Limit Theorems on a Computable Topological Space

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背景

「ほとんど確実に」はもうやめよう

本講演の結論

「ほとんど確実に」はもうやめよう
(by some researchers on algorithmic randomness)

確率論で「ほとんど確実に (almost surely)」とは「確率 1 で」ということである。

例 1. コイントスを繰り返した時、ほとんど確実にいつかは表が出る。

この表現はランダムネスの理論を使えば取り除け、さらなる発展の道が開けてくる。

大数の法則とは

「確率 1 で」という表現は極限定理でよく出てくる。最も基本的な定理が大数の法則である。

大数の法則の一例

コイントスの試行回数を増やせば、表が出る回数と裏が出る回数の比率はどちらも $\frac{1}{2}$ に近づく。

もう少し正確に厳密な数学的な定式化を、大数の法則の発展と共に見る。

Bernoulli の大数の弱法則

定理 2 (Bernoulli の大数の弱法則)**.** 確率 p の独立な事象において、n 回の試行のうち起った回数を r とする。任意に与えられた ϵ, δ に対して、

$$P\left\{\left|\frac{r}{n} - p\right| < \epsilon\right\} > 1 - \delta$$

となる。

- 成功か失敗かのみを考えている。
- 大数の法則 (law of large numbers) の命名は Poisson による。
- 弱法則と言われるのは Borel による強法則との比較による。

必要な大きさの推定

定理 3 (De Moivre (1730s)).

$$P\left\{\left|\frac{r}{n}-p\right| \leq c\sqrt{\frac{p(1-p)}{n}}\right\} \approx \frac{1}{\sqrt{2\pi}}\int_{-c}^{c} e^{-u^2/2}du$$

定理 4 (Laplace). 平均が μ、分散が σ^2 の独立試行の測定値を $x_1,\ldots,x_n$ とすると、

$$P\left\{\left|\frac{\sum_{k=1}^{n} x_k}{n}-\mu\right| \leq \frac{c\sigma}{\sqrt{n}}\right\} \approx \frac{1}{\sqrt{2\pi}}\int_{-c}^{c} e^{-u^2/2}du$$

- Laplace の証明は、x_k に有限個の可能な場合がありうる場合にのみ正当なものだった。
- 中心極限定理の一例である。

Borel の大数の強法則

測度論の到来後の発展。

定理 5 (Borel (1909)). 成功の確率が $\frac{1}{2}$ の独立な事象を n 回行ったときの成功の回数を r_n とする。ある N より大きいすべての n に対して確率 *1* で

$$\left|\frac{r_n}{n} - \frac{1}{2}\right| \leq \frac{\ln(n/2)}{\sqrt{2n}}$$

- Borel の表現の形とは少し変えてある。
- Borel の証明には食い違いがあり、完全な証明は 1910 年 Georg Faber による。

Kolmogorov の大数の強法則

定理 6 (Kolmogorov の大数の強法則 (1930))**.** 独立な確率変数 $X_1, X_2, \dots, X_n, \dots$ が

$$E(X_n) = 0, \ \sum_n \frac{Var(X_n)}{n^2} < \infty$$

を満たすとする。この時、

$$\frac{\sum_{k=1}^n X_k}{n} \to 0, \ a.s.$$

が成り立つ。

- 取りうる値が実数値に拡張された。
- 同分布の場合には成り立たないことに注意。

Kolmogorov の大数の強法則 (i.i.d.)

定理 7 (Kolmogorov の大数の強法則 (1933)). 独立同分布の確率変数 $X_1, X_2, \ldots, X_n, \ldots$ に対して、$S_n = \sum_{k=1}^{n} X_k$ と置く。この時、

$$E(|X_1|) < \infty \Rightarrow \frac{S_n}{n} \to E(X_1) \; a.s.,$$

$$E(|X_1|) = \infty \Rightarrow \limsup_{n\to\infty} \frac{|S_n|}{n} = +\infty \; a.s.$$

が成り立つ。

- 有名な「確率論の基礎概念」の中で示された。

その後、大数の強法則の収束速度も調べられ、重複対数の法則と呼ばれる。

ランダムな列の定義

問い

では具体的にどんな列ならば大数の法則が成り立つだろうか。

とりあえず２進無限列を考える。
1919 年、von Mises は大数の法則がすべての適当な部分列で成り立つような列をランダムな列と考えてはどうかと提案した。
Kolmogorov が測度論的確率論という定式化を与える前のことであり、von Mises はこのランダムな列を基盤として確率論の定式化を与えようとしたがうまく行かなかった。
1966 年、Martin-Löf は大数の法則が成り立たない列とは測度が計算可能に小さくなる部分に含まれる列であることを見抜き Martin-Löf ランダムネスを提案した。
現在、Martin-Löf ランダムネスは最も自然なランダムネスであると考えられている。

ランダムな列に対する大数の法則

Martin-Löf ランダムな列に対しては大数の法則が成り立つ。（論文としては出版されていないが、多くの人に知られている。）

定理 8. $A \in 2^\omega$ を *Martin-Löf* ランダムな列とすると、

$$\frac{\sum_{k=1}^{n} A(k)}{n} \to \frac{1}{2} \ (n \to \infty)$$

「確率 1 で」ではなく、すべてのランダムな列についてであることに注意 !!
しかしそれぞれの試行に関しては有限個の値しか取ることができず、その意味で Borel の大数の法則に相当する。
1997 年には Vovk により有理数の列に拡張された。
実数のランダムな列の定義が必要であり、それは大数の法則を満たすはずである。
本講演ではその定義を与え、その定義の元で大数の法則などを示す。

本講演の概要

(i) 位相空間の計算可能性の理論 (computable analysis) を元に、

(ii) その位相空間上の測度の計算可能性を議論する。

(iii) その測度空間でランダムな点を定義する。

(iv) そのランダムな点に関して大数の法則を示す。

TTE

Computable Analysis

Computability on $\mathbb{N}$ was defi ned by Church, Turing and others in 1930s.
Computable analysis studies computability on $\mathbb{R}$.
There are many approaches to this problem.

- Via representations by Hauck, Kreitz and Weihrauch.
- Via sequential computability and effective uniform continuity by Pour-El and Richards (and this is generalized to metric spaces by Yasugi).
- Ko's approach, Domain Theory, Markov's approach, etc.

We will mostly use the approach via representations, which is called Type-2 Theory of Effectivity (TTE).
This is because this approach can be naturally adapted to randomness.

Representation Approach

- A real is (seen as) a sequence of a fi nite alphabet, and
- a computable function from reals to reals is a computable function from sequences to sequences.

We must be careful about the choise of a representation.

例 9. *Consider the infinite decimal fraction as a representation of a real.*
Then $x \mapsto 3x$ *is not computable because machine does not make an output with an input*

$$\frac{1}{3} = 0.3333333\ldots.$$

Usually we use a sequence of intervals with rational endpoints containing the real as a representation of the real.

Notation

We will use the terminology from the work by Weihrauch and Grubba 2009.

- $f : A \rightrightarrows B$ denotes a multi-function.
- $f :\subseteq A \to B$ denotes a partial function.
- $f : A \to B$ denotes a total function.

$|w|$ is the length of the word $w \in \Sigma^*$.
$x \sqsubseteq y$ iff x is a prefi x of y.
The “wrapping function” $\iota : \Sigma^* \to \Sigma^*$ is the one such that

$$\iota(a_1 a_2 \ldots a_k) = 110a_1 0a_2 0 \ldots a_k 011.$$

$u \ll w$ iff $i(u)$ is a subword of w for $u \in \Sigma^*$ and $w \in \Sigma^* \cup \Sigma^\omega$.

Computability on infinite sequences

Let $Y_0, \ldots, Y_n \in \{\Sigma^*, \Sigma^\omega\}$ and $Y = Y_1 \times \ldots \times Y_n$.
A function $f :\subseteq Y \to Y_0$ is *computable* if it is computed by Type-2 machine.
Informally, a *Type-2 machine* is a Turing machine, which reads from input tapes with fi nite or infi nite inscription, operates on work tapes and write one-way to an output tape.
On Σ^* we consider the discrete topology.
On Σ^ω we consider the topology generated by the base $\{w\Sigma^\omega \;:\; w \in \Sigma^*\}$ of open sets.
Every computable function is continuous.

Representation of a Set

A *multi-representation* of a set M is a surjective function $\gamma : Y \rightrightarrows M$ where $Y \in \{\Sigma^*, \Sigma^\omega\}$.

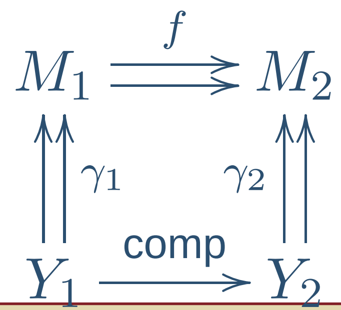
例 10. (i) $\nu_{\mathbb{N}} :\subseteq \Sigma^* \to \mathbb{N}$

(ii) $\nu_{\mathbb{Q}} :\subseteq \Sigma^* \to \mathbb{Q}$

(iii) $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$

A point is γ-*computable* if it has a computable representation by γ.

A function $f : M_1 \rightrightarrows M_2$ is (γ_1, γ_2)-*computable* if it has a computable realization.



Reducibility and Product

$\gamma_1 \leq \gamma_0$ if $M_1 \subseteq M_0$ and the identigy $\mathrm{id} : M_1 \to M_0$ is (γ_1, γ_0)-computable.

$\gamma_1 \equiv \gamma_0$ iff $\gamma_1 \leq \gamma_0$ and $\gamma_0 \leq \gamma_1$.

$[\gamma_1, \gamma_2]\langle y_1, y_2 \rangle = \gamma_1(y_1) \times \gamma(y_2)$.

Representations of computable functions

定義 11 (Notations of computable functions)**.** *For all* $a, b \in \{*, \omega\}$,

(i) $P^{ab} = \{f :\subseteq \Sigma^a \to \Sigma^b \ : \ f$ *is computable*$\}$ *and,*

(ii) *a notation* $\xi^{ab} : \Sigma^* \to P^{ab}$ *is defined as* $\xi^{ab}(w)$ *is the function* $f \in P^{ab}$ *computed by the Type-2 machine with code* w *if the machine is recursive.*

(i) F^{**} is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^*$.

(ii) $F^{*\omega}$ is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^\omega$.

(iii) $F^{\omega *}$ is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^*$ with open domain.

(iv) $F^{\omega\omega}$ is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^*$ with G_δ-domain.

Representations of functions

定義 12 (Representations of the sets F^{ab}). *Define* $\eta^{ab} : \Sigma^\omega \to F^{ab}$ *by*

$$\eta^{ab}(\langle x, p\rangle)(y) = \xi_x^{\omega b}\langle p, y\rangle$$

for all $x \in \Sigma^*$, $p \in \Sigma^\omega$ *and* $y \in \Sigma^a$.

定義 13. *Let* $\gamma_1 : \Sigma^a \rightrightarrows M_1$ *and* $\gamma_2 : \Sigma^b \rightrightarrows M_2$ *be multi-representations.*
We define a multi-representation $[\gamma_1 \rightrightarrows \gamma_2]$ *of the* (γ_1, γ_2)*-continuous multi-functions* $f : M_1 \rightrightarrows M_2$ *as* $f \in [\gamma_1 \rightrightarrows \gamma_2](p)$ *iff* $\eta_p^{ab} = \eta^{ab}(p)$ *realizes* f *w.r.t.* (γ_1, γ_2).
The restriction ot the single-valued functions is $[\gamma_1 \to_p \gamma_2]$.
The restriction to the total (γ_1, γ_2)*-continuous functions is* $[\gamma_1 \to \gamma_2]$.

Computable Topology

General Topology

Let X be a set.

A topology τ on X is a set of subsets of X (the set of open sets) that is closed under union and finite intersection.

A topological space (X, τ) is a T_0-space if whenever x and y are distinct points in X, there is an open set containing one and not the other.

A base is a subset $\beta \subseteq \tau$ such that every $U \in \tau$ is a union of base sets.

A subbase is a subset $\beta \subseteq \tau$ such that the set of all finite intersections forms a base.

A space is said to be second-countable if its topology has a countable base.

A space is called separable if it contains a countable dense subset.

Computable Topological Spaces

定義 14 (Hertling and Weihrauch 2009)**.** *A computable topological space is a 4-tuple* $\mathbf{X} = (X, \tau, \beta, \nu)$ *such that*

- (X, τ) *is a topological* T_0*-space,*
- $\nu :\subseteq \Sigma^* \to \beta$ *is a notation of a base* β *of* τ*,*
- $dom(\nu)$ *is recursive and*
- $\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \ : \ (u, v, w) \in S\}$ *for all* $u, v \in dom(\nu)$ *for some r.e. set* $S \subseteq (dom(\nu))^3$*.*

例 **15.** (i) *(real line) Define* $\mathbf{R} = (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ *such that* $\tau_{\mathbb{R}}$ *is the real line topology and* ν *is a canonical notation of the set of all open intervals with rational endpoints.*

(ii) *(lower unit interval) Define* $\mathbf{I}_< = (\mathbb{I}, \tau_<, \beta_<, \nu_<)$ *such that* $\nu_<(w) = \{x \ : \ 0 \leq q < x \leq 1$ *and* $q \in \nu_{\mathbb{Q}}\}$*.*

Representations on a Computable Topological Space

定義 16. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be an effective topological space. Define a representation $\delta :\subseteq \Sigma^\omega \to X$ of the points as*

$$x = \delta(p) \iff (\forall w \in \Sigma^*)(w \ll p \iff x \in \nu(w))$$

and a representation $\theta :\subseteq \Sigma^\omega \to \tau$ of the set of open sets as

$$W = \theta(p) \iff \begin{cases} w \ll p \Rightarrow w \in \mathit{dom}(\nu) \\ W = \bigcup\{\nu(w) \ : \ w \ll p\} \end{cases} .$$

Computable Operators

定理 17. (i) *eval* $: (f, x) \mapsto f(x)$ *is* $([\delta_1 \to_p \delta_2], \delta_1, \delta_2)$*-computable.*

(ii) $(f, g) \mapsto g \circ f$ *is* $([\delta_1 \to_p \delta_2], [\delta_2 \to_p \delta_3], [\delta_1 \to_p \delta_3])$*-computable.*

(iii) *The multi-function* $(f, W) \rightrightarrows T$ *mapping every continuous function* $f :\subseteq X_1 \to X_2$ *and every open set* $W \subseteq X_2$ *to some open set* $T \subseteq X_1$ *such that* $f^{-1}[W] = T \cap dom(f)$ *is* $([\delta_1 \to_p \delta_2], \theta_2, \theta_1)$*-computable.*

The Space of Continuous Functions

定義 18. *Define multi-representations of the set* $CP(X_1, X_2)$ *of all partial continuous functions* $f :\subseteq X_1 \to X_2$ *as follows:*

(i) $f \in \vec{\delta_1}(p)$ *iff* $f \circ \delta_1(q) = \delta_2 \circ \eta_p^{\omega\omega}(q)$ *for all* $q \in$ *dom*$(f \circ \delta_1)$,

(ii) $f \in \vec{\delta_2}(p)$ *iff* $f^{-1}[\theta_2(q)] = \theta_1 \circ \eta_p^{\omega\omega}(q) \cap$ *dom*(f) *for all* $q \in$ *dom*(θ_2),

(iii) $f \in \vec{\delta_3}(p)$ *iff* $f^{-1}[\nu_2(v)] = \theta_1 \circ \eta_p^{*\omega}(v) \cap$ *dom*(f) *for all* $v \in$ *dom*(ν_2),

In the paper they defi ned$\vec{\delta_1}, \ldots, \vec{\delta_8}$.

定理 19.

$$\vec{\delta_1} \equiv \vec{\delta_2} \equiv \vec{\delta_3} \equiv \vec{\delta_4} \equiv \vec{\delta_5} \equiv \vec{\delta_6} \le \vec{\delta_7} \le \vec{\delta_8}.$$

Computability of Measures

The Space of Measures

Let X be a topological space.
$P(X)$ denotes the space of probability Borel measures on X.
The weak topology τ_w on $\mathcal{M}(X)$ is defi ned as the weakest topology such that, for all lower semicontinuous functions $f : X \to \mathbb{I}_<$, the function $\mu \mapsto \int f d\mu$ is lower semicontinuous.
It is known that

- X is second-countable iff $P(X)$ is second-countable,
- X is separable iff $P(X)$ is separable,
- X is compact iff $P(X)$ is compact,
- and other similar relations of the descriptive complexity.

定理 **20** (Hoyrup and Rojas 2009)**.** *If X is a computable metric space then $P(X)$ is a computable metric space.*

Representations for Borel Measures

Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space.

定理 **21** (Schröder 2007)**.** *If X is second-countable, then the family of sets $\{\mu \in P(X) \,:\, \mu(B) > q\}$ is a subbase of the weak topology on $P(X)$ where B is the finite union of base elements and q is rational.*

定理 **22** (Miyabe)**.** *$\mathcal{P}(\mathbf{X}) = (P(X), \tau_w, \beta^P, \nu^P)$ is a computable topological space.*

定義 **23.** *A measure is computable if it is a computable point in the computable topological space.*

Natural Properties

定理 24. *A measure μ is computable iff its restriction to open sets is $[\theta \to \rho_{\mathbb{I}_<}]$-computable.*

定理 25. *The integral operator $\int : \mathcal{C}(X, \mathbb{I}_<) \times P(X) \to \mathbb{I}_<$ is $([\delta \to \rho_{\mathbb{I}_<}], \delta^P, \rho_{\mathbb{I}_<})$-computable.*

定理 26. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space. A measure μ is computable iff $\int d\mu : \mathcal{C}(X, \mathbb{I}_<) \to \mathbb{I}_<$ is $([\delta \to \rho_{\mathbb{I}_<}], \rho_{\mathbb{I}_<})$-computable.*

Algorithmic Randomness

Martin-Löf randomness

There are three approaches to defi ne randomness.
We consider a Cantor space 2^ω and the natural measure μ on it.

定義 27 (Martin-Löf 1966)**.** *A Martin-Löf test is a uniformly c.e. open sets U_n with $\mu(U_n) \leq 2^{-n}$. A sequence $A \in 2^\omega$ is Martin-Löf random if $A \notin \bigcap_n U_n$ for all Martin-Löf tests.*

定理 28 (Schnorr 1971)**.** *The following are equivalent.*

(i) *A sequence $A \in 2^\omega$ is Martin-Löf random.*

(ii) $\sup_n d(A \restriction n) < \infty$ *for all c.e. (super)martingales.*

(iii) $(\exists c)(\forall n) K(A \restriction n) \geq n - c$.

Generalization

We generalize it to a computable topological space.

定義 29 (Hertling and Weihrauch, Miyabe)**.** *A measure test over* $\mathbf{X}$ *is a sequence* $\{U_n\}$ *of uniformly* θ*-computable open sets with* $\mu(U_n) \leq 2^{-n}$ *for all* n*. A point* x *is measure* μ*-random if* $x \notin \bigcap_n U_n$ *for each measure test* $\{U_n\}$*.*

定理 30 (Miyabe)**.** *A point* x *is measure* μ*-random iff* $\sup_n f_n(x) < \infty$ *for each non-negative* $(\nu_{\mathbb{N}}, \delta, \overline{\rho}_<)$*-computable (super)martingale* $\{f_n\}$*.*

A characterization by Kolmogorov complexity on a computable metric space was given by Hoyrup and Rojas in 2009.I give a characterization on a computable topological space.

The space of sequences of reals

命題 31 (Miyabe)**.** $\mathbf{R}^{\mathbb{N}} = (\mathbb{R}^{\mathbb{N}}, \tau, \beta, \nu)$ *is a computable topological space.*

We consider a computable measure $\mu = \prod \mu_i$ on $\mathbb{R}^{\mathbb{N}}$ such that μ_i are continuous computable measures on $\mathbf{R}$ uniformly in i.

定理 32. *A point $x \in \mathbf{R}^{\mathbb{N}}$ is μ-random iff*

$$x \in \nu(u) \Rightarrow K(u) \geq -\log \mu(\nu(u)) - O(1).$$

Randomness version of some lemmas

Fix a computable topological space.

定義 **33** (essentially by Hoyrup and Rojas 2009). *A Borel-Cantteli test (BC-test) is a sequence $\{E_n\}$ of uniformly θ-computable open sets such that $\sum_n \mu(E_n) < \infty$.*

補題 **34.** *A point x is μ-random iff $x \in E_n$ for only finite n for each BC-test.*

We also give randomness version of BC2 but it is a little complicated.

定理 **35** (non-negative convergence theorem). *Let $\mu = \prod \mu_n$ be a computable measure. If $y = \{y_i\}$ is μ-random, then $f_n(y)$ converges for each non-negative $(\nu_{\mathbb{N}}, \delta, \overline{\rho})$-computable martingale.*

SLLN for randomness

定理 36 (Randomness version of SLLN). *Let $\mu = \prod \mu_i$ be a computable mesure on $\mathbf{R}^{\mathbb{N}}$ such that μ_i are continuous except at 0 and $m_i = \int z\mu_i(dz)$ and $v_i = \int (z - m_i)^2 \mu_i(dz)$ are uniformly computable.Then*

$$\{y_i\} \text{ is } \mu\text{-random and } \sum_i \frac{v_i}{i^2} < \infty \Rightarrow \frac{\sum_{i=1}^n (y_i - m_i)}{n} \to 0.$$

The proof is almost same as that in probability theory except checking computability.

SLLN for randomness 2

定理 **37.** *Let μ_1 be a continuous computable measure on $\mathbf{R}$ such that $m = \int |z| d\mu_1 < \infty$ is computable.Let $\mu = \prod \mu_1$ be the products of infinite same measures.Then*

$$\{y_i\} \text{ is } \mu\text{-random} \Rightarrow \frac{\sum_{i=1}^n y_i}{n} \to \int x d\mu.$$

定理 **38.** *Let μ_1 be a continuous computable measure on $\mathbf{R}$ such that $m = \int |z| d\mu_1 = \infty$.Let $\mu = \prod \mu_1$ be the products of infinite same measures.Then*

$$\{y_i\} \text{ is } \mu\text{-random} \Rightarrow \limsup_n \frac{|\sum_{i=1}^n y_i|}{n} = \infty.$$

History of LIL

The Law of the Iterated Logarithm tells us the asymptotic behaviour sums of random sequences.

定理 **39** (Khinchin 1924)**.**

$$\limsup_n \frac{S_n}{\sqrt{2n \log \log n}} = 1 \ a.s.$$

Kolmogorov in 1929 generalized to non identically case with a variance condition.
Hartman and Wintner in 1941 obtained for i.i.d. case.

LIL for randomness

定理 **40** (LIL with a variance condition)**.** *Let* $\mu = \prod \mu_i$ *be a computable measure on* $\mathbf{R}^{\mathbb{N}}$ *such that* μ_i *are continuous except at 0,* $m_i = \int z\mu_i(dz)$ *and* $v_i = \int (z-m_i)^2 \mu_i(dz)$ *are uniformly computable and* $B_n = \sum_{i=1}^n v_i \to \infty$*.If there exists a positive sequence* $\{M_n\}$ *such that*

$$M_n = o\left(\sqrt{\frac{B_n}{\ln\ln B_n}}\right) \text{ and } \mu_i(\{z \ : \ |z| \le M_n\}) = 1,$$

then

$$\{x_i\} \text{ is } \mu\text{-random} \Rightarrow \limsup_n \frac{\sum(x_i - m_i)}{\sqrt{2B_n \ln\ln B_n}} = 1.$$

Hartman-Wintner LIL for randomness

定理 41 (Hartman-Wintner LIL). *Let* $\mu = \prod \mu_1$ *be a computable measure on* $\mathbf{R}^{\mathbb{N}}$ *such that* μ_1 *is continous,* $\int z\mu(dz) = 0$ *and* $\int z^2\mu(dz) = \sigma^2$ *is computable. Then*

$$\{x_i\} \text{ is } \mu\text{-random} \Rightarrow \limsup_n \frac{\sum x_i}{\sqrt{2n \ln \ln n}} = \sigma.$$

End

Thank you!