On quantum Kolmogorov complexity and their relationships

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Preface

- Why QKC?
  - It can be a new tool to analyze quantum state in information theoritical point of view (ex. Miyadera[2012])
  - It can be a basic notion to consider “quantum randomness”
- Today I will talk about
  - Two definition of QKC by Berthiaume et al. and Gács and some known facts about them,
  - My conjecture (quantum Levin’s coding theorem).

Axioms for quantum mechanics

- To every quantum system ***S*** uniquely associated is a Hilbert space ***H*** called the state space of ***S***. States of ***S*** are represented by unit vectors $|\psi\rangle$ in ***H***.

Ex.) Qubit system … $\mathbb{C}^2$, $\{|0\rangle, |1\rangle\}$

- For every unit vector in ***H***, there exists a state represented by that vector.

Ex.) $\dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$

Axioms for quantum mechanics

- The state space of the composite system $\boldsymbol{S} = \boldsymbol{S_1} + \boldsymbol{S_2}$ consisting of the system $\boldsymbol{S_1}$ with the state space $\boldsymbol{H_1}$ and of the system $\boldsymbol{S_2}$ with the state space $\boldsymbol{H_2}$ is the tensor product $\boldsymbol{H_1} \otimes \boldsymbol{H_2}$.

Ex.) the space of n-letter qubit strings … $\{\mathbb{C}^2\}^{\otimes n}$

$$\{|0...0\rangle, \ldots, |1\ldots1\rangle\}$$

→ the space of all qubit strings … $\bigoplus_{n=0}^{\infty} \{\mathbb{C}^2\}^{\otimes n}$

Axioms for quantum mechanics

- (discrete) Time evolution of quantum states is expressed by a unitary operator ***U***:

$$|\psi\rangle \quad \rightarrow \quad U|\psi\rangle \rightarrow \quad U^2|\psi\rangle \quad \rightarrow \quad \ldots$$

Several definitions of QKC

- Use quantum Turing machine
  - Vitányi [2001]
  - Berthiaume et al. [2001]
- "Quantize" universal semimeasure
  - Gács [2001]

Quantum Turing machine (QTM)

- M = (Q, ∑, δ)
- δ : Q × ∑ × Q × ∑ × {L , R} → $\tilde{\mathbb{C}}$
- δ uniquely defines an operator ***U*** on $\boldsymbol{H_{qtm}}$:

$$U\left|q,T,\zeta\right\rangle = \sum_{\substack{(p,\tau,d)\\ \in Q\times\Sigma\times\{-1,1\}}} \delta\left(q,T(\zeta),p,\tau,d\right)\left|p,T_{\zeta}^{\tau},\zeta+d\right\rangle$$

- We call this machine QTM only if ***U*** is unitary.

Existence of universal QTM

<u>Theorem (Bernstein – Vazirani, 1997)</u> *There exists a QTM* M_0 *such that*

$$\forall M\ \exists s_m\ \forall T\ \forall \delta > 0\ \forall |\psi\rangle \quad \left\| M_0\left(s_m, T, \delta, |\psi\rangle\right) - M\left(|\psi\rangle\right) \right\|_{Tr} < \delta.$$

This machine simulates each QTM with polynomial slowdown.

Qubit complexity (Berthiaume et al. 2001)

- $QC^{\delta}\left(|\psi\rangle\right) = \min\left\{ l\left(|\varphi\rangle\right) \;\middle|\; \left\| M_0\left(|\varphi\rangle\right) - |\psi\rangle \right\|_{Tr} < \delta \right\}$
- $QC^{\downarrow}\left(|\psi\rangle\right) = \min\left\{ l\left(|\varphi\rangle\right) \;\middle|\; \forall k \left\| M_0\left(|\varphi\rangle, k\right) - |\psi\rangle \right\|_{Tr} < \frac{1}{k} \right\}$

Problem (invariance of QC)

- (Classical case) for universal TM M_0 and any TM M, there exists a constant c_M such that

$$C_M(w) \leq C_{M_0}(w) + c_M .$$

(Just input $\langle s_M, w \rangle$ to M_0)

- Bernstein – Vazirani UQTM requires the Halting time as an input … can't show the invariance of qubit complexity in the same manner as above.

Strongly UQTM (Müller, 2007)

Theorem (Müller, 2007) *There exists a QTM* M_1 *such that*

$$\forall M\ \forall|\psi\rangle\ \exists|\psi_M\rangle\ \forall\delta>0\quad \left\| M_1\left(|\psi_M\rangle,\delta\right)-M\left(|\psi\rangle\right)\right\|_{Tr}<\delta$$

for each $|\psi\rangle$ *for which* $M\left(|\psi\rangle\right)$ *is defined, where*

$$l\left(|\psi_M\rangle\right)\le l\left(|\psi\rangle\right)+c_M.$$

$$\rightarrow\begin{cases} QC^{\Delta}_{M_0}\left(|\psi\rangle\right)\le QC^{\delta}_{M}\left(|\psi\rangle\right)+c_{\Delta,\delta,M} & (\delta<\Delta) \\ QC^{\downarrow}_{M_1}\left(|\psi\rangle\right)\le QC^{\downarrow}_{M}\left(|\psi\rangle\right)+c_M & \end{cases}$$

Relation between classical complexity

Theorem (Müller, 2009)

$$QC^{\downarrow}(|x\rangle) = C(x) + O(1).$$

Question: Is the notion of QTM indispensable for the definition of qubit complexity?
Is there an equivalent definition of qubit complexity on the notion of classical Turing machine?

Density matrix

- $\rho \in \mathbb{M}_N$ is called *Density matrix* if

$$\rho \geq 0, \quad Tr(\rho) = 1.$$

→ $\rho = \sum_{i=1}^{N} \lambda_i |\psi_i\rangle\langle\psi_i|,$ where $\lambda_i \geq 0, \ \sum_{i=1}^{N} \lambda_i = 1.$

→ Density matrix represents a probability distribution of quantum states,

$$\{\lambda_1, \ldots, \lambda_N; |\psi_1\rangle, \ldots, |\psi_N\rangle\}.$$

Universal semi-density matrix (Gács, 2001)

- Semi-density matrix ρ (that is, $\rho \in \mathbb{M}_N$ with $\rho \geq 0, \quad Tr(\rho) \leq 1$) is *lower-semicomputable* if there is a sequence $\{\rho_n\}$ of elementary matrix such that

$$\rho_n \leq \rho_{n+1}, \; \rho_n \to \rho \; (n \to \infty).$$

- Lower-semicomputable semi-density matrix μ is *universal* if for any LSC-SDM ρ there is a constant $c_\rho > 0$ such that

$$\rho \leq c_\rho \mu.$$

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