Truth-table Schnorr randomness and truth-table reducibly randomness

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Abstract

Schnorr randomness and computably randomness are natural concepts of random sequences. However van Lambalgen's Theorem fails for both randomnesses. In this paper we define truth-table Schnorr randomness (defined in [6] too only by martingales) and truth-table reducibly randomness, for which we prove that van Lambalgen's Theorem holds. We also show that the classes of truth-table Schnorr random reals relative to a high set contain reals Turing equivalent to the high set. It follows that each high Schnorr random real is half of a real for which van Lambalgen's Theorem fails. Moreover we establish the coincidence between triviality and lowness notions for truth-table Schnorr randomness.

1 Introduction

Martin-Löf randomness was a first concept that succeeded in defining a natural randomness. However Schnorr criticized that randomness should be concerned with computable strategies. Schnorr defined Schnorr randomness that we call now and computably randomness.

Martin-Löf random sequences have many properties that we think "random" sequences should have. Van Lambalgen [25] proved van Lambalgen's Theorem we call now. This theorem says that intuitively any part of a Martin-Löf random sequence should not have any information of the other part. In [15], they proved that Kolmogorov-Loveland randomness has a similar property . Hence van Lambalgen's theorem is a criterion for a proper random concept.

In contrast many properties about Schnorr randomness and computably randomness were proved like those about Martin-Löf randomness. However van Lambalgen's theorem fails for Schnorr randomness and computably randomness. See [15, 26] or Kjos-Hanssen's proof in [18]. Schnorr randomness or computably randomness seem to be very natural concepts and the failure of their van Lambalgen's theorem is unnatural. This suggests that Schnorr randomness and computably randomness have another relativizations for which van Lambalgen's Theorem holds.

There is another reason that we should consider another relativizations of Schnorr randomness. One of the major achievement in the study of Martin-Löf randomness is the discovery of the equivalence between triviality, lowness notions and the basis notion. However Schnorr trivial sets are not equal to the sets that are low for Schnorr randomness [4]. Franklin and Stephan [6] defined truth-table Schnorr randomness by martingales and proved that a set is Schnorr trivial iff it is tt-low for Schnorr randomness.

In section 3 we discuss the reason why we need another relativization of Schnorr randomness.

In section 4 we define truth-table Schnorr randomness, which is another relativization of Schnorr randomness. We show that truth-table Schnorr randomness can be characterized by tests, machines and martingales similar to Schnorr randomness and Martin-Löf randomness. We also show that van Lambalgen's Theorem holds for truth-table Schnorr randomness.

In section 5 we define truth-table reducibly randomness which is another relativization of computably randomness. We show that van Lambalgen's Theorem holds for truth-table reducibly randomness.

In section 6 we study a difference between Schnorr randomness, truth-table Schnorr randomness. We show that the classes of truth-table Schnorr random reals relative to a high real contain a real Turing equivalent to the high real. It follows that each high Schnorr random real is half of a real for which van Lambalgen's Theorem fails.

In section 7 we define lowness for tt-reducible measure machine and proved that tt-degree of a set is computably traceable iff it is low for tt-reducible measure machine. Then it follows that a set is Schnorr trivial iff it is low for tt-reducible measure machine iff it is low for tt-Schnorr randomness.

2 Preliminaries

Now we fix notations we use in this paper and recall some basic definitions and results. For a more complete introduction, see Soare [23] or Odifreddi [20, 21] for computability theory and Li and Vitányi [13], Downey and Hirschfeldt [5] or Nies [18] for algorithmic randomness.

2.1 Basics of Algorithmic randomness

We use $\langle\cdot,\cdot\rangle$ to denote Cantor's pairing function $\langle x, y\rangle = \frac{1}{2}(x+y)(x+y+1)+y$. For a set of natural numbers A, the set $A' = \{e : \Phi_e^A(e)\}$ is called the *jump* of A where Φ_e^A is an e-th partial computable function with A as an oracle. A set A is *c.e.* if $A = \{n : \Phi(n)\downarrow\}$ for some Turing functional Φ. Let $K = \phi' = \{e : \Phi_e(e)\downarrow\}$ and $K_s = \{e \le s : \Phi_{e,s}(s)\}$ a approximation of K by steps. Note that K is a noncomputable c.e. set. We say that A is *T-reducible to B*, write $A \le_T B$, if $A = \Phi_e^B$ for some e. We also say that A is *truth-table reducible to B*, write $A \le_{tt} B$, if there is a Turing functional Φ_e such that $A = \Phi_e^B$ and Φ_e^Z is total for each oracle Z.

We can regard a set A as an infinite binary sequence such that i-th bit of the sequence is 1 if $i \in A$ and 0 if $i \notin A$. Let 2^ω denote the set of all infinite binary sequences and $2^{<\omega}$ the set of all finite binary strings. We also identify real numbers with their inifinite binary expansion. Elements of Cantor space 2^ω are sometimes called *reals*. For $\sigma \in 2^{<\omega}$, $|\sigma|$ denotes the length of σ. Let D_n be the effective listing of the finite subsets of ω satisfying the followings. Let $D_0 = \phi$. If $n > 0$ has the form $2^{x_1} + 2^{x_2} + \cdots + 2^{x^r}$ where $x_1 < \cdots < x_r$, then let $D_n = \{x_1, \cdots, x_r\}$. We say that n is a *strong index* for D_n. We write $\sigma \prec \tau$ to mean that σ is prefix of τ, that is $(\exists\rho)\sigma\rho = \tau$. Here τ can be infinite. Let $[\sigma] = \{Z \in 2^\omega : \sigma \prec Z\}$ be the class of infinite binary sequences extending σ. We use λ to denote the empty string. We say that a set A is a *B-c.e. real* if A is the limit of a B-computable rational approximation. A function $f : \omega \to \mathbb{R}$ is *c.e.* if $f(n)$ are uniformly c.e. reals. A open set A is *c.e.* if the corresponding set of strings $\{\sigma : [\sigma] \subseteq A\}$ is a c.e. set.

A *Martin-Löf test* is a sequence of uniformly c.e. open sets $\{U_n\}$ such that $\mu(U_n) \le 2^{-n}$. A real A *passes* a Martin-Löf test U_n if $A \notin \bigcap_n U_n$. A real A is *Martin-Löf random* or *1-random* if A passes all Martin-Löf tests. We say that a Martin-Löf test $\{U_n\}$ is *universal* if $\bigcap_n U_n$ contains $\bigcap_m V_m$ for any Martin-Löf test $\{V_m\}$.

A set X is *prefix-free* if whenever $\sigma, \tau \in X$, then σ is not a proper prefix of τ. A partial function (or a machine) $M : 2^{<\omega} \to 2^{<\omega}$ is called a *prefix-free machine* if $\mathrm{dom}(M)$ is prefix-free. There is a *universal* prefix-free machine, i.e., a prefix-free machine U such that for each prefix-free machine M there is a string $\tau \in 2^{<\omega}$ for which $(\forall\sigma)U(\tau\sigma) = M(\sigma)$ or both $U(\tau\sigma)$ and $M(\tau)$ diverge. Then *prefix-free Kolmogorov complexity* K of a string σ is defined as $K(\sigma) = \min\{\tau : U(\tau) = \sigma\}$. Let $\Omega_U^A = \sum_\sigma 2^{-|\sigma|} [\![U^A(\sigma)\downarrow]\!]$. This is called halting probability relative to A.

Lemma 2.1 (Kraft-Chaitin Theorem, see [5])**.** *A Kraft-Chaitin set is a a computable list of pairs of a natural number and a string $(d_0, \tau_0), (d_1, \tau_1), \cdots$ such that $\sum_{i<\omega} 2^{-d_i} \le 1$. Then there is a partial computable machine N such that for a computable list $\{\sigma_i\}$, $|\sigma_i| = d_i$ and $N(\sigma_i) = \tau_i$ for all i.*

A *martingale* is a function $M : 2^{<\omega} \to \mathbb{R}^+ \cup \{0\}$ that satisfies for every $\sigma \in 2^{<\omega}$ the average condition $2M(\sigma) = M(\sigma 0) + M(\sigma 1)$. A martingale M *succeeds* on a real A if $\limsup_n M(A \upharpoonright n) = \infty$.

We can characterize Martin-Löf randomness by prefix-free Kolmogorov complexity and c.e. martingales.

Theorem 2.2 (Schnorr [22], see [5] or [18])**.** *The followings are equivalent.*

(i) *A real A is Martin-Löf random.*

(ii) *$K(A \upharpoonright n) > n - O(1)$ for all n.*

(iii) *No c.e. martingale succeeds on a real A.*

Martin-Löf randomness is a natural concepts of randomness. In particular the following theorem holds for Martin-Löf randomness.

Theorem 2.3 (van Lambalgen [25])**.** *$X \oplus Y$ is Martin-Löf random iff X is Martin-Löf random and Y is Martin-Löf random.*

Intuitively this theorem says any part of a random sequence should not have any information of the other part.

2.2 Schnorr randomness

Definition 2.4. *We call $\{U_n\}$ a Schnorr test if it is a Martin-Löf test and $\{\mu(U_n)\}$ is uniformly computable. A real A passes a Schnorr test U_n if $A \notin \bigcap_n U_n$. A real A is Schnorr random if A passes all Schnorr tests.*

The followings are known results about Schnorr randomness.

Theorem 2.5 (Schnorr [22]). *For each Schnorr test $\{U_n\}$, one may effectively find a Schnorr test $\{V_n\}$ such that $(\forall n)U_n \subseteq V_n$ and $(\forall n)\mu(V_n) = 2^{-n}$.*

Theorem 2.6 (Schnorr [22]). *There is no universal Schnorr test.*

As Martin-Löf randomness has characterizations by machines and martingales 2.2, Schnorr randomness also has similar characterizations.

Definition 2.7 (Downey and Griffiths [3]). *A prefix-free machine M is a computable measure machine if Ω_M is computable.*

Theorem 2.8 (Downey and Griffiths [3]). *The following are equivalent.*

(i) *A set A is Schnorr random.*

(ii) *For all computable measure machine M, $(\exists c)(\forall n)K_M(A \upharpoonright n) > n - c$.*

It is clear by the proof that we can effectively construct a computable measure machine by a Schnorr test and vice versa.

Theorem 2.9 (Franklin and Stephan [6], after Schnorr [22]). *The followings are equivalent for a set A.*

(i) *A set A is not Schnorr random.*

(ii) *There is a computable martingale F strongly succeeds on A: i.e., there is a computable unbounded nondecreasing function $h : \mathbb{N} \to \mathbb{N}$ such that $F(A \upharpoonright n) \geq h(n)$ infinitely often.*

(iii) *For each computable function r there is a computable martingale G and a strictly increasing computable function f such that $G(A \upharpoonright f(n)) \geq r(n)$ infinitely often.*

Again the effectiveness of the construction is clear by the proof.

In contrast with Martin-Löf randomness, [15, 26] proved that van Lambalgen's Theorem fails for Schnorr randomness. Here we cite the following result whose proof is in [18].

Theorem 2.10 (Kjos-Hanssen, see [18]). *Van Lambalgen's Theorem fails for Schnorr randomness. In particular there exists a Schnorr random set $A = A_0 \oplus A_1$ such that $A_0 \equiv_T A_1$.*

On the other hand the other direction holds.

Theorem 2.11 (Yu [26]). *If A is Schnorr random and B is A-Schnorr random then $A \oplus B$ is Schnorr random.*

2.3 Computably randomness

We say that a real is *computably random* if no computable martingale succeeds on it.

Definition 2.12 (Merkle, Mihailovic, and Slaman [16]). *A computable rational probability distribution (or measure) is a computable function $\nu : 2^{<\omega} \to \mathbb{Q}$, with $\nu(\lambda) = 1$ and $\nu(\sigma) = \nu(\sigma 0) + \nu(\sigma 1)$. A bounded Martin-Löf test is a Martin-Löf test for which there exists a computable rational probability distribution ν with*

$$\mu(V_n \cap [\sigma]) \le \frac{\nu(\sigma)}{2^n},$$

for all $n \in \mathbb{N}$ and $\sigma \in 2^{<\omega}$.

Definition 2.13 (Downey, Griffiths, and LaForte [4]). *A Martin-Löf test $\{V_n\}$ is computably graded if there exists a computable map $f : 2^{<\omega} \times \omega \to \mathbb{Q}$ such that, for any $n \in \omega$, $\sigma \in 2^{<\omega}$, and any finite prefix-free set of strings $\{\sigma_i\}_{i \le I}$ with $\bigcup_{i=0}^{I} [\sigma_i] \subseteq [\tau]$, the following conditions are satisfied:*

(i) $\mu(V_n \cap [\sigma]) \le f(\sigma, n)$,

(ii) $\sum_{i=0}^{I} f(\sigma_i, n) \le 2^{-n}$,

(iii) $\sum_{i=0}^{I} f(\sigma_i, n) \le f(\tau, n)$.

We say that a real A withstands a computably graded test $\{V_n\}$ iff $A \notin \bigcap_n V_n$.

Theorem 2.14 (Merkle, Mihailovic, and Slaman [16], Downey, Griffiths, and LaForte [4]). *A real is computably random iff it withstands all bounded Martin-Löf tests (iff it withstands all computably graded tests).*

Theorem 2.15 (Merkle, Miller, Nies, Reimann, and Stephan [15]). *Van Lambalgen's Theorem fails for computably randomness. In particular there exist A, B such that $A \oplus B$ is computable random but A is not B-computable random.*

Similarly to the case of Schnorr randomness, the other direction holds.

Theorem 2.16 (Yu [26]). *If A is computably random and B is A-computably random then $A \oplus B$ is computably random.*

2.4 Trivial sets and lowness notions

A set A is *K-trivial* if there is $c \in \mathbb{N}$ such that

$$(\forall n) K(A \upharpoonright n) \le K(n) + c.$$

This class was introduced by Chaitin [1] and further studied by Solovay (unpublished).

A set A is *low for Martin-Löf randomness* if each Martin-Löf random set is already Martin-Löf random relative to A: i.e.,

$$\mathrm{MLR}^A = \mathrm{MLR}.$$

This class was defined in Zambella [27], and studied by Kučera and Terwijn [12].

A set A is *low for K* if

$$(\forall n)K(n) \leq K^A(n) + c.$$

This class was introduced by Andrej A. Muchnik in 1999, who proved that there is a c.e. noncomputable set in this class.

A set A is *a basis for Martin-Löf randomness* if

$$A \leq_T Z \text{ for some } Z \in \text{MLR}^A.$$

This class was introduced by Kučera [11]. Kučera proved that this class is different from the class of computable reals: i.e., there is a c.e. non-computable set A that is a basis for Martin-Löf randomness.

Actually these four classes are equal.

Theorem 2.17 (A. Nies [17], Hirschfeldt, Nies and Stephan [9])**.** *The following statements about a set A are equivalent.*

(i) *A set A is K-trivial.*

(ii) *A set A is low for Martin-Löf randomness.*

(iii) *A set A is low for K.*

(iv) *A set A is a basis for Martin-Löf randomness.*

These studies were also carried out for Schnorr randomness. We say that a set A is *computably traceable* if there is a computable function $h(x)$ such that for all functions $g \leq_T A$, there is a computable collection of canonical finite sets $D_{r(x)}$ with $|D_{r(x)}| \leq h(x)$ and such that $g(x) \in D_{r(x)}$. A set A is *low for Schnorr randomness* if each Schnorr random set is already Schnorr random relative to A.

Theorem 2.18 (Kjos-Hanssen, Nies and Stephan [10])**.** *A set A is low for Schnorr randomness iff A is computably traceable.*

A set A is *low for computable measure machines* if for each computable measure machine M relative to A, there is a computable measure machine N such that $(\exists c)(\forall n)K^A_M(n) \geq K_N(n) - c$.

Theorem 2.19 (Downey, Greenberg, Mikhailovich and Nies [2])**.** *A set A is low for computable measure machines iff A is computably traceable.*

A set A is *Schnorr trivial* if for every computable measure machine N there is a computable measure machine M such that $(\exists c)(\forall n)K_M(A \upharpoonright n) \leq K_N(n) + c$. Downey, Griffiths and LaForte [4] showed that this class does not coincide with the reals that are low for Schnorr randomness.

Theorem 2.20 (Downey, Griffiths and LaForte [4])**.** *There is a c.e. complete Schnorr trivial real.*

Franklin and Stephan [6] defined truth-table Schnorr randomness which is more suitable for a notion of relativized Schnorr randomness in the context of truth-table reducibility. A set R is *truth-table Schnorr random* relative to A if there is no martingale $d \leq_{tt} A$ and no computable bound function b such that $(\exists^\infty n) d(R \upharpoonright b(n)) \geq n$. A set A is *low for tt-Schnorr randomness* if every Schnorr random set R is truth-table Schnorr random relative to A.

Theorem 2.21 (Franklin and Stephan [6]). *A set A is Schnorr trivial iff A is truth-table low for Schnorr randomness.*

Moreover they obtain a theorem similar to those involving bases for randomness, although the reducibility is not a commonly accepted one. We say that $A \leq_{snr} B$ if

$$(\exists \text{ computable } h)(\forall f \leq_t tA)(\exists g \leq_t tB)(\forall n)(\exists m \leq h(n)) f(n) = g(m).$$

Theorem 2.22 (Franklin and Stephan [6]). *A set A is Schnorr trivial iff there is a set B such that B is truth-table Schnorr random relative to A and $A \leq_{snr} B$.*

3 Motivation

In this section we shall states formally why another relativizations of Schnorr randomness is necessary and investigate the reason that van Lambalgen's Theorem fails for Schnorr randomness.

Schnorr randomness has many properties natural as a random concept like Theorem 2.5, 2.6, 2.8 and 2.9. Moreover one direction of van Lambalgen's Theorem holds for Schnorr randomness. However the other direction does not. This means that relativization of Schnorr randomness is not proper.

For each sequence A, let $\mathrm{Sch}(A)$ be a set of A-Schnorr random sequences. We can regard Sch as a function from 2^ω to $\mathcal{P}(2^\omega)$ where $\mathcal{P}(S)$ denotes the power set of S. Generally a random concept induces a function $S : 2^\omega \to \mathcal{P}(2^\omega)$. Hence we identify a random concept with its induced function. We say that S is another relativization of T if $S(\phi) = T(\phi)$. Note that $S(A)$ may not equal to $T(A)$ for some $A \in 2^\omega$. We say that van Lambalgen's Theorem holds for S if $A \oplus B \in S(\phi) \iff A \in S(\phi)\ \&\ B \in S(A)$.

Let $C \in \mathcal{P}(2^\omega)$ be a class of reals. Suppose that $S(\phi) = C$ and van Lambalgen's Theorem holds for S. Then $A \oplus B \in C \iff A \in C\ \&\ B \in S(A)$. It follows that $S(A) = \{B : A \oplus B \in C\}$ for $A \in C$. Hence $S(A)$ is unique when $A \in C$. Suppose that S is another relativization of Schnorr randomness for which van Lambalgen's Theorem holds. Then $S(A)$ is unique when $A \in \mathrm{Sch}(\phi)$. This S is proper relativization of Schnorr randomness. Then another relativization of Schnorr randomness is necessary. In the next section we propose such a random concept.

To find this proper relativization of Schnorr randomness, recall the proof of van Lambalgen's Theorem. Here, we look at the one direction that fails for Schnorr randomness.

Theorem 3.1 (van Lambalgen [25]). *If $A \oplus B$ is Martin-Löf random then B is A-Martin-Löf random.*

Proof. This proof is in [5]. Suppose B is not A-Martin-Löf random. We have $B \in \bigcap_n V_n^A$ where V_n^A is A-Martin-Löf test and $\mu(V_n^A) \leq 2^{-n}$. Put

$$W_n = \{X \oplus Y \ : \ X \in 2^\omega, Y \in \tilde{V}_n^X\}$$

where $\tilde{V}_n^X$ is V_n^X enumerated so long as its measure is less than 2^{-n}. Note that W_n is Martin-Löf test. Moreover $\tilde{V}_n^A = V_n^A$, hence $A \oplus B \in \bigcap_n W_n$, contradicting the assumption that $A \oplus B$ is Martin-Löf random. □

This essentially says that the Martin-Löf test W_n is emulate X-Martin-Löf tests V_n^X for all $X \in 2^\omega$. This can not be adapted to Schnorr randomness. Even if $\mu(V_n^A)$ is A-computable uniformly in n, $\mu(V_n^X)$ may not be X-computable in n for some $X \in 2^\omega$. So we put the restriction that $\mu(V_n^X)$ must be X-computable uniformly in n for all $X \in 2^\omega$. Then a Schnorr test can emulate $\{V_n^X\}$ for all X. This also means that we do not allow the test $\{V_n^A\}$ such that $\mu(V_n^A)$ happens to be A-computable in n. Then $\{V_n^X\}$ can be computed by the same Turing machine with an oracle X. Hence we should say that this restriction is by machines and not by computability. Moreover this restriction coincides that of Schnorr tests if $X = \phi$.

4 Truth-table Schnorr randomness

In this section we define truth-table Schnorr randomness which is another relativization of Schnorr randomness. This is mainly because van Lambalgen's Theorem holds for this relativization. Our first definition is by test concept but later we found it equivalent to that in [6] in Theorem 4.18.

We restrict Schnorr tests so that for each test $\{V_n\}$, $\mu(V_n^X)$ is computable from X for each X. This enables us to emulate V_n^X for all X. Then the machines that we can use to calculate $\mu(V_n^X)$ are the same for all X. This is different from the usual relativization of Schnorr randomness, where we allow a Schnorr test $\{V_n^X\}$ that happens to be computable from X. Actually this restriction is the same as that by computable steps.

Now we extend truth-table reduction to allow sequences.

Definition 4.1. *A sequence of Turing functional $\{\Phi_n\}$ is a uniform truth-table reduction if Φ_n^Z is uniform in n and total for each oracle Z.*

From a different point of view, this restriction is by steps bounded by computable function.

Proposition 4.2 (see [18])**.** *For sequences A and B, $A \leq_{tt} B$ iff there is a Turing functional Φ and a computable function t such that $A = \Phi^B$ and the number of steps needed to compute $\Phi^B(n)$ is bounded by $t(n)$.*

Then for a uniform tt-reduction $\{\Phi_n\}$, the number of steps needed to compute $\Phi_n^Z(m)$ is bounded by a computable function $t(n, m)$ for all Z.

4.1 Truth-table Schnorr test

We shall give a precise definition of truth-table Schnorr randomness. As we will see later, truth-table Schnorr randomness is the same as Schnorr randomness when it does not use an oracle. Hence we define X-truth-table Schnorr randomness for each X.

TT-Schnorr randomness is defined by replacing uniformly computable in the definition of Schnorr randomness by uniformly truth-table reducible. Compare the following definition with Definition 2.4.

Definition 4.3. *Let X be a real. We call $\{U_n^X\}$ a truth-table Schnorr test relative to X or a X-tt-Schnorr test if*

(i) *$\{U_n^Z\}$ is a Z-Martin-Löf test for all Z,*

(ii) *there exsits a uniform tt-reduction Φ_n such that $\mu(U_n^Z) = \Phi_n^Z$ for all Z.*

We call a real A truth-table Schnorr random relative to X or X-tt-Schnorr random if A passes all X-truth-table Schnorr tests.

Intuitively the X-truth-table Schnorr test is a X-Martin-Löf test whose measure is computable from X and its step is bounded by a computable function.

Let $\mathrm{TTS}(X)$ be a set of truth-table Schnorr random sequences relative to X. We say that A is truth-table Schnorr random if A is ϕ-truth-table Schnorr random equivalently if $A \in \mathrm{TTS}(\phi)$.

We can easily see that A is truth-table Schnorr random iff A is Schnorr random.

Proposition 4.4. *Let A be a binary sequence. Then A is truth-table Schnorr random iff A is Schnorr random: i.e., $TTS(\phi) = Sch(\phi)$.*

Proof. Let $\{U_n\}$ be a ϕ-tt-Schnorr test. Then $\mu(U_n^Z) = \Phi_n^Z$ for some $\{\Phi_n\}$. Hence $\mu(U_n)$ is uniformly computable. It follows that $\{U_n\}$ is a Schnorr test. Conversely, a Schnorr test is a ϕ-tt-Schnorr test by the same way. □

Hence truth-table Schnorr randomness is another relativization of Schnorr randomness.

The expression "truth-table Schnorr random" is slightly confusing. As mentioned above, A is truth-table Schnorr random iff A is Schnorr random, which means that $\mathrm{TTS}(\phi) = \mathrm{Sch}(\phi)$. At the same time we will see in Corollary 4.11 that truth-table Schnorr randomness is not equal to Schnorr randomness, which means that $\mathrm{TTS} \neq \mathrm{Sch}$ as a function. However this expression is convenient.

Here we see an immediate implication of the definition of Schnorr randomness and tt-Schnorr randomness.

Proposition 4.5. *For each real X, a real is X-Schnorr random implies that it is X-truth-table Schnorr random: i.e., $Sch(X) \subseteq TTS(X)$.*

Proof. Let $\{U_n^X\}$ be a X-tt-Schnorr test. Then $\mu(U_n^X)$ is X-computable uniformly in n. Hence $\{U_n^X\}$ is a X-Schnorr test. □

We know that Schnorr tests can be defined by that the measure of n-th open set equals 2^{-n} by Theorem 2.5. Similarly we can replace $\mu(U_n^Z) = \Phi_n^Z$ for all Z in the definition of tt-Schnorr test with $\mu(V_n^Z) = 2^{-n}$ for all Z and n.

Proposition 4.6. *For each tt-Schnorr test U_n there exists a tt-Schnorr test V_n such that $\bigcap_n U_n \subseteq \bigcap_n V_n$ and $\mu(V_n^Z) = 2^{-n}$ for all Z.*

Proof. Since $\mu(U_n^Z)$ is computable from Z, we can effectively find n such that $\mu(U_n^Z) \leq 2^{-n}$ and construct V_n by adding some extra open sets. □

So we can assume that each X-tt-Schnorr test $\{U_n^X\}$ satisfies $\mu(U_n^Z) = 2^{-n}$ for all Z.

Here we give an example of difference between tt-Schnorr randomness and Schnorr randomness by changing the relativization. Note that $A \leq_T X$ implies that A is not X-Schnorr random because $\{[A \upharpoonright n]\}$ is a X-Schnorr test which A passes. In contrast $A \leq_T X$ does not imply that A is not X-tt-Schnorr random as we will see in Corollary 4.12. We can prove the following as a correspondence.

Proposition 4.7. *If $A \leq_{tt} X$ then A is not X-truth-table Schnorr random.*

Proof. For a truth-table reduction Φ of A to X, let $U_n = [\Phi^X \upharpoonright n]$. Then $\{U_n\}$ is a X-tt-Schnorr test and A passes this test. □

As there is no universal Schnorr test, neither is a universal truth-table Schnorr test. Before the proof we prepare the following lemmas.

Lemma 4.8. *Let $\{U^Z\}$ be the complements of open sets uniformly Z-c.e. for $Z \in 2^\omega$. If $\mu(U^Z)$ is uniformly computable in Z, then there exists a Turing functional Φ_f such that $\Phi_f^Z \in U^Z$ for each $Z \in 2^\omega$.*

Proof. For each $Z \in 2^\omega$, we construct a sequence of strings $\{\sigma_n^Z\}$ such that $\sigma_n^Z \prec \sigma_{n+1}^Z$ and $\mu(U^Z \cap [\sigma_n^Z]) > 0$ for all $n \in \omega$. Then let $\Phi_f^Z = \lim_n \sigma_n^Z$. Note that it follows that $\Phi_f^Z \in U^Z$.

At stage $n = 0$, let $\sigma_0^Z = \phi$.

At stage $n + 1$, suppose that σ_n^Z is defined such that $\mu(U^Z \cap [\sigma_n^Z]) > 0$. Since $\{U^Z\}$ is the complements of open sets uniformly Z-c.e., there exists approximations U_s^Z such that $U_{s+1}^Z \subseteq U_s^Z$ and $\lim_s U_s^Z = U^Z$. For each m there exists $s = s(Z)$ such that $\mu(U_s^Z) - \mu(U^Z) \leq 2^{-m}$ uniformly in Z. Then $\mu(U_s^Z \cap [\sigma^Z]) - \mu(U^Z \cap [\sigma^Z]) \leq 2^{-m}$. Hence $\mu(U^Z \cap [\sigma^Z])$ is uniformly computable in Z and $\sigma \in 2^{<\omega}$. If $\mu(U^Z \cap [\sigma_n^Z 0]) > 0$ then let $\sigma_{n+1} = \sigma_n^Z 0$. If $\mu(U^Z \cap [\sigma_n^Z 0]) = 0$ then $\mu(U^Z \cap [\sigma_n^Z 1]) = \mu(U^Z \cap [\sigma_n^Z]) > 0$ and let $\sigma_{n+1} = \sigma_n^Z 1$. This is end of the construction. □

Proposition 4.9. *For each X there is no universal X-truth-table Schnorr test.*

Proof. Let $\{U_n^X\}$ be a X-tt-Schnorr test. Then there is $\{V_n^Z\}$ such that $\{V_n^Z\}$ is a Z-tt-Schnorr test uniformly in Z and $V_n^X = U_n^X$ for all n. Now $\mu(V_1^Z) \leq 1/2$. Since V_1^Z is a open set uniformly in Z and $\mu(V_1^Z)$ is uniformly computable in Z, there exists a Turing functional Φ_f such that Φ_f^Z is in the complement of V_1^Z for each $Z \in 2^\omega$ by Lemma 4.8. Then $\Phi_f^X \notin \bigcap_n V_n^X = \bigcap_n U_n^X$. However $\{[\Phi_f^X \upharpoonright m]\}$ is a X-tt-Schnorr test and $\Phi_f^X \in \bigcap_m [\Phi_f^X \upharpoonright m]$. It follows that $\bigcap_n U_n^X$ does not contain $\bigcap_m [\Phi_f^X \upharpoonright m]$. Hence $\{U_n^X\}$ is not universal. □

4.2 Van Lambalgen's Theorem holds for truth-table Schnorr randomness

We will see later that truth-table Schnorr randomness has many good properties as Schnorr randomness does. In this subsection we prove that van Lambalgen's Theorem holds for truth-table Schnorr randomness.

As mentioned Theorem 2.10 and 2.11, the only one direction of van Lambalgen's Theorem holds for Schnorr randomness. This is because that van Lambalgen's Theorem holds for truth-table Schnorr randomness and that Schnorr randomness implies truth-table Schnorr randomness.

Theorem 4.10. *Van Lambalgen's Theorem holds for truth-table Schnorr randomness.*

The proof is almost straightforward modification of the proof of van Lambalgen's Theorem for Martin-Löf randomness.

Proof. First we shall prove that if $A \oplus B$ is tt-Schnorr random, then B is A-tt-Schnorr random. Suppose $B \in \bigcap_n U_n^A$ for some tt-Schnorr test U_n. We can assume $\mu(U_n^Z) = 2^{-n}$ for all Z. Let $V_n = \{X \oplus Y \mid X \in 2^\omega \text{ and } Y \in U_n^X\}$. Then $\mu(V_n) = \int U_n^X dX = 2^{-n}$ and V_n is c.e. and open uniformly in n. Hence V_n is a tt-Schnorr test. Moreover $A \oplus B \in \bigcap_n V_n$. This is a contradiction to the fact that $A \oplus B$ is tt-Schnorr random.

Next we shall prove that if A is tt-Schnorr random and B is A-tt-Schnorr random then $A \oplus B$ is tt-Schnorr random. Suppose $A \oplus B$ is not tt-Schnorr random. Since $A \oplus B$ is not Schnorr random, there exists a Schnorr test $\{U_n\}$ such that $A \oplus B \in \bigcap_n U_n$ and $\mu(U_n) = 2^{-n}$. By passing to a subsequence we may assume that $\mu(U_n) = 2^{-2n-1}$. Let

$$V_n^X = \{Y \mid X \oplus Y \in U_n\}.$$

We claim that $\mu(V_n^X)$ is computable from X uniformly in n. Fix n. For each m there exists a computable function g such that $\mu(U_n) - \mu(U_n[g(m)]) < 2^{-m}$. Then $\mu(V_n^X) - \mu(V_n^X[g(m)]) = \mu(\{Y \mid X \oplus Y \in U_n - U_n[g(m)]\}) < 2^{-m}$. Hence $\{\mu(V_n^X)\}$ is uniformly computable from X. Then by letting $\widetilde{V}_n^X$ be V_n^X as long as its measure is $\leq 2^{-n}$, we get a X-tt-Schnorr test $\widetilde{V}_n^X$.

Let

$$W_n = \{X \mid \mu(\widetilde{V}_n^X) \geq 2^{-n}\}$$

and

$$\widetilde{W}_n = \{X \mid \mu(\widetilde{V}_n^X[g(n+1)]) \geq 2^{-n-1}\}.$$

Since $\mu(\widetilde{V}_n^X) - \mu(\widetilde{V}_n^X[g(n+1)]) < 2^{-n-1}$, $W_n \subseteq \widetilde{W}_n$. Note that $\mu(\widetilde{W}_n) \leq 2^{-n}$ for all n, otherwise we would have $\mu(U_n) \geq \mu(W_n)2^{-n-1} > 2^{-2n-1}$ a contradiction. Since $\widetilde{V}_n^X[g(n+1)]$ can be computed from X in a finite step, X is used only at a finite initial segment. Hence $\mu(\widetilde{W}_n)$ is computable. It follows that $\widetilde{W}_n$ is a Schnorr test.

Since A is Schnorr random, there exists N such that $\mu(\widetilde{V}_n^A) \leq 2^{-n}$ for all $n \geq N$. Hence $V_n^A = \widetilde{V}_n^A$ for all $n \geq N$. Since $A \oplus B \in \bigcap_n U_n$, $B \in V_n^A = \{Y \mid A \oplus Y \in U_n\}$ for each n. It follows that $B \in \bigcap_n \widetilde{V}_n^A$ for all $n \geq N$. Since B is A-tt-Schnorr random and $\widetilde{V}_n$ is a tt-Schnorr test, this is a contradiction.

□

This shows that truth-table Schnorr randomness is different from Schnorr randomness.

Corollary 4.11. *There exists X such that X-truth-table Schnorr random does not imply X-Schnorr random.*

Moreover by combining with Proposition 2.10, it follows that $A \leq_T X$ does not imply that A is not X-truth-table Schnorr random.

Corollary 4.12. *There exist reals A, B such that $A \equiv_T B$ and A is B-truth-table Schnorr random.*

Proof. There exists a Schnorr random set $C = A \oplus B$ such that $A \equiv_T B$ by Proposition 2.10. Note that C is tt-Schnorr random by Proposition 4.5. Hence A is B-tt-Schnorr random by Theorem 4.10. □

We shall explore more the difference between Schnorr randomness and truth-table Schnorr randomness in Section 7.

4.3 Truth-table reducible measure machine

Next we shall prove that truth-table Schnorr randomness has good properties that Schnorr randomness has. Schnorr randomness has characterization by machines by Theorem 2.8. We shall prove that these characterizations also have similar relativizations.

Definition 4.13. *A prefix-free machine M is a truth-table reducible measure machine if there is a Turing functional Φ such that $\Omega_M^Z = \Phi^Z$ for all Z.*

Theorem 4.14. *The following are equivalent.*

(i) *A is X-truth-table Schnorr random.*

(ii) *For all truth-table reducible measure machine M, $(\exists c)(\forall n)K_M^X(A \upharpoonright n) > n - c$.*

It is enough to prove the following lemma.

Lemma 4.15. *Let $R_{M,n}^Z = \{\sigma \mid K_M^Z(\sigma) \leq |\sigma| - n\}$.*

(i) *For a truth-table reducible measure machine M, $\{R_{M,n}\}$ is a truth-table Schnorr test.*

(ii) *For a truth-table Schnorr test $\{U_n\}$, we can effectively obtain a truth-table reducible measure machine M such that $\bigcap_n U_n^Z \subseteq \bigcap_n R_{M,n}^Z$ for all Z.*

Proof. The proof is the almost same as that of Schnorr random. The following proof is based on [18].
(i) It is well-known that $\{R_{M,n}^Z\}$ is a Martin-Löf test relative to Z. Note that $\Omega_M^Z - \Omega_{M,s}^Z \geq 2^n(\mu(R_{M,n}^Z) - \mu(R_{M,n,s}^Z))$ for each Z and s. Since Ω_M^Z is computable uniformly in Z, this shows that $\mu(R_{M,n}^Z)$ is computable uniformly in n and Z.

(ii) We can assume that $\mu(U_n^Z) = 2^{-2n}$ for all Z and n. Since the following construction is effective, we abbreviate Z. Represent each U_n as a union of extensions $[\sigma_{n,i}]$ of a prefix-free set $\{\sigma_{n,i}\}$ such that $g(\langle n, i\rangle) = \sigma_{n,i}$ is a computable function from ω to $2^{<\omega}$.

Let $L = \{\langle|\sigma_{n,i}| - n + 1, \sigma_{n,i}\rangle\}$. Then

$$\sum_{n,i} 2^{-(|\sigma_{n,i}|-n+1)} = \sum_n 2^{n-1} \sum_i 2^{-|\sigma_{n,i}|} = \sum_n 2^{-n-1} = 1.$$

Since L is a bounded request set, by Kraft-Chaitin theorem there is a prefix-free machine M such that $K_M(\sigma_{n,i}) \leq |\sigma_{n,i}| - n + 1$ for all n, i. Moreover $\Omega_M = \text{weight}(L) = 1$. □

4.4 Truth-table reducible martingale

Next we give a characterization by martingales. Recall the characterization of Schnorr randomness by martingales by Theorem 2.9.

Definition 4.16. *F^X is a X-truth-table reducible martingale if F^Z is a martingale for all Z and there exists a uniform tt-reduction Φ_σ such that $F^Z(\sigma) = \Phi_\sigma^Z$ for all Z and σ.*

Definition 4.17. *X-truth-table reducible martingale F strongly succeeds on a real A if there is a computable unbounded nondecreasing function h such that $F(A \upharpoonright n) \geq h(n)$ infinitely often. We say that F h-succeeds for the particular computable order h.*

Theorem 4.18. *The followings are equivalent.*

(i) *A real A is not truth-table Schnorr random relative to X.*

(ii) *There is a X-truth-table reducible martingale strongly succeeds on A.*

(iii) *There is a X-truth-table reducible martingale F and a strictly increasing computable function f such that $G(A \upharpoonright f(n)) \geq n$ infinitely often.*

(iv) *For each computable function r there is a X-truth-table reducible martingale F and a strictly increasing computable function f such that $G(A \upharpoonright f(n)) \geq r(n)$ infinitely often.*

The third statement is the definition by Franklin and Stephan [6], so their definition is equivalent to ours.

Proof. This is immediate from 2.9.

The direction (iv)⇒(iii) is obvious.

For (iii)⇒(ii), let $h(n) = \max\{l : f(l) \leq n\}$, then G h-succeeds on A.

For (ii)⇒(i), suppose that a X-tt-reducible martingale G succeeds on A. By effectiveness of Theorem 2.9 one obtains X-tt-Schnorr test such that A does not pass.

For (i)⇒(iv), suppose that there is a X-tt-Schnorr test $\{U_n\}$ such that $A \in \bigcap_n U_n$. By Theorem 2.9 for a computable function r there is a X-tt-reducible martingale G^X and a strictly increasing X-computable function f^X such that $G^X(A \upharpoonright f^X(n)) \geq r(n)$ infinitely often. Let $f(n) = \min\{f^X(n) : X \in 2^\omega\}$ then f is a a strictly increasing computable function. Moreover $G^X(A \upharpoonright f(n)) \geq r(n)$ infinitely often. □

5 Truth-table reducibly randomness

In this section we define truth-table reducibly randomness which is another relativization of computably randomness. Motivation is the same as truth-table Schnorr randomness.

Definition 5.1. *A real is truth-table reducibly random relative to X or X-tt-reducibly random if no X-tt-reducible martingale succeeds on it.*

The following implication is immediate by Theorem 4.18.

Proposition 5.2. *For all X, X-Martin-Löf randomness implies X-tt-reducible randomness implies X-tt-Schnorr randomness.*

5.1 Characterization by test concept

To prove that van Lambalgen's Theorem holds for truth-table reducibly randomness, we shall characterize truth-table reducibly randomness by a test concept. Recall the characterization of Schnorr randomness by test concept by Theorem 2.14.

Definition 5.3. *A X-Martin-Löf test $\{V_n\}$ is X-tt-reducibly graded if there exists a truth-table reduction $f : 2^{<\omega} \times \omega \to \mathbb{Q}$ such that, for any $Z \in 2^{\omega}$, $n \in \omega$, $\sigma \in 2^{<\omega}$, and any finite prefix-free set of strings $\{\sigma_i\}_{i \le I}$ with $\bigcup_{i=0}^{I}[\sigma_i] \subseteq [\tau]$, the following conditions are satisfied:*

(i) $\mu(V_n \cap [\sigma]) \le f^Z(\sigma, n)$,

(ii) $\sum_{i=0}^{I} f^Z(\sigma_i, n) \le 2^{-n}$,

(iii) $\sum_{i=0}^{I} f^Z(\sigma_i, n) \le f(\tau, n)$.

We say that a real A withstands a X-tt-reducibly graded test $\{V_n\}$ iff $A \notin \bigcap_n V_n$.

For the characterization of truth-table reducibly randomness by a test concept, we use the following theorem Downey, Griffiths, and Laforte showed in [4].

Theorem 5.4. (i) *From a computable martingale $G : 2^{<\omega} \to \mathbb{Q}$ we can effectively define a computably graded test (V_n, f) such that for every real A, if $\limsup_j G(A \upharpoonright j) = \infty$, then $A \in \bigcap_n V_n$.*

(ii) *From a computably graded test (V_n, f) we can effectively define a computable martingale $G : 2^{<\omega} \to \mathbb{Q}$ such that for every real A, if $A \in \bigcap_n V_n$, then $\limsup_j G(A \upharpoonright j) = \infty$.*

Then we can show another relativized version.

Theorem 5.5. *A real A is X-reducibly random iff it withstands all X-tt-reducibly graded Martin-Löf tests.*

Proof. Suppose that a X-tt-reducible martingale G succeeds on a real A. By the effectiveness of Theorem 5.4(i) we can define a X-tt-reducibly graded test (V_n, f^X) such that $A \in \bigcap_n V_n$. Then A does not withstand X-tt-reducibly graded test V_n.

On the other hand suppose that $A \in \bigcap_n V_n$ for a real A and a X-tt-reducibly graded test (V_n, f^X). Again by the effectiveness of Theorem 5.4(ii) we can define a X-tt-reducible martingale $G : 2^{<\omega} \to \mathbb{Q}$ such that $\limsup_j G(A \upharpoonright j) = \infty$. Then G succeeds on A. □

Remark 5.6. *Mihailović gave a machine characterization of computably randomness (see [5]). Although we have not checked yet, it is likely that we can also make a machine characterization of tt-reducibly randomness straightforwardly.*

5.2 Van Lambalgen's Theorem holds for truth-table reducibly randomness

In this subsection we prove that van Lambalgen's Theorem holds for tt-reducibly randomness. Recall the situation of computably randomness at Theorem 2.15 and 2.16. In contrast we can prove that van Lambalgen's Theorem holds for tt-reducibly randomness. First we give definitions and a lemma.

Definition 5.7. *For a tt-reducible rational probability distribution ν, we define σ-approximation $\nu^\sigma(\tau)$ as*

$$\nu^\sigma(\tau) = \mu(\{X \oplus Y \mid X \in [\sigma], Y \leq \nu^X(\tau)\}).$$

Lemma 5.8. *For each σ, ν^σ is a computable probability distribution.*

Proof. First we shall prove that that ν^σ is a probability distribution.

$$\begin{aligned}\nu^\sigma(\tau) =& \mu(\{X \oplus Y \mid X \in [\sigma], Y \leq \nu^X(\tau)\})\\ =& \mu(\{X \oplus Y \mid X \in [\sigma 0], Y \leq \nu^X(\tau)\}) + \mu(\{X \oplus Y \mid X \in [\sigma 1], Y \leq \nu^X(\tau)\})\\ =& \nu^\sigma(\tau 0) + \nu^\sigma(\tau 1).\end{aligned}$$

Then ν^σ is a probability distribution.

Next we shall prove that ν^σ is computable. Since ν is a truth-table reduction, one can compute s such that $|\nu^Z(\tau) - \nu_s^{Z\upharpoonright s}(\tau)| \leq 2^{-m}$ uniformly in Z, τ and m. Let $T_m(\tau)$ be the set of strings η such that $|\nu^Z(\tau) - \nu^\eta(\tau)| > 2^{-m}$ for some $\eta \prec Z \in 2^\omega$. Note that $T_m(\tau)$ is a tree for each m and τ. If $T_m(\tau)$ is not finite, then it has an infinite path Z by König's Lemma and $|\nu^Z(\tau) - \nu_s^{Z\upharpoonright s}(\tau)| > 2^{-m}$ for each s which is a contradiction. Then $T_m(\tau)$ is finite.

Given τ and m, one can compute a strong index $\tilde{g}(\tau, m)$ for the finite set of strings η (under the prefix relation) such that $|\nu^Z(\tau) - \nu_s^{Z\upharpoonright s}(\tau)| \leq 2^{-m}$ Hence one can compute a strong index $g(\tau, m)$ for the set of strings η such that $|\nu^Z(\tau) - \nu_{|\eta|}^{\eta}(\tau)| \leq 2^{-m}$ for each $\eta \prec Z$. Then

$$\sum_{\eta \in D_{g(\tau,m)}} 2^{-|\eta|} = 1$$

and

$$\eta \in D_{g(\tau,m)} \Rightarrow |\nu^Z(\tau) - \nu^\eta(\tau)| \le 2^{-m}$$

for all $\eta \prec Z$.

Pick a real Z_η for each η such that $\eta \prec Z_\eta$. Then

$$\begin{aligned}
&|\nu^\sigma(\tau) - \sum_{\eta \in D_{g(\tau,m)} \cap [\sigma]} \nu^{Z_\eta}(\tau) 2^{-|\eta|}| \\
=& \sum_{\eta \in D_{g(\tau,m)} \cap [\sigma]} |\mu(\{X \oplus Y \mid X \in [\eta], Y \le \nu^X(\tau)\}) - \nu^{Z_\eta}(\tau) 2^{-|\eta|}| \\
\le& \sum_{\eta \in D_{g(\tau,m)} \cap [\sigma]} 2^{-m-|\eta|} \\
=& 2^{-m}.
\end{aligned}$$

Since $\sum_{\eta \in D_{g(\tau,m)} \cap [\sigma]} \nu^{Z_\eta}(\tau) 2^{-|\eta|}$ is computable, $\nu^\sigma(\tau)$ is also computable. □

Definition 5.9. *For strings σ and τ, $\sigma \oplus \tau$ denotes a partial string η such that $\eta(2i) = \sigma(i)$ and $\eta(2i+1) = \tau(i)$ for all i. For a martingale F, we define $F(\sigma \oplus \tau) = \sum_\eta F(\eta) 2^{|\sigma|+|\tau|-|\eta|} [\![|\eta| = 2\max\{|\sigma|, |\tau|\}$ and $\sigma \oplus \tau \le \eta]\!]$.*

Note that $G(\sigma) = F(\sigma \oplus \tau)$ is a martingale for each martingale F and each τ.

Theorem 5.10. *Van Lambalgen's Theorem holds for tt-reducibly randomness.*

Proof. We shall prove that if $A \oplus B$ is tt-reducibly random then B is A-tt-reducibly random. Suppose that there exists a A-tt reducible bounded A-Martin-Löf test $\{V_n\}$ such that $B \in \bigcap_n V_n^A$. Note that there exists a tt-reducible rational probability distribution ν with $\mu(V_n^Z \cap [\sigma]) \le \nu^Z(\sigma) 2^{-n}$, for all $n \in \mathbb{N}$, $\sigma \in 2^{<\omega}$ and Z. Let

$$W_n = \{X \oplus Y \mid X \in 2^\omega, Y \in V_n^X\}.$$

Note that $A \oplus B \in \bigcap_n W_n$. Since $\mu(W_n) \le \max\{\mu(V_n^X) \mid X \in 2^\omega\} \le 2^{-n}$, $\{W_n\}$ is a Martin-Löf test. We also let

$$\xi(\sigma \oplus \tau) = \nu^\sigma(\tau).$$

Then ξ is a computable probability distribution. Moreover for all $n \in \mathbb{N}$ and $\sigma \oplus \tau \in 2^{<\omega}$,

$$\begin{aligned}
\mu(W_n \cap [\sigma \oplus \tau]) =& \mu(\{X \oplus Y \mid X \in [\sigma], Y \in V_n^X \cap [\tau]\}) \\
\le& \mu(\{X \oplus Y \mid X \in [\sigma], Y \le \nu^X(\tau) 2^{-n}\}) \\
=& \nu^\sigma(\tau) 2^{-n} \\
=& \xi(\sigma \oplus \tau) 2^{-n}.
\end{aligned}$$

Hence W_n is a bounded Martin-Löf test. It follows that $A \oplus B$ is not computably random. This is a contradiction with the assumption that $A \oplus B$ is tt-reducibly random.

Next we shall prove the other direction. That is, if A is tt-reducibly random and B is A-tt-reducibly random then $A \oplus B$ is tt-reducibly random. Suppose there exists a bounded Martin-Löf test $\{W_n\}$ such that $A \oplus B \in \bigcap_n W_n$. Note that there exists a computable rational probability distribution ν with $\mu(W_n \cap [\sigma]) \leq \nu(\sigma)2^{-n}$, for all $n \in \mathbb{N}$ and $\sigma \in 2^{<\omega}$. By passing to a subsequence we can assume that

$$\mu(W_n \cap [\sigma]) \leq \nu(\sigma)2^{-2n}.$$

We define a c.e. relation P as

$$P(\sigma, \tau) \iff \mu(\{Y \mid [\sigma] \oplus Y \in W_n\} \cap [\tau]) > \frac{\nu(\sigma \oplus \tau)}{\nu(\sigma \oplus \lambda)} 2^{-n}.$$

Let

$$U_n = \{X \mid (\exists \sigma \prec X)(\exists \tau) P(\sigma, \tau)\}.$$

We shall prove that U_n is a bounded Martin-Löf test. It is clear that

$$\mu(U_n \cap [\sigma]) \times \mu(\{Y \ : \ [\sigma] \oplus Y \in W_n\} \cap [\sigma]) \leq \mu(W_n \cap [\sigma \oplus \tau]).$$

Hence if $P(\sigma, \tau)$ then

$$\mu(U_n \cap [\sigma]) \frac{\nu(\sigma \oplus \tau)}{\nu(\sigma \oplus \lambda)} 2^{-n} \leq \nu(\sigma \oplus \tau)2^{-2n}.$$

It follows that $\mu(U_n \cap [\sigma]) \leq \nu(\sigma \oplus \lambda)2^{-n}$. Hence for all $\sigma \in 2^{<\omega}$, $\mu(U_n \cap [\sigma]) \leq \nu(\sigma \oplus \lambda)2^{-n}$ by separating $[\sigma]$ into $\bigcup [\sigma']$ such that $(\exists \tau) P(\sigma', \tau)$. Since n and σ is arbitrary, U_n is a bounded Martin-Löf test.

Since A is computably random, it follows that $\{n \mid A \in U_n\}$ is finite. Thus for all but finitely many n we have $A \notin U_n$, i.e.,

$$\mu(\{Y \mid A \oplus Y \in W_n\} \cap [\tau]) \leq \frac{\nu(A \upharpoonright m \oplus \tau)}{\nu(A \upharpoonright m \oplus \lambda)} 2^{-n}$$

for all m. Especially by letting $m = 0$ we get $\mu(\{Y \mid A \oplus Y \in W_n\} \cap [\tau]) \leq \nu(\lambda \oplus \tau)2^{-n}$. Let $V_n^X = \{Y \mid X \oplus Y \in W_n\}$. Then $\mu(V_n^X \cap [\tau]) \leq \nu(\lambda \oplus \tau)2^{-n}$. Hence V_n^A is A-tt-reducibly bounded A-Martin-Löf test. Moreover $B \in \bigcap_n V_n^A$, contradicting the assumption that B is A-tt-reducibly random.

□

6 High degrees and separating notions of randomness

In this section we look at the difference between Schnorr randomness and truth-table Schnorr randomness.

Theorem 6.1. *For every set A and B, there is a set R such that $B \leq_T R$, R is not computably random and R is truth-table Schnorr random relative to A. If $A \oplus B$ is high, one also obtains $R \leq_T A \oplus B$.*

Corollary 6.2. *Let A be a high set. Then there is a set $R \equiv_T A$ such that R is not computably random and truth-table Schnorr random relative to A.*

It follows that if A is high, $\mathrm{TTS}(A) \subset \mathrm{Sch}(A)$. Here the inclusion is proper.

Proof of Theorem 6.1. The idea of the construction is similar to that of [8].

First we fix a strictly increasing function $F : \omega \to \omega$ which dominates all computable functions. For each such F we can construct a needed R. Later we let $F \leq_T A \oplus B$ to get $R \leq_T A \oplus B$.

We shall construct a martingale V that dominates all A-tt-reducibly martingale. To make such V we sum up an effective listing of A-computable martingale whose steps are bounded by F. Using N we shall construct R that is not computably random and A-tt-Schnorr random.

Let $F : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that F dominates all computable functions. We also let

$$a_m = \begin{cases} n & \text{if } F(n) = m \\ m & \text{otherwise.} \end{cases}$$

Since F is strictly increasing, $\{a_i\}$ is computable from F. It is clear that for each l there are at most 2 indices m with $a_m = l$. Divide the integers into intervals I_m of length $3a_m + 1$ such that $\min(I_0) = 0$ and $\min(I_{m+1}) = \max(I_m) + 1$ for every m.

Let $\{M_i\}$ be an effective listing of all A-partial computable martingales. We also let

$$N_i(\sigma) = \begin{cases} M_i(\sigma) & \text{if } (\forall \tau)|\tau| \leq |\sigma| \Rightarrow M_i(\tau)[F(i + |\tau|)] \downarrow \\ 0 & \text{otherwise.} \end{cases}$$

Then for each $\sigma \in 2^{<\omega}$ and k such that $|\sigma| \in I_k$ let

$$V(\sigma) = 2^{-k} + \sum_{i \leq k} 2^{-i} N_i(\sigma).$$

Note that V is computable from A and F.

Let $\{f_i\}$ be a effective listing of all A-partial computable functions such that

$$f_i(n) \uparrow \Rightarrow f_i(m) \uparrow$$

for all $n \leq m$ and i. Now let $E = \{x_0, x_1, x_2, \cdots\}$, where

$$x_n = \min\{y \ : \ (\forall m < n) x_m < y \wedge (f_m(y) < F(y) \vee f_m(y)[F(y)] \uparrow)\}.$$

We need to check that every every x_n can be defined. For a contradiction suppose that there is n such that

$$(\exists m < n) f_m(y) \geq F(y) \wedge f_m(y)[F(y)] \downarrow$$

for almost all y. Since n is finite, there is m such that $f_m(y) \geq F(y)$ and $f_m(y)[F(y)] \downarrow$ for almost all y. This contradicts the fact that F dominates all computable functions. Note that E is computable from A and F.

Using E, one can now define the set R inductively on all intervals I_m as follows.

- If there is $k > m$ with $a_m = a_k$ or if $a_m \notin E$, then choose R on I_m such that R is not 0 on all of the least $2a_m$ elements of I_m and V grows on I_m by at most the factor $2^{2a_m}/(2^{2a_m} - 1)$.

- Otherwise (that is, if there is no $k > m$ with $a_m = a_k$ and if $a_m \in E$), choose $R(\min(I_m) + u) = 0$ for $u \in \{0, 1, \cdots, 2a_m - 1\}$ and choose $R(\min(I_m) + u) = B(u - 2a_m)$ for $u \in \{2a_m, 2a_m + 1 \cdots, 3a_m\}$.

Note that R is computable from A, B and F Now it is shown that R has the desired properties.

We shall prove $B \leq_T R$. To compute $B(n)$, search for the first interval I_m such that $a_m \geq n+1$ and $R(\min(I_m)+u) = 0$ for all $u \in \{0, 1, \cdots, a_m-1\}$. As E contains a number larger than n, the search will terminate. It can be seen that $B(n) = R(\min(I_m)+2a_m+n)$.

We shall prove that R is not computably random. One can construct a computable martingale d that succeeds on R as follows. The initial capital of d is set as 3 and for each interval I_m, d invests 2^{-2a_m}, which is then bet on R begin 0 for the first $2a_m$ members of I_m. If all bets are true, then d doubles the invested capital $2a_m$ times and makes a profit of

$$2^{2a_m} \cdot 2^{-2a_m} - 2^{-2a_m} = 1 - 2^{-2a_m}.$$

Otherwise, d loses the invested 2^{-2a_m}. On one hand, all potential losses can be bounded by

$$\sum_m 2^{-2a_m} \leq \sum_{l \geq 0} 2 \cdot 2^{-2l} < 3.$$

and therefore the martingale never takes the value of 0. On the other hand, so the profit is at least 3/4 on these intervals and the value of d goes to infinity on R. Thus d witnesses that R is not computably random.

We shall prove that R is truth-table Schnorr random relative to A. To see this, consider the following function $\tilde{r}(n)$.

$$\tilde{r}(n) = n \cdot \left(\Pi_{m<n} 2^{2^{3m+1}}\right) \cdot \left(\Pi_{m>0} \left(\frac{2^{2m}}{2^{2m}-1}\right)^{m+2}\right).$$

Note that an infinite product $\Pi_k q_k$ such that $q_k > 1$ satisfies $\Pi_k q_k < \infty$ if and only if $\sum_k (q_k - 1) < \infty$. To adjust for the fact that some intervals I_m are copies of each other as described in the first component of the definition of R, let $q_k = 2^{2m}/(2^{2m} - 1)$ as appropriate. Since $2^{2m}/(2^{2m} - 1) - 1 = 1/(2^{2m} - 1)$, this inequality can be applied here. For each m, there are at most $m + 2$ values of k for which $q_k = 2^{2m}/(2^{2m} - 1)$. Hence

$$\sum_{m>0} (m+2) \cdot \frac{1}{2^{2m} - 1} \leq \sum_{m>0} \frac{2^{m+2}}{2^{2m}} \leq \sum_{m>0} 2^{2-m} = 4$$

and $(\Pi_{m>0}(\frac{2^{2m}}{2^{2m}-1})^{m+2})$ is a positive real number. Therefore, the function $\tilde{r}$ has a computable upper bound r such that $r(n) \in \mathbb{N}$ for all n.

For a contradiction assume that d' is a A-tt-reducible martingale and f is a computable function such that

$$d'(R \upharpoonright f(n)) > r(n)$$

infinitely many n and, in addition, $n < f_k(n) < f_k(n + 1)$ for all n. Let

$$h(n) = \max\{s : d'(\sigma)[s] \downarrow \text{ for all } |\sigma| \leq n\}.$$

Note that h is computable. Since there are infinitely many indices i such that $M_i = d'$, there is a index k such that $M_k = d'$ and $h(n) \leq F(k + n)$ for all n. Hence $N_k = d'$.

Now consider $n > x_0 + x_1 + \cdots + x_k$. Then for each $u < n$, there is at most one interval I_m such that $m \leq f(n)$, $a_m = u$, $F(a_m) = u$ and $u \in E$; for $u \geq n$ there is no interval I_m satisfying these conditions. On the intervals that satisfy these conditions, the martingale V can increase its capital by at most a factor of 2^{3a_m+1}; on all other intervals I_m below $f(n)$, mg can increase its capital by at most a factor of $2^{2a_m}/(2^{2a_m} - 1)$. Hence, one has that

$$V(R \upharpoonright f(n)) \leq r(n)/n.$$

Since

$$N_k(R \upharpoonright f(n)) \leq n \cdot V(R \upharpoonright n)$$

for almost all n, then

$$N_k(R \upharpoonright f(n)) \leq r(n)$$

which is a contradiction.

If $A \oplus B$ is high, one obtains $F \leq_T A \oplus B$. Hence $R \leq_T A \oplus B$ because R is computable from A, B and F. □

This has an interesting corollary. Franklin and Stephan [7] asked whether each Schnorr random real is half of a real for which van Lambalgen's Theorem fails. They proved a partial result, i.e., if A is a Schnorr random real and $A \not\geq_T \phi'$, then there is a real B such that $A \oplus B$ is Schnorr random and A is not B-Schnorr random. Now we have the following corollary immediately.

Corollary 6.3. *If A is a Schnorr random real, then there is a real B such that $A \oplus B$ is Schnorr random and A is not B-Schnorr random.*

Proof. Given A, choose B such that $B \geq_T A$ and B is truth-table Schnorr random relative to A. Such a B exists by Corollary 6.2. Then $A \oplus B$ is Schnorr random by Theorem 4.10. But as B is Turing reducible to A, B is not Schnorr random relative to A. □

7 Equivalent classes

In this section we study lowness notions for tt-Schnorr randomness.

Theorem 7.1 (Franklin and Stephan [6])**.** *The followings are equivalent.*

(i) *The tt-degree of A is computably traceable. That is, there is a computable function g such that for each $f \leq_{tt} A$, there are uniformly computable finite sets $S_0, S_1, \cdots$ with $|S_n| \leq g(n)$ and $f(n) \in S_n$ for all n.*

(ii) *A is Schnorr trivial.*

(iii) *A is low for tt-Schnorr randomness.*

(iv) *There is a set B such that B is A-tt-Schnorr randomness and $A \leq_{snr} B$.*

Franklin and Stephan use the expression "tt-low for Schnorr randomness". However we prefer the one "low for tt-Schnorr randomness" because this is not the change of lowness but the change of randomness.

The last piece in the picture is lowness for tt-reducible measure machine. The proof is a straightforward modification of that of Theorem 2.19.

Definition 7.2. *A set A is low for tt-reducible measure machine if for each A-tt-reducible measure machine M there is a tt-reducible measure machine N such that for a constant c, $K_N(\sigma) \leq K_M(\sigma) + c$ for all σ.*

Theorem 7.3. *A set A is low for tt-reducible measure machine iff the tt-degree of A is computably traceable.*

Proof. The "only if" direction follows from the fact that lowness for tt-reducible measure machine implies lowness for tt-Schnorr randomness.

Now suppose that the tt-degree of A is computably traceable. Let M be an A-tt-reducible measure machine. We need to define a tt-reducible measure machine N such that for a constant c, $K_N(\sigma) \leq K_M(\sigma) + c$ for all σ.

Let $D_0, D_1, \cdots$ be a canonical list of the finite subsets of $\{0,1\}^* \times \{0,1\}^*$. Recall that the domain of D_i is the set of all σ such that $(\sigma, \tau) \in D_i$ for some τ. Let t_n be the least t such that $\mu(\mathrm{dom}(M[t])) > 1 - 2^{-2n}$. Let G_s be the graph of $M[s]$, that is, the set of all (σ, τ) such that $M(\sigma)[s] = \tau$. Let c_n be such that $D_{c_0} = G_{t_0}$ and $D_{c_{n+1}} = G_{t_{n+1}} - G_{t_n}$. Note that $\mu(\mathrm{dom}(D_{c_{n+1}})) < 2^{-2n}$. Let $F_0, F_1, \cdots$ be a computable sequence of finite sets such that for each n we have $|F_n| \leq 2^n$ and $c_n \in F_n$. Such a sequence exists because the function $n \to c_n$ is tt-reducible to A and the degree of A is computably traceable. By removing elements if necessary, we can assume that for each $c \in F_{n+1}$, the domain of D_c is prefix-free and $\mu(\mathrm{dom}(D_c)) < 2^{-2n}$.

Let $L = \{(|\tau| + 1, \sigma) : \exists n \exists c \in F_n((\tau, \sigma) \in D_c)\}$. This set is c.e., and its weight is

$$\sum_n \sum_{c \in F_n} \frac{\mu(\mathrm{dom}(D_c))}{2} < \sum_n |F_n| 2^{-(2n+1)} \leq \sum_n 2^n 2^{-(2n+1)} = 1.$$

Thus L is a Kraft-Chaitin set. Furthermore, the weight of L is computable, since we can approximate it to within 2^{-m} by $\sum_{n \leq m} \sum_{c \in F_n} \mu(\mathrm{dom}(D_c))/2$. Now the Kraft-Chaitin Theorem gives us a prefix-free machine N such that for each request (d, σ) in L, there is a $\nu \in 2^d$ such that $N(\nu) = \tau$. In particular, if $M(\sigma) = \tau$ then $(d, \tau) \in D_{c_n}$ for some n, and hence there is a ν such that $|\nu| = |\sigma| + 1$ and $N(\nu) = \tau$. Furthermore, $\mu(\mathrm{dom}(N))$ is equal to the weight of L, and hence is computable. So N is a tt-reducible measure machine and $K_N(\sigma) \leq K_M(\sigma) + c$ for a constant c. □

8 Discussion

We defined tt-Schnorr randomness and tt-reducibly randomness which are another relativizations of Schnorr randomness and computably randomness respectively. We proved that van Lambalgen's Theorem holds for both randomnesses. We also established the complete equivalence of lowness notions for tt-Schnorr randomness. Hence these randomnesses are more proper relativizations than those of usual ones. We usually think of only relativization by computability, which is very natural but sometimes does not work well. This is because this relativization is essentially induced by Turing reducibility. Since Schnorr randomness and computably randomness fit truth-table reducibility well, another relativization is proper. We should say that the restriction of tt-Schnorr randomness and tt-reducibly randomness are by machines and not by computability. Then a machine restriction is a good way in order to define a new random concept for which van Lambalgen's Theorem holds. We also proved that the classes of tt-Schnorr random reals relative to a high set contain reals Turing equivalent to the high set. Then we gave the answer to the question whether each Schnorr random real is half of a real for which van Lambalgen's Theorem fails.

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