Differentiability and porosity

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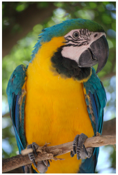

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“Almost everywhere” theorems

Several important theorems in analysis assert a property for almost every real z. Two examples:

Theorem (Lebesgue, 1904)

Let $E \subseteq [0,1]$ be measurable. Then for almost every $z \in [0,1]$: if $z \in E$, then E has density 1 at z.

Theorem (Lebesgue, 1904)

Let $f : [0,1] \to \mathbb{R}$ be of bounded variation. Then the derivative $f'(z)$ exists for almost every real z.

Variation of a function

Recall that for a function $g\colon [0,1] \to \mathbb{R}$ we let

$$V(g,[0,x]) = \sup \sum_{i=1}^{n-1} |g(t_{i+1}) - g(t_i)|,$$

where the sup is taken over all $t_1 \leq t_2 \leq \ldots \leq t_n$ in $[0,x]$.

We say that g is of bounded variation if $V(g,[0,1])$ is finite.

Complexity of the exception set

Theorem (Demuth 1975/Brattka, Miller, Nies 2011)

Let $r \in [0,1]$. Then
r is ML-random $\Longleftrightarrow$
$f'(r)$ exists, for each function f of bounded variation such that $f(q)$ is a computable real, uniformly in each rational q.

- The implication "$\Rightarrow$" is an effective version of the classical theorem.
- The implication "$\Leftarrow$" has no classical counterpart. To prove it, one builds a computable function f of bounded variation that is only differentiable at ML-random reals.

Computable randomness

Can you bet on this and make unbounded profit?

1010011100010111101010100001010110111101100001011110101 0
100101011000111110101100011001111111011000001110011110 00
001100110111101000111101000111001010110110010111000101 10
011001100011110000100110010111011001001010000011100011 11
111001000110001011111101000101111100110111001001100110 10
001111110110101011010011010101100000110000010011010111 00
010010010010110100010100001101000101000111000011000001 00
110001111101110010000110010110101001111011110101011111 11
000000010100111100100000000110110010100110101011010000 10 ...

We call a sequence of bits computably random if no computable betting strategy (martingale) has unbounded capital along the sequence.

ML-random $\Rightarrow$ computably random, but not conversely.

Computable randomness and differentiability

Theorem (Brattka, Miller, Nies, 2011)

Let $r \in [0,1]$. Then
r (in binary) is computably random $\iff$
$f'(r)$ exists, for each *nondecreasing* function f that is uniformly computable on the rationals.

- Full computability of a function $f\colon [0,1] \to \mathbb{R}$ means that with a Cauchy name for x as an oracle, one can compute a Cauchy name for $f(x)$.
- For *continuous* nondecreasing functions, full computability is equivalent to being computable on the rationals.

Other notions of effectiveness

Variants of the Demuth/ BMN theorems have been proved:

Theorem (Freer, Kjos, Nies, Stephan, 2012)

x is computably random $\Leftrightarrow$
each computable Lipschitz functions is differentiable at x.

Theorem (BMN, 2011)

z is weakly 2-random $\Leftrightarrow$
each a.e. differentiable computable function f is differentiable at z.

Theorem (Pathak, Rojas, Simpson 2011/ Freer, Kjos, Nies, Stephan, 2012/ Rute's thesis)

z is Schnorr random $\Leftrightarrow$
z is a weak Lebesgue point of each L_1-computable function.

In this talk, we will look (mostly) at nondecreasing functions, but
vary the notion of effectiveness.

Δ^1_1 randomness

Effectiveness "higher up"...

To be hyperarithmetical means to be computable in $\emptyset^{(\alpha)}$ for some recursive ordinal α. Another term for this is Δ^1_1.

We say that z is Δ^1_1 random if no hyperarithmetical martingale succeeds on z.

- This notion was proposed by Martin-Löf in a 1970 paper (4 years after his famous one). He showed there is no universal test for Δ^1_1-randomness.
- Higher randomness was later studied e.g. by Hjorth/N (2006), Chong, N and Yu (2008), and recently Bienvenu, Greenberg and Monin.

Hyperarithmetical functions

Theorem

z is Δ^1_1 random

$\Leftrightarrow$ each nondecreasing hyperarithmetical fcn f is differentiable at z

$\Leftrightarrow$ each hyperarithmetical f of bounded variation is differentiable at z.

- This is because $V(f, [0, x])$ can be evaluated by quantifying over rationals, and hence is also hyperarithmetical. So the Jordan decomposition $V(f) - (V(f) - f)$ into two non-decreasing functions of a hyperarithmetical function f is hyperarithmetical.
- It is hard to get past Δ^1_1 randomness. Even the a.e. differentiable hyperarithmetical functions only need that.
- However, there is a larger class, the interval Π^1_1 functions, where even higher ML-randomness is not sufficient to make all functions differentiable.

The two theorems

Firstly, we will look at feasibly computable nondecreasing functions. One obtains an analog of the Brattka, Miller, N 2011 result.

Theorem (Kawamura and Miyabe/ N independently)

$r \in [0,1]$ is polynomial time random $\Longleftrightarrow$
$g'(r)$ exists, for each nondecreasing function g
that is polynomial time computable.

Secondly, we look at a class of nondecreasing functions larger than computable. We say a nondecreasing function f is interval c.e. if $f(0) = 0$, and for any rational $q > p$, $f(q) - f(p)$ is a uniformly left-c.e. real.

Theorem

Let $z \in [0,1]$. Then z is a ML-random density-one point $\Longleftrightarrow$
$f'(r)$ exists, for each interval-c.e. function f

Porosity, density, and derivatives

Upper and lower derivatives

Let $f\colon [0,1] \to \mathbb{R}$. We define

$$\begin{aligned} \overline{D}f(z) &= \limsup_{h\to 0} \frac{f(z+h)-f(z)}{h} \\ \underline{D}f(z) &= \liminf_{h\to 0} \frac{f(z+h)-f(z)}{h} \end{aligned}$$

Then

$$f'(z) \text{ exists} \iff \overline{D}f(z) \text{ equals } \underline{D}f(z) \text{ and is finite.}$$

Notation for slopes, and for basic dyadic intervals

For a function $f\colon \subseteq \mathbb{R} \to \mathbb{R}$, the *slope* at a pair a, b of distinct reals in its domain is

$$S_f(a,b) = \frac{f(a) - f(b)}{a - b}.$$

For an interval A with endpoints a, b, we also write $S_f(A)$ instead of $S_f(a,b)$.

- Let $[\sigma]$ denote the closed basic dyadic interval $[0.\sigma, 0.\sigma + 2^{-|\sigma|}]$, for a string σ.
- The open basic dyadic interval is denoted (σ).
- We write $S_f([\sigma])$ with the expected meaning.

Pseudo-derivatives

- If f is only defined on the rationals in $[0, 1]$, we can still consider the upper and lower *pseudo*-derivatives defined by:

$$\underset{\sim}{D}f(x) = \liminf_{h \to 0^+} \{S_f(a, b) \mid a \leq x \leq b \wedge 0 < b - a \leq h\},$$
$$\widetilde{D}f(x) = \limsup_{h \to 0^+} \{S_f(a, b) \mid a \leq x \leq b \wedge 0 < b - a \leq h\}.$$

  where a, b range over rationals in $[0, 1]$.

- If f is total and continuous, or nondecreasing, this is the same as the usual derivatives.
- We will use the subscript 2 to indicate that all the limit operations are restricted to the case of basic dyadic intervals containing z. For instance,

$$\widetilde{D}_2 f(x) = \limsup_{|\sigma| \to \infty} \{S_f([\sigma]) \mid x \in [\sigma]\}.$$

Porosity

Definition (sukasuka)

We say that a set $\mathcal{C} \subseteq \mathbb{R}$ is porous at z via the porosity factor $\varepsilon > 0$ if there exists arbitrarily small $\beta > 0$ such that $(z - \beta, z + \beta)$ contains an open interval of length $\varepsilon\beta$ that is disjoint from $\mathcal{C}$.

Definition

We call z a porosity point if some effectively closed class to which it belongs is porous at z. Otherwise, z is a non-porosity point.

Theorem (Bienvenu, Hölzl, Miller, N, 2011)

Let z be ML-random. If z is a porosity point then z is Turing complete.

The converse is currently unknown.
We will see that porosity at a real z is closely related to non-differentiability at z. Both say that something bad happens for arbitrarily short intervals containing z.

Non-porosity and the Denjoy alternative

The short story how porosity came up in this research direction.

- The Denjoy-Saks-Young theorem for a function $f\colon \subseteq [0,1] \to \mathbb{R}$ defined on all the rationals in $[0,1]$ says that for almost every $z \in [0,1]$, the Denjoy alternative holds:
  either $\underset{\sim}{D}f(z) = -\infty$ and $\widetilde{D}f(z) = \infty$, or both are equal and finite (so f is pseudo-differentiable at z).
- f is Markov computable if it is uniformly computable on indices for computable reals.
- Demuth showed that Demuth randomness of z implies the DA for f at z. The following is much stronger.

Theorem (BHMN, 2011)

Let f be Markov computable. Let z be a computably random non-porosity point. Then the DA holds for f at z.

Density

The (lower Lebesgue) density of a set $\mathcal{C} \subseteq \mathbb{R}$ at a point z is the quantity

$$\varrho(\mathcal{C}|z) := \liminf_{z \in I \,\wedge\, |I| \to 0} \frac{\lambda(I \cap \mathcal{C})}{|I|},$$

where I ranges over intervals containing z.

Definition (Bienvenu, Hölzl, Miller, N, 2011)

We say that $z \in [0,1]$ is a density-one point if $\varrho(\mathcal{C}|z) = 1$ for every effectively closed class $\mathcal{C}$ containing z.

Recent solution of the covering problem via density

Theorem [Bienvenu, Greenberg, Kučera, N. Turetsky, Mar 2012]

Suppose some effectively closed (i.e., Π^0_1) class $\mathcal{P} \subseteq 2^{\mathbb{N}}$ has lower density < 1 at some ML-random set $Y \in \mathcal{P}$.

Then Y is Turing above each K-trivial set.

Theorem [Day and Miller, August 2012]

There is an effectively closed class $\mathcal{P}$ and a ML-random set $Y \in \mathcal{P}$ strictly Turing below the halting problem such that $\mathcal{P}$ has lower density < 1 at Y.

- Thus, there is a *single* Turing incomplete ML-random Δ^0_2 set Y above all the K-trivials!
- BGKNT '12 also showed that this Y must be close to the halting problem.

Dyadic versus full density

A (closed) basic dyadic interval has the form $[r2^{-n}, (r+1)2^{-n}]$ where $r \in \mathbb{Z}, n \in \mathbb{N}$. For the lower dyadic density of a set $\mathcal{C} \subseteq \mathbb{R}$ at a point z only consider basic dyadic intervals containing z:

$$\varrho_2(\mathcal{C}|z) := \liminf_{z \in I \,\wedge\, |I| \to 0} \frac{\lambda(I \cap \mathcal{C})}{|I|},$$

where I ranges over basic dyadic intervals containing z.

Theorem (Khan and Miller, 2012)

Let z be a ML-random dyadic density-one point. Then z is a full density-one point.

We know from Franklin and Ng (2010) and BHMN (2011) that z is a non-porosity point. The actual statement Miller and Khan proved:

Suppose z is a ML-random non-porosity point. Let $\mathcal{P}$ be a Π^0_1 class, $z \in \mathcal{P}$, and $\varrho_2(\mathcal{P} \mid z) = 1$. Then already $\varrho(\mathcal{P} \mid z) = 1$.

Khan has shown that ML-randomness is necessary here. See the 2013 Logic Blog available on my web site.

Khan/Miller: Suppose z is a non-porosity point. Let $\mathcal{P}$ be a Π^0_1 class, $z \in \mathcal{P}$, and $\varrho_2(\mathcal{P} \mid z) = 1$. Then already $\varrho(\mathcal{P} \mid z) = 1$.

Proof.

Consider an arbitrary interval I with $z \in I$ and $\lambda_I(\mathcal{P}) < 1 - \epsilon$. Let $\delta = \epsilon/4$.

Let n be such that $2^{-n+1} > |I| \geq 2^{-n}$. Cover I with three consecutive basic dyadic intervals A, B, C of length 2^{-n}.
Say $z \in B$. Since $\mathcal{P}$ is relatively sparse in I, but thick in B, this means it must be sparse in A or C.
Let the Π^0_1 class $\mathcal{Q}$ consist of the basic dyadic intervals where $\mathcal{P}$ is thick:

$$\mathcal{Q} = [0,1] - \bigcup\{L\colon\ \lambda_L(\mathcal{P}) < 1 - \delta\}$$

where L ranges over *open* basic dyadic intervals. Then $\mathcal{Q}$ is porous at z with porosity factor $1/3$: if $z \in B$, say, then one of A, C must be missing. □

Slopes and martingales

The basic connections:

- if f is nondecreasing then $M(\sigma) = S_f([\sigma])$ is a martingale.
- M succeeds on $z \Leftrightarrow \widetilde{D}_2 f(z) = \infty$
- M converges on $z \Leftrightarrow \underset{\sim}{D}_2 f(z) = \widetilde{D}_2 f(z) < \infty$

A useful lemma entirely in the classical analysis setting

High dyadic slopes lemma

Suppose $f\colon [0,1] \to \mathbb{R}$ is a nondecreasing function. Suppose for a real $z \in [0,1]$ we have

$$\widetilde{D}_2 f(z) < p < \widetilde{D} f(z).$$

Let $\sigma^* \prec Z$ be any string such that $\forall \sigma \, [Z \succ \sigma \succeq \sigma^* \Rightarrow S_f([\sigma]) \leq p]$. Then the closed set

$$\mathcal{C} = [\sigma^*] - \bigcup \{ (\sigma) \colon \sigma \succeq \sigma^* \, \wedge \, S_f([\sigma]) > p \},$$

which contains z, is porous at z.

Proof of high dyadic slopes lemma

Show:

$$\widetilde{D}_2 f(z) < p < \widetilde{D} f(z)$$

implies

$$\mathcal{C} = [\sigma^*] - \bigcup\{(\sigma) \colon \sigma \succeq \sigma^* \wedge S_f([\sigma]) > p\}$$

is porous at z.

Proof.

Suppose $k \in \mathbb{N}$ is such that $p(1 + 2^{-k+1}) < \widetilde{D} f(z)$. We show that can choose 2^{-k-2} as a porosity constant.

- There is an interval $I \ni z$ of arbitrarily short positive length such that $p(1 + 2^{-k+1}) < S_f(I)$. Let n be such that $2^{-n+1} > |I| \geq 2^{-n}$.
- Let a_0 be greatest of the form $v2^{-n-k}$, $v \in \mathbb{Z}$, such that $a_0 < \min I$.
- Let $a_v = a_0 + v2^{-n-k}$. Let r be least such that $a_r \geq \max I$.

By the averaging property of slopes and since f is nondecreasing, there must be i with $0 \leq i \leq r$ such that the slope at $[a_i, a_{i+1}]$ is $> p$. This interval does not contain z. □

Polynomial time randomness and differentiability

Special Cauchy names

A Cauchy name is a sequence of rationals $(p_i)_{i\in\mathbb{N}}$ such that $\forall k > i \, |p_i - p_k| \leq 2^{-i}$. We represent a real x by a Cauchy name converging to x.

For feasible analysis, we use a compact set of Cauchy names: the signed digit representation of a real. Such Cauchy names, called special, have the form

$$p_i = \sum_{k=0}^{i} b_k 2^{-k},$$

where $b_k \in \{-1, 0, 1\}$. (Also, $b_0 = 0, b_1 = 1$.)

So they are given by paths through $\{-1, 0, 1\}^\omega$, something a resource bounded TM can pro cess. We call the b_k the symbols of the special Cauchy name.

Polynomial time computable functions

The following has been formulated in equivalent forms by Ker-i-Ko (1989), Weihrauch (2000), Braverman (2008), and others.

Definition

A function $g\colon [0,1] \to \mathbb{R}$ is polynomial time computable if there is a polynomial time TM turning every special Cauchy name for $x \in [0,1]$ into a special Cauchy name for $g(x)$.

This means that the first n symbols of $g(x)$ can be computed in time poly(n), thereby using polynomially many symbols of the oracle tape holding x.

Functions such as $e^x, \sin x$ are polynomial time computable.

Analysis gives us rapidly converging approximation sequences, such as $e^x = \sum_n x^n/n!$. As Braverman points out, e^x is computable in time $O(n^3)$. Namely, from $O(n^3)$ symbols of x we can in time $O(n^3)$ compute an approximation of e^x with error $\leq 2^{-n}$.

Polynomial time randomness

A martingale $M: 2^{<\omega} \to \mathbb{R}$ is called polynomial time computable if from string σ and $i \in \mathbb{N}$ we can in time polynomial in $|\sigma| + i$ compute the i-th component of a special Cauchy name for $M(\sigma)$.

We say Z is polynomial time random if no polynomial time martingale succeeds on Z.

Fact

f is a nondecreasing polynomial time computable function

$$\Rightarrow$$

the slope $S_f([\sigma])$ determines a polynomial time computable martingale.

This is so because we can compute f with sufficiently high precision.

The first theorem

Theorem (Kawamura and Miyabe/ N independently)

The following are equivalent.

(I) $z \in [0,1]$ is polynomial time random

(II) $f'(z)$ exists, for each nondecreasing function f that is polynomial time computable.

(II) → (I)

One actually shows: z not polytime random $\Rightarrow$

$$\underline{D}f(z) = \infty \text{ for some polynomial time computable function } f.$$

I use machinery from the Figueira/N (2013) paper 'Randomness, feasible analysis, and base invariance'.

Proof.

- If z is not polytime random, some polytime martingale M with the savings property succeeds on z.
- Then the function $\mathsf{cdf}_M : [0,1] \to \mathbb{R}$ given by $\mathsf{cdf}_M(x) = \mu_M[0,x)$ is polytime computable (using the almost Lipschitz property).
- And the lower derivative $\underline{D}\mathsf{cdf}_M(z) = \infty$.

□

(I) $\rightarrow$ (II)

We need to show:

$$z \in [0,1] \text{ is polynomial time random} \Rightarrow f'(z) \text{ exists,}$$

for each nondecreasing function f that is polynomial time computable.

- Consider the polynomial time computable martingale
$$M(\sigma) = S_f(0.\sigma, 0.\sigma + 2^{-|\sigma|}) = S_f([\sigma]) \ .$$
- $\lim_n M(Z \upharpoonright_n)$ exists and is finite for each polynomially random Z. This is a version of Doob martingale convergence.
- Returning to the language of slopes, the convergence of M on Z means that $\underset{\sim}{D}_2 f(z) = \widetilde{D}_2 f(z) < \infty$.

Assume for a contradiction that $f'(z)$ fails to exist. First suppose that

$$\widetilde{D}_2 f(z) < p < \widetilde{D} f(z).$$

We may suppose $S_f(A) < p$ for all dyadic intervals containing z. Choose k with $p(1 + 2^{-k+1}) < \widetilde{D} f(z)$.

By the "high dyadic slopes" lemma and its proof, there exists arbitrarily large n such that some basic dyadic interval $[\tau_n]$ of length 2^{-n-k} has slope $> p$ and is contained in $[z - 2^{-n+2}, z + 2^{-n+2}]$. Let $0.Z = z$ where $Z \in 2^{\mathbb{N}}$.

Lucky case: there are infinitely many n with $\eta = Z\restriction_{n-4} \prec \tau_n$. Then the martingale that from η on bets everything on the strings of length $n + k$ other than τ_n gains a fixed factor $2^{k+4}/(2^{k+4} - 1)$.

Unlucky case: for almost all n we have $Z\restriction_{n-4} \not\prec \tau_n$. That means $0.\tau_n$ is on the left side of z, and the martingale betting along Z can't use τ_n, as it may be far from Z in Cantor space! E.g. $Z = 1000000000000\ldots$, $n - 4 = 9$, $\tau_n = 0111111111111$.

Morayne-Solecki trick

The following was used in a paper by Morayne and Solecki (1989). They gave a martingale proof of Lebesgue differentiation theorem. For $m \in \mathbb{N}$ let $\mathcal{D}_m$ be the collection of intervals of the form

$$[k2^{-m}, (k+1)2^{-m}]$$

where $k \in \mathbb{Z}$. Let $\mathcal{D}'_m$ be the set of intervals $(1/3) + I$ where $I \in \mathcal{D}_m$.

Fact

Let $m \geq 1$. If $I \in \mathcal{D}_m$ and $J \in \mathcal{D}'_m$, then the distance between an endpoint of I and an endpoint of J is at least $1/(3 \cdot 2^m)$.

To see this: assume that $k2^{-m} - (p2^{-m} + 1/3) < 1/(3 \cdot 2^m)$. This yields $(3k - 3p - 2^m)/(3 \cdot 2^m) < 1/(3 \cdot 2^m)$, and hence $3|2^m$, a contradiction.

Using this trick

So, in the unlucky case, we instead bet on the dyadic expansion Y of $z - 1/3$. (We may assume that $z > 1/2$).

Given $\eta' = Y \upharpoonright_{n-4}$, where n is as above, we look for an extension $\tau' \succ \eta'$ of length $n + k + 1$, such that $1/3 + [\tau'] \subseteq [\tau]$ for a string $[\tau]$ with $S_f([\tau]) > p$. If it is found, we bet everything on the other extensions of η' of that length. We gain a fixed factor $2^{k+5}/(2^{k+5} - 1)$.

So we get a polytime martingale that wins on the dyadic expansion of $z - 1/3$. Since polytime randomness is base invariant, this gives a contradiction.

The case $\underset{\sim}{D} f(z) < \underset{\sim}{D}_2 f(z)$ is analogous, using the symmetric "low dyadic slopes lemma" instead.

Ambos-Spies et al., 1996 called a martingale "weakly simple" if it has only have finitely many, rational, betting factors. The martingales showing that dyadic derivative = full derivative are such. So being polynomially stochastic is sufficient for this.

Martin-Löf random density-one points

and differentiability

The second theorem

Theorem

Let $f\colon [0,1] \to \mathbb{R}$ be an interval-c.e. function. Let z a be ML-random density-one point. Then $f'(z)$ exists.

Interval-c.e. functions

A real z is called left-c.e. if the set of rationals $< z$ is c.e.

Definition

A non-decreasing function f on $[0,1]$ with $f(0) = 0$ is called interval-c.e. if $f(q) - f(p)$ is a left-c.e. real uniformly in rationals $p < q$.

If f is continuous, this implies lower semicomputable.
Recall that for $g\colon [0,1] \to \mathbb{R}$ we let

$$V(g,[0,x]) = \sup \sum_{i=1}^{n-1} |g(t_{i+1}) - g(t_i)|,$$

where the sup is taken over all $t_1 \leq t_2 \leq \ldots \leq t_n$ in $[0,x]$.

Theorem (Freer, Kjos-Hanssen, N, Stephan, Rute 2012)

A continuous function f is interval-c.e. $\Leftrightarrow$
there is a computable function g such that $f(x) = \mathrm{Var}(g,[0,x])$.

Left-c.e. martingales

Definition

A martingale $M: 2^{<\omega} \to \mathbb{R}$ is called left-c.e. if $M(\sigma)$ is a left-c.e. real uniformly in string σ.

Z is ML-random iff no left-c.e. martingale succeeds on Z.

Definition

A martingale M converges on $Z \in 2^{\mathbb{N}}$ if $lim_n M(Z\restriction_n)$ exists and is finite.

$Z \in 2^{\mathbb{N}}$ is a convergence point for left-c.e. martingales if each left-c.e. martingale converges on Z.

- The computably randoms are the convergence points for all computable martingales.
- The Martin-Löf randoms that are density-one points are the convergence points for all left-c.e. martingales (Andrews, Cai, Diamondstone, Lempp, Miller; 2012).

The actual theorem

Theorem

Let $f\colon [0,1] \to \mathbb{R}$ be an interval-c.e. function. Let z be a convergence point for left-c.e. martingales. Then $f'(z)$ exists.

The basic connection:

- if f is interval-c.e., then $M(\sigma) = S_f([\sigma])$ is a left-c.e. martingale.
- Convergence of M on Z means that $\underset{\sim}{D}_2 f(z) = \widetilde{D}_2 f(z)$, i.e., f is dyadic differentiable at z.

The theorem says that we can get full differentiability for convergence points for left-c.e. martingales (but also looking at other left-c.e. martingales).

Recall: High dyadic slopes lemma

Suppose $f\colon [0,1] \to \mathbb{R}$ is a nondecreasing function. Suppose for a real $z \in [0,1]$ we have

$$\widetilde{D}_2 f(z) < p < \widetilde{D} f(z).$$

Let $\sigma^* \prec Z$ be any string such that $\forall \sigma\, [Z \succ \sigma \succeq \sigma^* \Rightarrow S_f([\sigma]) \leq p]$. Then the closed set

$$\mathcal{C} = [\sigma^*] - \bigcup \{(\sigma)\colon\, \sigma \succeq \sigma^* \,\wedge\, S_f([\sigma]) > p\},$$

which contains z, is porous at z.

Proposition

Let $f\colon [0,1] \to \mathbb{R}$ be interval-c.e. Then $\widetilde{D}_2 f(z) = \widetilde{D} f(z)$ for each non-porosity point z.

Proof.

Assume $\widetilde{D}_2 f(z) < \widetilde{D} f(z)$. Since f is interval c.e., the class $\mathcal{C}$ defined in the Lemma is effectively closed. This class is porous at z. Contradiction. □

Proof that $f'(z)$ exists for left-c.e. convergence points z

We may assume $z > 1/2$, else we work with $f(x + 1/2)$ instead of f.

- The real z is a a dyadic density one point, hence a (full) density-one point by the Khan-Miller Theorem.
- Then $z - 1/3$ is also a ML-random density-one point, so using the work of the Madison group discussed earlier, $z - 1/3$ is also a convergence point for left-c.e. martingales.
- In particular, both z and $z - 1/3$ are non-porosity points.

To complete the proof ...:

Let M be the martingale associated with the dyadic slopes of f.

- Note that M converges on z by hypothesis. Thus $\underset{\sim}{D}_2 f(z) = \widetilde{D}_2 f(z) = M(z)$.
- By the Proposition above we have $\widetilde{D}_2 f(z) = \widetilde{D} f(z)$.
- It remains to be shown that

  $$\underset{\sim}{D} f(z) = \underset{\sim}{D}_2 f(z).$$

  Since f is nondecreasing, $\underline{D} = \underset{\sim}{D}$ etc., so this will establish that $f'(z)$ exists.

Shifting by 1/3 yields the same dyadic derivative

Let $\widehat{f}(x) = f(x + 1/3)$, and let M' the martingale associated with the dyadic slopes of $\widehat{f}$.

Claim

$M(z) = M'(z - 1/3)$.

Proof.

Since $z - 1/3$ is a convergence point for c.e. martingales, M' converges on $z - 1/3$.

If $M(z) < M'(z - 1/3)$ then $\widetilde{D}_2 f(z) < \widetilde{D} f(z)$. However z is a non-porosity point, so this contradicts the Proposition.

If $M'(z - 1/3) < M(z)$ we argue similarly, using that $z - 1/3$ is a non-porosity point. □

Choosing some rational parameters

Assume for a contradiction that

$$\underset{\sim}{D} f(z) < \underset{\sim}{D}_2 f(z).$$

Then we can choose rationals p, q such that

$$\underset{\sim}{D} f(z) < p < q < M(z) = M'(z - 1/3).$$

Let $k \in \mathbb{N}$ be such that $p < q(1 - 2^{-k+1})$.

Let u, v be rationals such that

$$q < u < M(z) < v \text{ and } v - u \leq 2^{-k-3}(u - q).$$

Two Π^0_1 classes

Let $n^* \in \mathbb{N}$ be such that we have $S_f(A) \geq u$, for each $n \geq n^*$ and any interval A of length $\leq 2^{-n^*}$ that is basic dyadic or basic dyadic $+1/3$.

$$\begin{aligned} \mathcal{E} &= \{X \in 2^{\mathbb{N}} \colon \forall n \geq n^* M(X \restriction_n) \leq v\} \\ \mathcal{E}' &= \{W \in 2^{\mathbb{N}} \colon \forall n \geq n^* M'(W \restriction_n) \leq v\} \end{aligned}$$

- Let $0.Z$ be as usual the binary expansion of z. Let $0.Y$ be the binary expansion of $z - 1/3$.
- We have $Z \in \mathcal{E}$ and $Y \in \mathcal{E}'$.

We will show that $\mathcal{E}$ is porous at Z, or $\mathcal{E}'$ is porous at Y.

Low dyadic slopes for both types of intervals

Consider an interval $I \ni z$ of positive length $\leq 2^{-n^*-3}$ such that $S_f(I) \leq p$.

- Let n be such that $2^{-n+1} > |I| \geq 2^{-n}$.
- Let a_0 $[b_0]$ be least of the form $j2^{-n-k}$ $[j2^{-n-k} + 1/3]$, where $j \in \mathbb{Z}$, such that a_0 $[b_0] \geq \min(I)$.
- Let $a_v = a_0 + v2^{-n-k}$ and $b_v = b_0 + v2^{-n-k}$. Let r, s be greatest such that $a_r \leq \max(I)$ and $b_s \leq \max(I)$.

Since f is nondecreasing and $a_r - a_0 \geq |I| - 2^{-n-k+1} \geq (1 - 2^{-k+1})|I|$, we have $S_f(I) \geq S_f(a_0, a_r)(1 - 2^{-k+1})$, and therefore $S_f(a_0, a_r) < q$. (Slope at I is low, slope at $[a_0, a_r]$ can only be slightly larger.) Then there is an $i < r$ such that $S_f(a_i, a_{i+1}) < q$.
Similarly, there is $j < s$ such that $S_f(b_j, b_{j+1}) < q$.

Claim (Morayne-Solecki trick)

One of the following is true.

(i) z, a_i, a_{i+1} are all contained in a single interval taken from $\mathcal{D}_{n-3}$.

(ii) z, b_j, b_{j+1} are all contained in a single interval taken from $\mathcal{D}'_{n-3}$.

Proving porosity of one of the Π^0_1 classes

Let $\eta = Z \upharpoonright_{n-3}$ and $\eta' = Y \upharpoonright_{n-3}$.

If (i) holds for this I then there is α of length $k+3$ (where $[\eta\alpha] = [a_i, a_{i+1}]$) such that $M(\eta\alpha) < q$.

- So by the choice of $q < u < v$ and since $M(\eta) \geq u$ there is β of length $k+3$ such that $M(\eta\beta) > r$.
- This yields a hole in $\mathcal{E}$, large and near $z = 0.Z$ on the scale of I, which is required for porosity of $\mathcal{E}$ at Z.

Similarly, if (ii) holds for this I, then there is α of length $k+3$ (where $[\eta'\alpha] = [b_j, b_{j+1}]$) such that $M'(\eta'\alpha) < q$. This yields a hole large and near $z - 1/3 = 0.Y$ on the scale of I required for porosity of $\mathcal{E}'$ at Y.

Thus, if case (i) applies for arbitrarily short intervals I, then $\mathcal{E}$ is porous at Z, whence z is a porosity point. Otherwise (ii) applies for intervals below a certain length. Then $\mathcal{E}'$ is porous at Y, whence $z - 1/3$ is a porosity point. Either case is a contradiction.

Some open questions

Question

Study effective analogs of Rademacher's theorem that every Lipschitz function on $\mathbb{R}^n$ is a.e. differentiable.

Question

If Z is a ML-random density-one point, is it Oberwolfach random? Equivalently, does it fail to compute some K-trivial?

Full proofs of the two theorems are on the 2013 Logic blog, available on my web site.