Bitwise Addition and Lowness for Randomness

Takayuki Kihara

Japan Advanced Institute of Science and Technology (JAIST)
JSPS research fellow PD

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For infinite binary strings x, y, $(x + y)(n) \equiv x(n) + y(n) \bmod 2$.

Main Theorem

The following are equivalent for $x \in 2^\omega$.

1. x is low for uniform Kurtz randomness.
2. $x + y$ is Kurtz random whenever y is Kurtz random.

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2. $x + y$ is Kurtz random whenever y is Kurtz random.
3. x is low for uniform weak 1-genericity.
4. $x + y$ is weakly 1-generic whenever y is weakly 1-generic.

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5. x is $\mathcal{K}^h$-null for every computable h.

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Key Idea

- Pawlikowski's characterization of strong measure zero [1]
- Characterization of meager-additivity [2]

[1] J. Pawlikowski, *A characterization of strong measure zero sets*, 1993.

[2] T. Bartoszyński and H. Judah, *Set Theory: On the Structure of the Real Line*, 1995.

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The concept of $\mathcal{K}^h$-nullness is introduced by K.-Miyabe [3] as a Kurtz version of effective Hausdorff dimension.

Definition (K.-Miyabe [3])

For an order $h : \omega \to \omega$, a set $E \subseteq 2^\omega$ is *Kurtz h-null* ($\mathcal{K}^h$-null) if there is a computable sequence $\{C_n\}_{n\in\omega}$ of finite sets of strings such that

$$E \subseteq [C_n] \text{ and } \sum_{\sigma\in C_n} 2^{-h(|\sigma|)} \leq 2^{-n} \text{ for all } n \in \omega.$$

We also say that $A \in 2^\omega$ is *Kurtz h-null* if $\{A\}$ is Kurtz h-null.

[3] T. Kihara and K. Miyabe, *Uniform Kurtz randomness*, 2013.

Fine Structure inside "Probability **0**"

$\mathcal{H}^1$-null	$\leftarrow$	$\mathcal{K}^1$-null	$\leftarrow$	$\mathcal{P}^1$-null	
$\uparrow$		$\uparrow$		$\uparrow$	
$\dim_H \leq 1$	$\leftarrow$	$\dim_K \leq 1$	$\leftarrow$	$\dim_P \leq 1$	
$\uparrow$		$\uparrow$		$\uparrow$	
$\dim_H \leq s$	$\leftarrow$	$\dim_K \leq s$	$\leftarrow$	$\dim_P \leq s$	$(s \in (0,1])$
$\uparrow$		$\uparrow$		$\uparrow$	
$\dim_H = 0$	$\leftarrow$	$\dim_K = 0$	$\leftarrow$	$\dim_P = 0$	
$\uparrow$		$\uparrow$		$\uparrow$	
$\mathcal{H}^h$-null	$\leftarrow$	$\mathcal{K}^h$-null	$\leftarrow$	$\mathcal{P}^h$-null	$(h \to \infty)$
$\uparrow$		$\uparrow$		$\uparrow$	
$(\forall h)\ \mathcal{H}^h$-null	$\leftarrow$	$(\forall h)\ \mathcal{K}^h$-null	$\leftarrow$	$(\forall h)\ \mathcal{P}^h$-null	
$\updownarrow$		$\updownarrow$		$\updownarrow$	
strong measure zero	$\leftarrow$	meager-additive	$\leftarrow$	null-additive	
$\updownarrow$ (eff.)		$\updownarrow$ (eff.)		$\updownarrow$ (eff.)	
uniLow(Sch, Kur)	$\leftarrow$	**uniLow(Kur)**	$\leftarrow$	**uniLow(Sch)**	
(uniLow(ML,Kur))		**uniLow(W1G)**		(uniLow(ML,Sch))	

Bitwise Sum

For sequences $x, y \in 2^\omega$, the *bitwise addition* $x + y$ is defined by $(x + y)(n) \equiv x(n) + y(n) \bmod 2$.

$$
\begin{array}{c}
1\,1\,0\,0\,1\,0\,0\,1\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,0 \\
+ \\
1\,0\,1\,0\,1\,1\,0\,1\,1\,1\,1\,1\,0\,1\,0\,1\,0\,1\,0 \\
\parallel \\
0\,1\,1\,0\,0\,1\,0\,0\,1\,1\,1\,1\,1\,0\,1\,0\,1\,0\,0
\end{array}
$$

Bitwise Sum

For sets $X, Y \subseteq 2^{\mathbb{N}}$, their *bitwise addition* is defined by $X + Y = \{A + B : (A, B) \in X \times Y\}$.

$$
\begin{aligned}
A_0 &= 1\,1\,0\,0\,1\,0\,0\,1\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,0\\
A_1 &= 0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\\
&\qquad\qquad\qquad +\\
B_0 &= 1\,0\,1\,0\,1\,1\,0\,1\,1\,1\,1\,1\,0\,1\,0\,1\,0\,1\,0\\
B_1 &= 1\,1\,0\,0\,1\,0\,0\,1\,0\,0\,0\,0\,1\,1\,1\,1\,1\,1\,0\\
B_2 &= 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\\
&\qquad\qquad\qquad \Vert\\
A_0 + B_0 &= 0\,1\,1\,0\,0\,1\,0\,0\,1\,1\,1\,1\,1\,0\,1\,0\,1\,0\,0\\
A_0 + B_1 &= 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\\
A_0 + B_2 &= 0\,0\,1\,1\,0\,1\,1\,0\,1\,1\,1\,1\,0\,0\,0\,0\,0\,0\,1\\
A_1 + B_0 &= 1\,1\,1\,1\,1\,0\,0\,0\,1\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\\
A_1 + B_1 &= 1\,0\,0\,1\,1\,1\,0\,0\,0\,1\,0\,1\,1\,0\,1\,0\,1\,0\,0\\
A_1 + B_2 &= 1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1
\end{aligned}
$$

- The union of countably many μ-null set is μ-null.

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- If $\mathcal{I}$ is a σ-ideal of $\mathbf{2}^\omega$, the cardinal $\mathbf{add}(\mathcal{I})$ is defined by

$$\min\{\kappa : \{N_\theta\}_{\theta<\kappa} \subseteq \mathcal{I}, \ \& \ \bigcup_{\theta<\kappa} N_\theta \notin \mathcal{I}\}.$$

- $\aleph_1 \leq \mathbf{add}(\mathcal{N}) \leq \mathbf{add}(\mathcal{M}) \leq 2^{\aleph_0}$.

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- If N is null (meager, resp.), $X + N = \bigcup_{x \in X}(x + N)$ is the union of $|X|$ many null (meager, resp.) sets.

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- If $\mathcal{I}$ is a σ-ideal of 2^ω, the cardinal $\mathbf{add}(\mathcal{I})$ is defined by

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Definition

- $X \subseteq 2^\omega$ is *null-additive* if $X + N$ is null whenever N is null.
- $X \subseteq 2^\omega$ is *meager-additive* if $X + M$ is meager whenever M is meager.

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Theorem (Shelah [4])

null-additive ⇒ meager-additive ⇒ strongly measure zero

[4] S. Shelah, *Every null-additive set is meagre additive*, 1995.

Fine Structure inside “Probability **0**”

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$\uparrow$		$\uparrow$		$\uparrow$	
$\mathbf{dim}_H \leq s$	$\leftarrow$	$\mathbf{dim}_K \leq s$	$\leftarrow$	$\mathbf{dim}_P \leq s$	$(s \in (0, 1])$
$\uparrow$		$\uparrow$		$\uparrow$	
$\mathbf{dim}_H = 0$	$\leftarrow$	$\mathbf{dim}_K = 0$	$\leftarrow$	$\mathbf{dim}_P = 0$	
$\uparrow$		$\uparrow$		$\uparrow$	
$\mathcal{H}^h$-null	$\leftarrow$	$\mathcal{K}^h$-null	$\leftarrow$	$\mathcal{P}^h$-null	$(h \to \infty)$
$\uparrow$		$\uparrow$		$\uparrow$	
$(\forall h)\ \mathcal{H}^h$-null	$\leftarrow$	$(\forall h)\ \mathcal{K}^h$-null	$\leftarrow$	$(\forall h)\ \mathcal{P}^h$-null	
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Let $\mathcal{I}$ be an ideal of $\mathbf{2}^\omega$. A partial function $\rho :\subseteq \omega^\omega \to \mathcal{I}$ is a *representation* of $\mathcal{I}$ if the image of ρ generates $\mathcal{I}$.

1. The *Martin-Löf representation* of the null sets $\mathcal{N}$ is a partial function $\rho_{\mathrm{ML}} :\subseteq \omega^\omega \to \mathcal{N}$ defined by

$$\rho_{\mathrm{ML}}(p) = \bigcap_n \bigcup_m [\sigma_{p(n,m)}],$$
$$\mathrm{dom}(\rho_{\mathrm{ML}}) = \{p : (\forall n)\ \mu(\bigcup_m [\sigma_{p(n,m)}]) \leq 2^{-n}\}.$$

2. The *Kurtz representation* of the closed null sets $\mathcal{E}$ is a partial function $\rho_{\mathrm{Kur}} :\subseteq \omega^\omega \to \mathcal{E}$ defined by

$$\rho_{\mathrm{Kur}}(p) = \bigcap_n \bigcup_{m<|p(n)|} [\sigma_{p(n)(m)}],$$
$$\mathrm{dom}(\rho_{\mathrm{Kur}}) = \{p : (\forall n)\ \mu(\bigcup_{m<|p(n)|} [\sigma_{p(n)(m)}]) \leq 2^{-n}\}.$$

3. The *weakly 1-generic representation* of the nowhere dense sets $\mathcal{ND}$ is a partial function $\rho_{\mathrm{W1G}} :\subseteq \omega^\omega \to \mathcal{ND}$ defined by

$$\rho_{\mathrm{W1G}}(p) = \bigcap_n \bigcup_{m<|p(n)|} [\sigma_{p(n)(m)}],$$
$$\mathrm{dom}(\rho_{\mathrm{W1G}}) = \{p : \bigcap_n \bigcup_{m<|p(n)|} [\sigma_{p(n)(m)}] \in \mathcal{ND}\}.$$

Oracle Tests

$(\mathcal{I}, \rho)$: a represented ideal in $\mathbf{2}^\omega$.
An *oracle $(\mathcal{I}, \rho)$-test* is a partial function $N :\subseteq \mathbf{2}^\omega \to \mathcal{I}$ realized by a computable function $f_N : \mathbf{dom}(N) \to \mathbf{dom}(\rho)$ such that $N = \rho \circ f_N$. If N is total, it is called a *uniform $(\mathcal{I}, \rho)$-test*.

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Remark

If we think of a represented ideal $(\mathcal{I}, \rho)$ as a (multi-)represented space in the sense of Computable Analysis (TTE),

- an oracle $(\mathcal{I}, \rho)$-test is a partial computable function $N :\subseteq \mathbf{2}^\omega \to \mathcal{I}$ (w.r.t. ρ)
- a uniform $(\mathcal{I}, \rho)$-test is a total computable function $N : \mathbf{2}^\omega \to \mathcal{I}$ (w.r.t. ρ)

Examples of Uniform Tests

1. An universal oracle Martin-Löf test is a uniform $(\mathcal{N}, \rho_{\mathbf{ML}})$-test.
2. The ***tt***-Schnorr tests $=$ the uniform $(\mathcal{N}, \rho_{\mathbf{Sch}})$-tests.
3. The ***tt***-Kurtz test $=$ the uniform $(\mathcal{E}, \rho_{\mathbf{Kur}})$-tests.
4. The Demuth$_{\mathbf{BLR}}$-tests $=$ the uniform $(\mathcal{N}, \rho_{\mathbf{Demuth}})$-tests.

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4. The Demuth$_{\mathbf{BLR}}$-tests = the uniform $(\mathcal{N}, \rho_{\mathbf{Demuth}})$-tests.
5. If N is a Schnorr test, $x \mapsto x + N$ is a uniform $(\mathcal{N}, \rho_{\mathbf{Sch}})$-test.
6. If E is a Kurtz test, $x \mapsto x + E$ is a uniform $(\mathcal{E}, \rho_{\mathbf{Kur}})$-test.
7. If M is nowhere dense Π^0_1,
   $x \mapsto x + N$ is a uniform $(\mathcal{M}, \rho_{\mathbf{W1G}})$-test.

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7. If M is nowhere dense Π^0_1,
   $x \mapsto x + N$ is a uniform $(\mathcal{M}, \rho_{\mathrm{W1G}})$-test.
8. (Since $+ : 2^\omega \times \mathcal{I} \to \mathcal{I}$ is computable, for the above represented ideals $(\mathcal{I}, \rho)$).

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$\dim_H \leq s$	$\leftarrow$	$\dim_K \leq s$	$\leftarrow$	$\dim_P \leq s$	$(s \in (0,1])$
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$\dim_H = 0$	$\leftarrow$	$\dim_K = 0$	$\leftarrow$	$\dim_P = 0$	
$\uparrow$		$\uparrow$		$\uparrow$	
$\mathcal{H}^h$-null	$\leftarrow$	$\mathcal{K}^h$-null	$\leftarrow$	$\mathcal{P}^h$-null	$(h \to \infty)$
$\uparrow$		$\uparrow$		$\uparrow$	
$(\forall h)\ \mathcal{H}^h$-null	$\leftarrow$	$(\forall h)\ \mathcal{K}^h$-null	$\leftarrow$	$(\forall h)\ \mathcal{P}^h$-null	
$\updownarrow$		$\updownarrow$		$\updownarrow$	
strong measure zero	$\leftarrow$	meager-additive	$\leftarrow$	null-additive	
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$(\forall \text{comp. } h)\ V \text{ is } \mathcal{K}^h\text{-null} \iff (\forall \text{Kurtz test } E)\ V + E \text{ is Kurtz null.}$

Lemma (K.-Miyabe [3])

Assume that V is $\mathcal{K}^h$-null for every computable h. Then, $\bigcup_{y \in V} E(y)$ is Kurtz null for every uniform Kurtz test $E : 2^\omega \to \mathcal{E}$.

(∀comp. h) V is $\mathcal{K}^h$-null $\iff$ (∀Kurtz test E) $V + E$ is Kurtz null.

Lemma

Assume that $V + E$ is Kurtz null for every Kurtz null set E.
Then, V is $\mathcal{K}^h$-null for every computable h.

1. h: given. $g(n) = g(n-1) + h(n) + 2^{h(n)}$.
2. $E_k \subseteq 2^{g(k)}$: strings of the form $\tau^\frown\sigma_i{}^\frown\rho$ s.t.

$$|\tau| = g(k-1),\ |\sigma_i| = h(k),\ |\rho| = 2^{h(k)},\ \text{and}\ \rho(i) = 0,$$

   where $\{\sigma_i : i < 2^{h(k)}\}$ is an enumeration of $2^{h(k)}$.
3. By assumption, $V + E$ is covered by a Kurtz test $D = \bigcap_n D_n$.
4. $D_{e(k)}$: $\mu(D_{e(k)}|\tau) < 1/8$ for any $\tau \in 2^{g(k-1)}$.

$(\forall \text{comp. } h)$ V is $\mathcal{K}^h$-null $\Leftarrow$ $(\forall \text{Kurtz test } E)$ $V + E$ is Kurtz null.

1. $d(k) = e(k)$ if $E_{d(k-1)} \subseteq 2^{\leq g(k-1)}$; $d(k) = d(k-1)$ o.w.
2. Given $\tau \in 2^{g(k-1)}$, $\sigma \in 2^{h(k)+2^{h(k)}}$ *gets k-closer to D/τ* if
$$(1 - 2^{-k-1})\, \mu(D_{d(k)}|\tau\sigma) > \mu(D_{d(k)}|\tau).$$
$D_\tau[k]$: all σ which get k-closer to D/τ.
3. (Remark) $\mu(D_\tau[k]) \leq 1 - 2^{-k-1}$.
4. $V_\tau[k] = \{\sigma \in 2^{h(k)} : (\exists \sigma' \succeq \sigma)\ \sigma' + E \in D_\tau[k]\}$.
5. $V[k] = \{\tau\sigma : \sigma \in D_\tau[k]\}$.

Claim

$\#V[k] \leq (k+1) \cdot 2^{h(k)}$.

1. Note that $V_\tau[k] + E_k \subseteq D_\tau[k]$.
2. By probability independence, $\mu(V_\tau[k] + E_k) = 1 - 2^{-|V_\tau[k]|}$.
3. However, $\mu(D_\tau[k]) \leq 1 - 2^{-k-1}$.
4. Hence, $|V_\tau[k]| \leq k + 1$.

Claim

Let $\{k(l)\}_{l\in\omega}$ be the list of all k s.t. $d(k) \neq d(k-1)$.
Then, $V \subseteq \bigcap_l \bigcup_{j=k(l-1)}^{k(l)-1} V[j]$.

1. Otherwise, there is $x \in V$ s.t. $x \notin \bigcup_{j=k(l-1)}^{k(l)-1} V[j]$ for some l.
2. $a := k(l-1)$, $b = k(l)-1$
3. By def., $\mu(D_{d(a)}|x+\tau) < 1/8$ for any $\tau \in 2^{g(a-1)}$.
4. $\mu(D_{d(a)}|x+\tau) = 1$ for $\forall\tau \in E \upharpoonright g(b)$, since $V + E \subseteq D_{d(a)}$.
5. This is impossible, since $x \notin \bigcup_{j=a}^{b} V[j]$ implies we can find sequence $\tau_a, \tau_{a+1}, \dots \in E$ s.t.
$\mu(D_{d(a)}|x+\tau_a) \geq \prod_{j=a}^{b}(1-2^{-j-1}) \cdot \mu(D_{d(a)}|x+\tau_b) > 1/8$.

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$\dim_H \leq s$	$\leftarrow$	$\dim_K \leq s$	$\leftarrow$	$\dim_P \leq s$	$(s \in (0,1])$
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Question I

Are the following equivalent for $x \in 2^\omega$?

1. x is low for uniform Schnorr randomness.
2. $x + y$ is Schnorr random whenever y is Schnorr random.

Question II

Are the following equivalent for $x \in 2^\omega$?

1. x is low for Martin-Löf randomness.
2. $x + y$ is Martin-Löf random whenever y is Martin-Löf random.

Question I

Are the following equivalent for $x \in 2^\omega$?

1. x is low for uniform Schnorr randomness.
2. $x + y$ is Schnorr random whenever y is Schnorr random.

Question II

Are the following equivalent for $x \in 2^\omega$?

1. x is low for Martin-Löf randomness.
2. $x + y$ is Martin-Löf random whenever y is Martin-Löf random.

Question III

$\mathcal{R}, \mathcal{S}$: randomness notions.

Are the following equivalent for $x \in 2^\omega$?

1. x is low for $\mathcal{R}$-randomness versus uniform $\mathcal{S}$-randomness.
2. $x + y$ is $\mathcal{S}$-random whenever y is $\mathcal{R}$-random.