

Bitwise Addition and Lowness for Randomness

Takayuki Kihara

Japan Advanced Institute of Science and Technology (JAIST)
JSPS research fellow PD

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For infinite binary strings \mathbf{x}, \mathbf{y} , $(\mathbf{x} + \mathbf{y})(n) \equiv \mathbf{x}(n) + \mathbf{y}(n) \pmod{2}$.

Main Theorem

The following are equivalent for $\mathbf{x} \in 2^\omega$.

- 1 \mathbf{x} is low for uniform Kurtz randomness.
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- 3 \mathbf{x} is low for uniform weak **1**-genericity.
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Key Idea

- Pawlikowski's characterization of strong measure zero [1]
- Characterization of meager-additivity [2]

[1] J. Pawlikowski, *A characterization of strong measure zero sets*, 1993.

[2] T. Bartoszyński and H. Judah, *Set Theory: On the Structure of the Real Line*, 1995.

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The concept of \mathcal{K}^h -nullness is introduced by K.-Miyabe [3] as a Kurtz version of effective Hausdorff dimension.

Definition (K.-Miyabe [3])

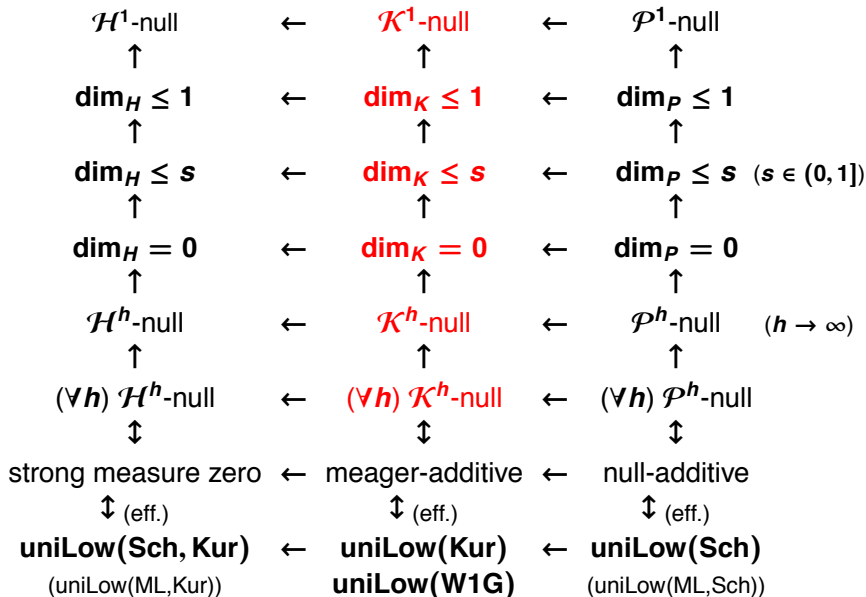
For an order $h : \omega \rightarrow \omega$, a set $E \subseteq 2^\omega$ is *Kurtz h -null* (\mathcal{K}^h -null) if there is a computable sequence $\{\mathbf{C}_n\}_{n \in \omega}$ of finite sets of strings such that

$$E \subseteq [\mathbf{C}_n] \text{ and } \sum_{\sigma \in \mathbf{C}_n} 2^{-h(|\sigma|)} \leq 2^{-n} \text{ for all } n \in \omega.$$

We also say that $\mathbf{A} \in 2^\omega$ is *Kurtz h -null* if $\{\mathbf{A}\}$ is Kurtz h -null.

[3] T. Kihara and K. Miyabe, *Uniform Kurtz randomness*, 2013.

Fine Structure inside “Probability 0”



Bitwise Sum

For sequences $\mathbf{x}, \mathbf{y} \in 2^\omega$, the *bitwise addition* $\mathbf{x} + \mathbf{y}$ is defined by $(\mathbf{x} + \mathbf{y})(n) \equiv \mathbf{x}(n) + \mathbf{y}(n) \pmod{2}$.

```
  1 1 0 0 1 0 0 1 0 0 0 0 1 1 1 1 1 1 0
      +
  1 0 1 0 1 1 0 1 1 1 1 1 0 1 0 1 0 1 0
      ||
  0 1 1 0 0 1 0 0 1 1 1 1 1 0 1 0 1 0 0
```

Bitwise Sum

For sets $X, Y \subseteq 2^{\mathbb{N}}$, their *bitwise addition* is defined by $X + Y = \{A + B : (A, B) \in X \times Y\}$.

$$A_0 = 1100100100001111110$$

$$A_1 = 0101010101010101010$$

+

$$B_0 = 1010110111110101010$$

$$B_1 = 1100100100001111110$$

$$B_2 = 1111111111111111111$$

||

$$A_0 + B_0 = 0110010011111010100$$

$$A_0 + B_1 = 0000000000000000000$$

$$A_0 + B_2 = 0011011011110000001$$

$$A_1 + B_0 = 1111100010100000000$$

$$A_1 + B_1 = 1001110001011010100$$

$$A_1 + B_2 = 1010101010101010101$$

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$$\min\{\kappa : \{N_\theta\}_{\theta < \kappa} \subseteq \mathcal{I}, \& \bigcup_{\theta < \kappa} N_\theta \notin \mathcal{I}\}.$$

- $\aleph_1 \leq \mathbf{add}(\mathcal{N}) \leq \mathbf{add}(\mathcal{M}) \leq 2^{\aleph_0}$.

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- If N is null (meager, resp.), $X + N = \bigcup_{x \in X} (x + N)$ is the union of $|X|$ many null (meager, resp.) sets.

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Definition

- $X \subseteq 2^\omega$ is *null-additive* if $X + N$ is null whenever N is null.
- $X \subseteq 2^\omega$ is *meager-additive* if $X + M$ is meager whenever M is meager.

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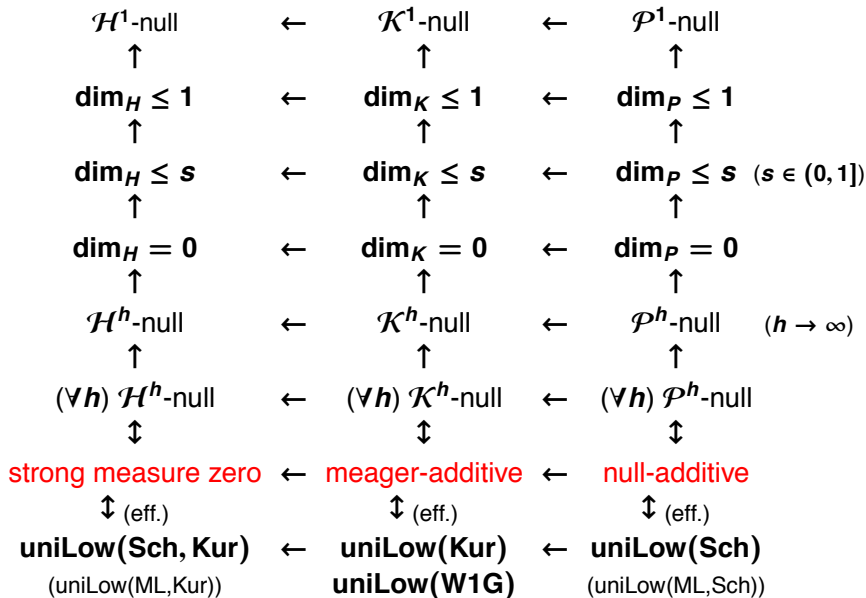
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- $X \subseteq 2^\omega$ is *meager-additive* if $X + M$ is meager whenever M is meager.

Theorem (Shelah [4])

null-additive \Rightarrow meager-additive \Rightarrow strongly measure zero

[4] S. Shelah, *Every null-additive set is meagre additive*, 1995.

Fine Structure inside “Probability 0”



Let \mathcal{I} be an ideal of 2^ω . A partial function $\rho : \subseteq \omega^\omega \rightarrow \mathcal{I}$ is a *representation* of \mathcal{I} if the image of ρ generates \mathcal{I} .

- 1 The *Martin-Löf representation* of the null sets \mathcal{N} is a partial function $\rho_{\text{ML}} : \subseteq \omega^\omega \rightarrow \mathcal{N}$ defined by

$$\rho_{\text{ML}}(\mathbf{p}) = \bigcap_n \bigcup_m [\sigma_{\mathbf{p}(n,m)}],$$

$$\text{dom}(\rho_{\text{ML}}) = \{\mathbf{p} : (\forall n) \mu(\bigcup_m [\sigma_{\mathbf{p}(n,m)}]) \leq 2^{-n}\}.$$

- 2 The *Kurtz representation* of the closed null sets \mathcal{E} is a partial function $\rho_{\text{Kur}} : \subseteq \omega^\omega \rightarrow \mathcal{E}$ defined by

$$\rho_{\text{Kur}}(\mathbf{p}) = \bigcap_n \bigcup_{m < |\mathbf{p}(n)|} [\sigma_{\mathbf{p}(n)(m)}],$$

$$\text{dom}(\rho_{\text{Kur}}) = \{\mathbf{p} : (\forall n) \mu(\bigcup_{m < |\mathbf{p}(n)|} [\sigma_{\mathbf{p}(n)(m)}]) \leq 2^{-n}\}.$$

- 3 The *weakly 1-generic representation* of the nowhere dense sets \mathcal{ND} is a partial function $\rho_{\text{W1G}} : \subseteq \omega^\omega \rightarrow \mathcal{ND}$ defined by

$$\rho_{\text{W1G}}(\mathbf{p}) = \bigcap_n \bigcup_{m < |\mathbf{p}(n)|} [\sigma_{\mathbf{p}(n)(m)}],$$

$$\text{dom}(\rho_{\text{W1G}}) = \{\mathbf{p} : \bigcap_n \bigcup_{m < |\mathbf{p}(n)|} [\sigma_{\mathbf{p}(n)(m)}] \in \mathcal{ND}\}.$$

Oracle Tests

(I, ρ) : a represented ideal in 2^ω .

An *oracle (I, ρ) -test* is a **partial** function $N : \subseteq 2^\omega \rightarrow I$ realized by a computable function $f_N : \text{dom}(N) \rightarrow \text{dom}(\rho)$ such that $N = \rho \circ f_N$. If N is **total**, it is called a *uniform (I, ρ) -test*.

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Remark

If we think of a represented ideal (I, ρ) as a (multi-)represented space in the sense of Computable Analysis (TTE),

- an *oracle (I, ρ) -test* is a **partial** computable function $N : \subseteq 2^\omega \rightarrow I$ (w.r.t. ρ)
- a *uniform (I, ρ) -test* is a **total** computable function $N : 2^\omega \rightarrow I$ (w.r.t. ρ)

Examples of Uniform Tests

- 1 An universal oracle Martin-Löf test is a uniform $(\mathcal{N}, \rho_{\text{ML}})$ -test.
- 2 The *tt*-Schnorr tests = the uniform $(\mathcal{N}, \rho_{\text{Sch}})$ -tests.
- 3 The *tt*-Kurtz test = the uniform $(\mathcal{E}, \rho_{\text{Kur}})$ -tests.
- 4 The Demuth_{BLR}-tests = the uniform $(\mathcal{N}, \rho_{\text{Demuth}})$ -tests.

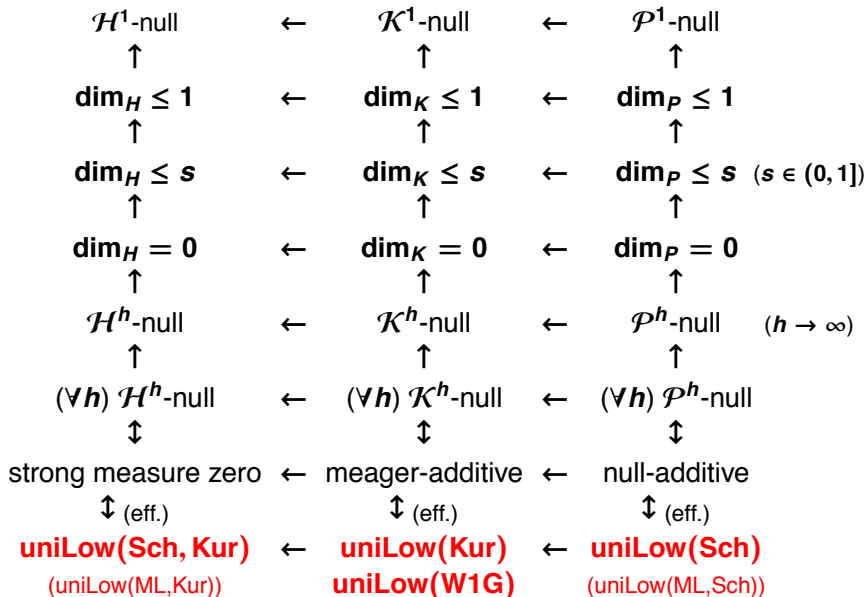
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- 3 The *tt*-Kurtz test = the uniform $(\mathcal{E}, \rho_{\text{Kur}})$ -tests.
- 4 The Demuth_{BLR}-tests = the uniform $(\mathcal{N}, \rho_{\text{Demuth}})$ -tests.
- 5 If \mathbf{N} is a Schnorr test, $\mathbf{x} \mapsto \mathbf{x} + \mathbf{N}$ is a uniform $(\mathcal{N}, \rho_{\text{Sch}})$ -test.
- 6 If \mathbf{E} is a Kurtz test, $\mathbf{x} \mapsto \mathbf{x} + \mathbf{E}$ is a uniform $(\mathcal{E}, \rho_{\text{Kur}})$ -test.
- 7 If \mathbf{M} is nowhere dense Π_1^0 ,
 $\mathbf{x} \mapsto \mathbf{x} + \mathbf{N}$ is a uniform $(\mathcal{M}, \rho_{\text{W1G}})$ -test.

Examples of Uniform Tests

- 1 An universal oracle Martin-Löf test is a uniform $(\mathcal{N}, \rho_{\text{ML}})$ -test.
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- 3 The tt -Kurtz test = the uniform $(\mathcal{E}, \rho_{\text{Kur}})$ -tests.
- 4 The Demuth_{BLR}-tests = the uniform $(\mathcal{N}, \rho_{\text{Demuth}})$ -tests.
- 5 If N is a Schnorr test, $x \mapsto x + N$ is a uniform $(\mathcal{N}, \rho_{\text{Sch}})$ -test.
- 6 If E is a Kurtz test, $x \mapsto x + E$ is a uniform $(\mathcal{E}, \rho_{\text{Kur}})$ -test.
- 7 If M is nowhere dense Π_1^0 ,
 $x \mapsto x + N$ is a uniform $(\mathcal{M}, \rho_{\text{W1G}})$ -test.
- 8 (Since $+ : 2^\omega \times \mathcal{I} \rightarrow \mathcal{I}$ is computable, for the above represented ideals (\mathcal{I}, ρ)).

Fine Structure inside “Probability 0”



$(\forall \text{comp. } h) V \text{ is } \mathcal{K}^h\text{-null} \iff (\forall \text{Kurtz test } E) V + E \text{ is Kurtz null.}$

Lemma (K.-Miyabe [3])

Assume that V is \mathcal{K}^h -null for every computable h . Then,
 $\bigcup_{y \in V} E(y)$ is Kurtz null for every uniform Kurtz test $E : 2^\omega \rightarrow \mathcal{E}$.

$(\forall \text{comp. } h) V \text{ is } \mathcal{K}^h\text{-null} \iff (\forall \text{Kurtz test } E) V + E \text{ is Kurtz null.}$

Lemma

Assume that $V + E$ is Kurtz null for every Kurtz null set E .
Then, V is \mathcal{K}^h -null for every computable h .

① h : given. $g(n) = g(n-1) + h(n) + 2^{h(n)}$.

② $E_k \subseteq 2^{g(k)}$: strings of the form $\tau \hat{\ } \sigma_i \hat{\ } \rho$ s.t.

$|\tau| = g(k-1)$, $|\sigma_i| = h(k)$, $|\rho| = 2^{h(k)}$, and $\rho(i) = 0$,

where $\{\sigma_i : i < 2^{h(k)}\}$ is an enumeration of $2^{h(k)}$.

③ By assumption, $V + E$ is covered by a Kurtz test $D = \bigcap_n D_n$.

④ $D_{e(k)}$: $\mu(D_{e(k)}|\tau) < 1/8$ for any $\tau \in 2^{g(k-1)}$.

(\forall comp. h) V is \mathcal{K}^h -null \Leftrightarrow (\forall Kurtz test E) $V + E$ is Kurtz null.

- 1 $d(k) = e(k)$ if $E_{d(k-1)} \subseteq 2^{\leq g(k-1)}$; $d(k) = d(k-1)$ o.w.
- 2 Given $\tau \in 2^{g(k-1)}$, $\sigma \in 2^{h(k)+2^{h(k)}}$ gets k -closer to D/τ if $(1 - 2^{-k-1}) \mu(D_{d(k)}|\tau\sigma) > \mu(D_{d(k)}|\tau)$.

$D_\tau[k]$: all σ which get k -closer to D/τ .

- 3 (Remark) $\mu(D_\tau[k]) \leq 1 - 2^{-k-1}$.
- 4 $V_\tau[k] = \{\sigma \in 2^{h(k)} : (\exists \sigma' \succeq \sigma) \sigma' + E \in D_\tau[k]\}$.
- 5 $V[k] = \{\tau\sigma : \sigma \in D_\tau[k]\}$.

Claim

$$\#V[k] \leq (k + 1) \cdot 2^{h(k)}.$$

- 1 Note that $V_\tau[k] + E_k \subseteq D_\tau[k]$.
- 2 By probability independence, $\mu(V_\tau[k] + E_k) = 1 - 2^{-|V_\tau[k]|}$.
- 3 However, $\mu(D_\tau[k]) \leq 1 - 2^{-k-1}$.
- 4 Hence, $|V_\tau[k]| \leq k + 1$.

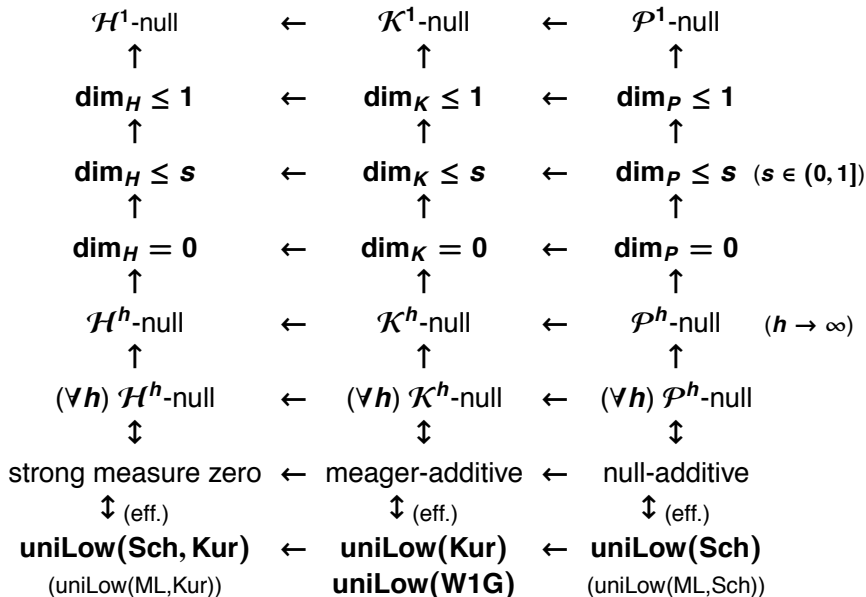
Claim

Let $\{k(l)\}_{l \in \omega}$ be the list of all k s.t. $d(k) \neq d(k-1)$.

Then, $V \subseteq \bigcap_l \bigcup_{j=k(l)-1}^{k(l)-1} V[j]$.

- 1 Otherwise, there is $x \in V$ s.t. $x \notin \bigcup_{j=k(l)-1}^{k(l)-1} V[j]$ for some l .
- 2 $a := k(l-1)$, $b = k(l) - 1$
- 3 By def., $\mu(D_{d(a)}|x + \tau) < 1/8$ for any $\tau \in 2^{g(a-1)}$.
- 4 $\mu(D_{d(a)}|x + \tau) = 1$ for $\forall \tau \in E \upharpoonright g(b)$, since $V + E \subseteq D_{d(a)}$.
- 5 This is impossible, since $x \notin \bigcup_{j=a}^b V[j]$ implies we can find sequence $\tau_a, \tau_{a+1}, \dots \in E$ s.t.
$$\mu(D_{d(a)}|x + \tau_a) \geq \prod_{j=a}^b (1 - 2^{-j-1}) \cdot \mu(D_{d(a)}|x + \tau_b) > 1/8.$$

Fine Structure inside “Probability 0”



Question I

Are the following equivalent for $x \in 2^\omega$?

- 1 x is low for uniform Schnorr randomness.
- 2 $x + y$ is Schnorr random whenever y is Schnorr random.

Question II

Are the following equivalent for $x \in 2^\omega$?

- 1 x is low for Martin-Löf randomness.
- 2 $x + y$ is Martin-Löf random whenever y is Martin-Löf random.

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Question III

\mathcal{R}, \mathcal{S} : randomness notions.

Are the following equivalent for $x \in 2^\omega$?

- 1 x is low for \mathcal{R} -randomness versus uniform \mathcal{S} -randomness.
- 2 $x + y$ is \mathcal{S} -random whenever y is \mathcal{R} -random.