a **term.**

Examples of terms are:

1. $f(f(x, y), z)$, $f(x, f(y, z))$, $f(x, g(x))$, $f(f(x, a), f(a, a))$, and
2. a, $f(a, b)$, $f(a, f(b, g(a)))$, $f(g(a), a)$.

Terms in Example (2) are called *ground terms*.

The term algebra $\mathcal{T}_G$

The term algebra $\mathcal{T}_G$ is defined as follows:

1. The domain is T_G, the set of all ground terms.
2. For a basic operation symbol f of arity k, its value on the k-tuple $(t_1, \ldots, t_k)$ of ground terms is the ground term

$$f(t_1, \ldots, t_k).$$

The word problem

Let $\mathcal{A}$ be a universal algebra generated by the constants. Then there exists a homomorphism

$$h : \mathcal{T}_G \to \mathcal{A}.$$

Definition

The **word problem** for $\mathcal{A}$ is the set

$$WP(\mathcal{A}) = \{(p, q) \mid h(p) = h(q)\}.$$

Definition

The universal algebra $\mathcal{A}$ is **computably enumerable** if $WP(\mathcal{A})$ is a computably enumerable set.

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Equational specifications

Definition

An **equation** is of the form $t = t'$, where t and t' are terms. An **equational specification** S is a finite set of equations.

Examples:

- *GroupAxioms*, *RingAxioms*, *BooleanAlgebraAxioms*.

Finitely presented universal algebras

Let S be an equational specification. Consider:

$$E(S) = \{(t,p) \mid t,p \in T_G \text{ and the equality } t = p \text{ can be deduced from the equations in } S\}.$$

Define:

$$\mathcal{T}(S) = \mathcal{T}_G / E(S).$$

Definition

A universal algebra $\mathcal{A}$ is **finitely presented** if there is an S such that $\mathcal{A}$ is isomorphic to $\mathcal{T}(S)$.

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Properties of $\mathcal{T}(S)$

A computability-theoretic property:

- $\mathcal{T}(S)$ is a c.e. algebra.

Algebraic and logical properties:

- $\mathcal{T}(S)$ is finitely generated and satisfies S.
- If $\mathcal{A} \models S$ and $\mathcal{A}$ is finitely generated by the constants then $\mathcal{A}$ is a homomorphic image of $\mathcal{T}(S)$.
- $\mathcal{T}(S)$ is unique.

C.e. does not imply f.p.

Examples:

- The universal algebra $(\omega; S, 2^x, 0)$ is not finitely presented.
- (Baumslag) The restricted wreath product of any non-trivial group G by Z is not finitely presented. In particular, the restricted wreath product of Z_2 by Z is not finitely presented (The lamplighter group).

Making $(\omega; S, 2^x, 0)$ finitely presented

Consider the following **expansion** of $(\omega; S, 2^x, 0)$:

$$(\omega; S, 2^x, 0, +, \times).$$

This expansion is finitely presented. The presentation is this:

$$x + 0 = x, \; x + S(y) = S(x + y)$$

$$x \times 0 = 0, \; x \times S(0) = x, \; x \times S(y) = x \times y + x,$$

$$2^0 = S(0), \; 2^{x+S(0)} = 2^x \times S(S(0)).$$

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The theorem of Bergstra and Tucker

Definition

An **expansion** of $\mathcal{A} = (A; f_1, \ldots, f_n)$ is $\mathcal{A}' = (A; f_1, \ldots, f_n, h_1, \ldots, h_k)$, where all h are new operations.

Theorem (Bergstra-Tucker, 1980)

Every computably enumerable universal algebra with decidable word problem has finitely presented expansion.

Bergstra and Tucker problem

In the 1980s Bergstra and Tucker, and independently Goncharov, posed the following question:

> *Does every finitely generated computably enumerable universal algebra possess a finitely presented expansion?*

Theorem (Kassymov (1987), Khoussainov (1994))

There exist finitely generated c.e. universal algebras that fail to possess finitely presented expansions.

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Bergstra and Tucker problem revisited

Do there exist standard finitely generated algebraic structures that fail to posses finitely presented expansions?

More precisely, do there exist finitely generated c.e. semigroups, groups, and algebras without finitely presented expansions?

Effectively infinite universal algebras

Definition

A universal algebra $\mathcal{A}$ is **effectively infinite** if there exists an algorithm that enumerates a sequence $t_0, t_1, \ldots$ of ground terms such that they all represent pairwise distinct elements of $\mathcal{A}$.

Non-effectively infinite universal algebras

If $\mathcal{A}$ is not effectively infinite then:

1. Every expansion of $\mathcal{A}$ is not effectively infinite.
2. All infinite sub-algebras of $\mathcal{A}$ are not effectively infinite.
3. For every term $t(x)$ and element a of $\mathcal{A}$, the sequence

   $$a, t(a), t(t(a)), \ldots$$

   is eventually periodic. In particular if a group is not effectively infinite then it is a periodic group.
4. All infinite homomorphic images of $\mathcal{A}$ are not effectively infinite.

Non-effectively infinite semigroups

Theorem (Khoussainov, Hirschfeldt)

There exists a c.e., f.g. and non effectively infinite semigroup.

Proof. Consider $\mathcal{A} = (\{x, y\}^{\star}; \cdot)$. Take $Z \subseteq \{x, y\}^{\star}$, and define:

$$R(Z) = \{u \mid u \text{ contains a subword from } Z\}.$$

The relation $\eta(Z)$

$$(u, v) \in \eta(Z) \leftrightarrow u = v \ \vee (u \in R(Z) \& v \in R(Z)).$$

is a congruence relation on $\mathcal{A}$. Set:

$$\mathcal{A}(Z) = \mathcal{A}/\eta(Z).$$

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Proof of the theorem

Lemma

There exists a simple subset $Z \subseteq \{x, y\}^\star$ such that

1. *The semigroup $\mathcal{A}(Z) = \mathcal{A}/\eta(Z)$ is infinite.*
2. *For each n, r_n the number of strings in Z of length n is bounded by n and $r_0 = r_1 = \ldots = r_5 = 0$.*

The lemma is proved by a careful Post type of construction of a simple set. We use Joe Miller's lemma.

This lemma proves the theorem.

Residually finite universal algebras

Definition

A universal algebra $\mathcal{A}$ from a class K is **residually finite** if for any two distinct elements a and b of $\mathcal{A}$, there exists a homomorphism of $\mathcal{A}$ onto a finite universal algebra from the class K such that the images of a and b under the homomorphism are distinct.

Separation Lemma

Lemma

Let $\mathcal{A} = \mathcal{T}_G/E$ be a residually finite universal algebra. For all distinct elements x and y of $\mathcal{A}$ there exists a set $S(x, y) \subseteq T_G$ such that each of the following is satisfied:

1. *$S(x, y)$ is a computable set.*
2. *$x \in S(x, y)$ and $y \notin S(x, y)$.*
3. *The set $S(x, y)$ is a union of E-equivalence classes.*

Non-effective infinity and residual finiteness

Theorem (Khoussainov and Miasnikov)

Let $\mathcal{A} = \mathcal{T}_G/E$ be a f.g. and c.e. universal algebra.

If $\mathcal{A}$ is residually finite and non-effectively infinite, then all c.e. expansions of $\mathcal{A}$ are residually finite.

In other words, residual finiteness is invariant under expansions if non-effective infinity is assumed.

Ingredients of the proof I

Proof. Let $\mathcal{A}'$ be a c.e. expansion of $\mathcal{A}$.

- Set $C_{(a,b)}$ be the minimal congruence relation generated by the pair (a, b).
- There exists a procedure that given (a, b) extracts an algorithm generating $C_{(a,b)}$.
- Select $x \neq y$ from $\mathcal{A}'$. Want to show that we can distinguish them in a finite homomorphic image of $\mathcal{A}'$.

Ingredients of the proof II

Define $E_{(x,y)}$:

$$(a,b) \in E_{(x,y)} \text{ iff } \forall u \in S(x,y)\ \forall v \notin S(x,y)((u,v) \notin C_{(a,b)}).$$

Lemma

The binary relation $E_{(x,y)}$ is a congruence relation of $\mathcal{A}'$.

This is based on the following two lemmas (by Malcev):

1. $(u,v) \in C_{(a,b)}$ iff $\exists e_0, \ldots, e_n \in \mathcal{A}$ and $\exists t_0, \ldots, t_{n-1}$ such that $u = e_0$, $v = e_n$, and $\{e_i, e_{i+1}\} = \{t_i(a), t_i(b)\}$ for $i \leq n-1$.
2. E is a congruence of $\mathcal{A}$ iff all algebraic terms respect E.

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2. E is a congruence of $\mathcal{A}$ iff all algebraic terms respect E.

Lemma

The relation $E_{(x,y)}$ is a co-c.e.congruence relation.

The transversal of every co-c.e. equivalence relation is a c.e. set. Hence, the transversal must be a finite set.

Thus, $E(x, y)$ has a finite index. Therefore $\mathcal{A}'/E(x, y)$ is the desired homomorphic image. □

Ingredients of the proof III

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The NFP theorem

Theorem (Khoussainov and Miasnikov)

If $\mathcal{A}$ is an infinite, finitely generated, computably enumerable, non-effectively infinite and residually finite universal algebra then no expansion of $\mathcal{A}$ is finitely presented.

Proof.

Assume $\mathcal{A}'$ is a f.p. expansion of $\mathcal{A}$. Then $\mathcal{A}'$ is residually finite as we have proved. The next lemma implies the theorem:

Lemma (Malcev)

If a finitely presented algebra is residually finite then the word problem for the algebra is decidable.

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Application of the NFP to semigroups

Corollary

There exists a finitely generated, non-effectively infinite, residually finite and computably enumerable semigroup that fails to possess finitely presented expansions.

Proof.

It suffices to note that the semigroup $\mathcal{A}(Z)$ constructed above is residually finite. □

Free algebras

Consider the algebra of polynomials over the field k:

$$\mathcal{F} = k\{x_1, \ldots, x_d\},\ d > 1.$$

Represent $\mathcal{F}$ as

$$\mathcal{F}_0 \oplus \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \ldots.$$

Write each polynom p in $\mathcal{F}$ as the sum of **homogeneous polynomials**:

$$p = p_1 + p_2 + \ldots + p_k,$$

where $p_1 \in \mathcal{F}_{n_1}$, ..., $p_k \in F_{n_k}$ and $n_1 < n_1 < \ldots < n_k$.

Homogeneous ideals

Definition

A ideal is **homogeneous** if it is generated by homogeneous polynomials.

Lemma

An ideal I of $\mathcal{F}$ is homogenous if and only if it contains homogeneous components of all of its members.

Lemma

If I is homogeneous then $\mathcal{A} = \mathcal{F}/I$ can be represented as

$$\mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \dots,$$

where $\mathcal{A}_n = (\mathcal{F}_n + I)/I$ for all n. □

Let I be an ideal generated by homogeneous polynomials:

$$h_1, h_2, h_3, \ldots, \text{ where } 2 \leq deg(h_1) \leq deg(h_2) \leq \ldots$$

Set: $r_n = \#$(polynomials of degree n in this sequence).

Theorem (Golod-Shafarevich Theorem)

Assume that for some ϵ such that $0 < \epsilon \leq d/2$ we have the following inequality:

$$r_n \leq \epsilon^2 (d - 2\epsilon)^{n-2} \text{ for all } n \geq 2.$$

Then the algebra $\mathcal{A} = \mathcal{F}/I$ has an infinite dimension. □

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Construction of an ideal H

Let $W_0, W_1, \ldots$ be an enumeration of all c.e. subsets of $\mathcal{F}$. Uniformly on i proceed as follows. Wait for two polynomials f and g that appear in W_i such that:

1. Both f and g are of the form $f = f_1 + h_1$, $g = g_1 + h_2$, and $f_1 = g_1$.
2. The degrees of all homogeneous components occurring in h_1 and h_2 are greater than $i + 10$.

Put the homogeneous components present in h_1 and h_2 into H.

Set $I =< H >$ and consider the algebra

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Application of the NFP theorem to algebras

Theorem (Khoussainov and Miasnikov)

The algebra $\mathcal{A} = \mathcal{F}/I$ is finitely generated, infinite, computably enumerable, non-effectively infinite, residually finite, and nil-algebra.

For the proof one needs to check that the conditions of the Golod-Shafarevich theorem are satisfied. Hence,

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Application of the NFP theorem to groups

Theorem (Khoussainov and Miasnikov)

For every prime number p there exists a finitely generated, non-effectively infinite, residually finite and computably enumerable p-group that fails to possess finitely presented expansions.

Proof. Consider the muliplicative substructure G of $\mathcal{A}$ generated by $1 + x$ and $1 + y$. One can show the following:

1. G is infinite.
2. G is a group.
3. G is residually finite.

The group G is non-effectively infinite finite. Apply the NFP theorem. □

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Is being non-effectively infinite essential?

Theorem (Khoussainov and Miasnikov)

There exists a finitely generated, residually finite, computably enumerable and effectively infinite algebra that fails to possess finitely presented expansions.

Let I be the homogenious ideal generated by set Z constructed for the semigroup. The algebra $\mathcal{A} = \mathcal{F}/I$ is the desired algebra.

Open questions

1. Does there exist effectively infinite groups that fail to possess finitely presented expansions?
2. Are there examples of c.e. finitely generated groups, semigroups, and algebras that fail to possess quasi-equational presentations in all expansions?
3. Does there exist a finitely presented universal algebra which is ML-random?
4. Does there exist an effectively infinite and ML-random algebra?