Algorithmic randomness over general spaces

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Background

Martin-Löf randomness

We have three approaches to define randomness.
A *Cantor space* 2^ω is the set of all infinite binary sequences.
The topology is the one generated by the cylinder sets $[w] = \{A \in 2^\omega \ : \ w \preceq A\}$.
The measure μ is induced by $\mu([w]) = 2^{-|w|}$.
A open set W is *c.e.* if $W = \bigcup_{w \in V} [w]$ for some c.e.

Definition 1 (Martin-Löf 1966)**.** *A Martin-Löf test is a uniformly c.e. open set U_n with $\mu(U_n) \leq 2^{-n}$.*
A sequence A is Martin-Löf random if it passes all Martin-Löf test, that is, $A \notin \bigcap_n U_n$.

Theorem 2. *There is a universal Martin-Löf test.*
The class of Martin-Löf random sequences has measure 1.

Complexity and martingales

The *prefix-free Kolmogorov complexity* K of w is defined as $K(w) = \{|u| : U(u) = w\}$ where U is the universal Turing machine.

Theorem 3 (Schnorr 1971). *A sequence A is Martin-Löf random iff $K(A \upharpoonright n) \geq n - O(1)$.*

A martingale is a function $d : 2^* \to \mathbb{R}^+$ satisfying $2d(w) = d(w0) + d(w1)$ for all $w \in 2^*$.

Theorem 4 (Schnorr 1971). *A sequence A is Martin-Löf random iff no c.e. martingale succeeds on A, that is, $\sup_n d(A \upharpoonright n) < \infty$ for all d.*

This coincidence is one of the reasons of the fact that Martin-Löf randomness is considered a natural randomness.

Generalization of computability

Computability on $\mathbb{N}$ was defined by Church, Turing and others in 1930s.
Computable analysis studies computability on $\mathbb{R}$.
There are many approaches to this problem.

- Via representations by Hauck, Kreitz and Weihrauch.
- Via sequential computability and effective uniform continuity by Pour-El and Richards (and this is generalized to metric spaces by Yasugi).
- Ko's approach, Domain Theory, Markov's approach, etc.

We will mostly use the approach via representations, which is called Type-2 Theory of Effectivity (TTE).
This is because this approach can be naturally adapted to randomness.

Generalization of randomness

History of generalization of randomness is the following.

	measure	complexity	martingales
$2^\omega, \mathbb{R}$	Martin-Löf 1966	Schnorr 1971	Schnorr 1971
compact CMS	-	Gács 2005	?
CMS	-	Hoyrup&Rojas 2009	?
CTS	Hertling&Weihrauch 1998	?	?

Here "CMS" means a computable metric space and "CTS" means a computable topological space.

Our goal is to define randomness by complexity and martingales on a computable topological space and to confirm the naturalness for randomness.

TTE

Representation Approach

- A real is (seen as) a sequence of a finite alphabet, and
- a computable function from reals to reals is a computable function from sequences to sequences.

We must be careful about the choise of a representation.

Example 5. Consider the infinite decimal fraction as a representation of a real.
Then $x \mapsto 3x$ is not computable because machine does not make an output with an input

$$\frac{1}{3} = 0.3333333\ldots.$$

Usually we use a sequence of intervals with rational endpoints containing the real as a representation of the real.

Notation

We will use the terminology from the work by Weihrauch and Grubba 2009.

- $f : A \rightrightarrows B$ denotes a multi-function.
- $f :\subseteq A \to B$ denotes a partial function.
- $f : A \to B$ denotes a total function.

$|w|$ is the length of the word $w \in \Sigma^*$.
$x \sqsubseteq y$ iff x is a prefix of y.
The "wrapping function" $\iota : \Sigma^* \to \Sigma^*$ is the one such that

$$\iota(a_1 a_2 \ldots a_k) = 110 a_1 0 a_2 0 \ldots a_k 011.$$

$u \ll w$ iff $i(u)$ is a subword of w for $u \in \Sigma^*$ and $w \in \Sigma^* \cup \Sigma^\omega$.

Computability on infinite sequences

Let $Y_0, \ldots, Y_n \in \{\Sigma^*, \Sigma^\omega\}$ and $Y = Y_1 \times \ldots \times Y_n$.
A function $f :\subseteq Y \to Y_0$ is *computable* if it is computed by Type-2 machine.
Informally, a *Type-2 machine* is a Turing machine, which reads from input tapes with finite or infinite inscription, operates on work tapes and write one-way to an output tape.
On Σ^* we consider the discrete topology.
On Σ^ω we consider the topology generated by the base $\{w\Sigma^\omega \ : \ w \in \Sigma^*\}$ of open sets.
Every computable function is continuous.

Representation of a Set

A *multi-representation* of a set M is a surjective function $\gamma : Y \rightrightarrows M$ where $Y \in \{\Sigma^*, \Sigma^\omega\}$.

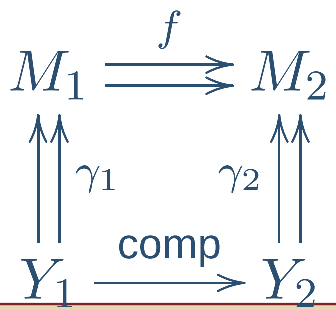
Example 6. (i) $\nu_{\mathbb{N}} :\subseteq \Sigma^* \to \mathbb{N}$

(ii) $\nu_{\mathbb{Q}} :\subseteq \Sigma^* \to \mathbb{Q}$

(iii) $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$

A point is γ-*computable* if it has a computable representation by γ.

A function $f : M_1 \rightrightarrows M_2$ is (γ_1, γ_2)-*computable* if it has a computable realization.



Reducibility and Product

$\gamma_1 \le \gamma_0$ if $M_1 \subseteq M_0$ and the identigy $\mathrm{id} : M_1 \to M_0$ is (γ_1, γ_0)-computable.
$\gamma_1 \equiv \gamma_0$ iff $\gamma_1 \le \gamma_0$ and $\gamma_0 \le \gamma_1$.
$[\gamma_1, \gamma_2]\langle y_1, y_2\rangle = \gamma_1(y_1) \times \gamma(y_2)$.

Definition 7 (Notations of computable functions)**.** *For all* $a, b \in \{*, \omega\}$,

(i) $P^{ab} = \{f :\subseteq \Sigma^a \to \Sigma^b \ : \ f$ *is computable*$\}$ *and,*

(ii) *a notation* $\xi^{ab} : \Sigma^* \to P^{ab}$ *is defined as* $\xi^{ab}(w)$ *is the function* $f \in P^{ab}$ *computed by the Type-2 machine with code* w *if the machine is recursive.*

(i) F^{**} is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^*$.

(ii) $F^{*\omega}$ is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^\omega$.

(iii) $F^{\omega *}$ is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^*$ with open domain.

(iv) $F^{\omega\omega}$ is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^*$ with G_δ-domain.

Representations of functions

Definition 8 (Representations of the sets F^{ab})**.** *Define* $\eta^{ab} : \Sigma^\omega \to F^{ab}$ *by*

$$\eta^{ab}(\langle x, p\rangle)(y) = \xi_x^{\omega b}\langle p, y\rangle$$

for all $x \in \Sigma^*$, $p \in \Sigma^\omega$ *and* $y \in \Sigma^a$.

Definition 9. *Let* $\gamma_1 : \Sigma^a \rightrightarrows M_1$ *and* $\gamma_2 : \Sigma^b \rightrightarrows M_2$ *be multi-representations.*
We define a multi-representation $[\gamma_1 \rightrightarrows \gamma_2]$ *of the* (γ_1, γ_2)*-continuous multi-functions* $f : M_1 \rightrightarrows M_2$ *as* $f \in [\gamma_1 \rightrightarrows \gamma_2](p)$ *iff* $\eta_p^{ab} = \eta^{ab}(p)$ *realizes* f *w.r.t.* (γ_1, γ_2)*.*
The restriction ot the single-valued functions is $[\gamma_1 \to_p \gamma_2]$*.*
The restriction to the total (γ_1, γ_2)*-continuous functions is* $[\gamma_1 \to \gamma_2]$*.*

Computable Topology

General Topology

Let X be a set.
A topology τ on X is a set of subsets of X (the set of open sets) that is closed under union and finite intersection.
A topological space (X, τ) is a T_0-space if whenever x and y are distinct points in X, there is an open set containing one and not the other.
A base is a subset $\beta \subseteq \tau$ such that every $U \in \tau$ is a union of base sets.
A subbase is a subset $\beta \subseteq \tau$ such that the set of all finite intersections forms a base.
A space is said to be second-countable if its topology has a countable base.
A space is called separable if it contains a countable dense subset.

Computable Topological Spaces

Definition 10 (Hertling and Weihrauch 2009)**.** *A computable topological space is a 4-tuple $\mathbf{X} = (X, \tau, \beta, \nu)$ such that*

- (X, τ) *is a topological* T_0*-space,*
- $\nu :\subseteq \Sigma^* \to \beta$ *is a notation of a base* β *of* τ*,*
- $dom(\nu)$ *is recursive and*
- $\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \ : \ (u, v, w) \in S\}$ *for all* $u, v \in dom(\nu)$ *for some r.e. set* $S \subseteq (dom(\nu))^3$*.*

Example 11. (i) *(real line) Define* $\mathbf{R} = (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ *such that* $\tau_{\mathbb{R}}$ *is the real line topology and* ν *is a canonical notation of the set of all open intervals with rational endpoints.*

(ii) *(lower unit interval) Define* $\mathbf{I}_< = (\mathbb{I}, \tau_<, \beta_<, \nu_<)$ *such that* $\nu_<(w) = \{x \ : \ 0 \leq q < x \leq 1$ *and* $q \in \nu_{\mathbb{Q}}\}$*.*

Representations on a Computable Topological Space

Definition 12. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be an effective topological space.*
Define a representation $\delta :\subseteq \Sigma^\omega \to X$ of the points as

$$x = \delta(p) \iff (\forall w \in \Sigma^*)(w \ll p \iff x \in \nu(w))$$

and a representation $\theta :\subseteq \Sigma^\omega \to \tau$ of the set of open sets as

$$W = \theta(p) \iff \begin{cases} w \ll p \Rightarrow w \in \mathit{dom}(\nu) \\ W = \bigcup\{\nu(w) \ : \ w \ll p\} \end{cases} .$$

Computable Operators

Theorem 13. (i) *eval* $: (f, x) \mapsto f(x)$ *is* $([\delta_1 \to_p \delta_2], \delta_1, \delta_2)$*-computable.*

(ii) $(f, g) \mapsto g \circ f$ *is* $([\delta_1 \to_p \delta_2], [\delta_2 \to_p \delta_3], [\delta_1 \to_p \delta_3])$*-computable.*

(iii) *The multi-function* $(f, W) \rightrightarrows T$ *mapping every continuous function* $f :\subseteq X_1 \to X_2$ *and every open set* $W \subseteq X_2$ *to some open set* $T \subseteq X_1$ *such that* $f^{-1}[W] = T \cap dom(f)$ *is* $([\delta_1 \to_p \delta_2], \theta_2, \theta_1)$*-computable.*

The Space of Continuous Functions

Definition 14. *Define multi-representations of the set $CP(X_1, X_2)$ of all partial continuous functions $f :\subseteq X_1 \to X_2$ as follows:*

(i) $f \in \vec{\delta_1}(p)$ *iff* $f \circ \delta_1(q) = \delta_2 \circ \eta_p^{\omega\omega}(q)$ *for all* $q \in$ *dom*$(f \circ \delta_1)$,

(ii) $f \in \vec{\delta_2}(p)$ *iff* $f^{-1}[\theta_2(q)] = \theta_1 \circ \eta_p^{\omega\omega}(q) \cap$ *dom*(f) *for all* $q \in$ *dom*(θ_2),

(iii) $f \in \vec{\delta_3}(p)$ *iff* $f^{-1}[\nu_2(v)] = \theta_1 \circ \eta_p^{*\omega}(v) \cap$ *dom*(f) *for all* $v \in$ *dom*(ν_2),

In the paper they defined $\vec{\delta_1}, \ldots, \vec{\delta_8}$.

Theorem 15.

$$\vec{\delta_1} \equiv \vec{\delta_2} \equiv \vec{\delta_3} \equiv \vec{\delta_4} \equiv \vec{\delta_5} \equiv \vec{\delta_6} \le \vec{\delta_7} \le \vec{\delta_8}.$$

Computability of Measures

The Space of Measures

Let X be a topological space.
$P(X)$ denotes the space of probability Borel measures on X.
The weak topology τ_w on $\mathcal{M}(X)$ is defined as the weakest topology such that, for all lower semicontinuous functions $f : X \to \mathbb{I}_<$, the function $\mu \mapsto \int f d\mu$ is lower semicontinuous.
It is known that

- X is second-countable iff $P(X)$ is second-countable,
- X is separable iff $P(X)$ is separable,
- X is compact iff $P(X)$ is compact,
- and other similar relations of the descriptive complexity.

Theorem 16 (Hoyrup and Rojas 2009)**.** *If X is a computable metric space then $P(X)$ is a computable metric space.*

Representations for Borel Measures

Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space.

Theorem 17 (Schröder 2007). *If X is second-countable, then the family of sets $\{\mu \in P(X) \,:\, \mu(B) > q\}$ is a subbase of the weak topology on $P(X)$ where B is the finite union of base elements and q is rational.*

Theorem 18 (Miyabe). *$\mathcal{P}(\mathbf{X}) = (P(X), \tau_w, \beta^P, \nu^P)$ is a computable topological space.*

Definition 19. *A measure is computable if it is a computable point in the computable topological space.*

Natural Properties

Theorem 20. *A measure μ is computable iff its restriction to open sets is $[\theta \longrightarrow \rho_{\mathbb{I}_<}]$-computable.*

Theorem 21. *The integral operator $\int : \mathcal{C}(X, \mathbb{I}_<) \times P(X) \longrightarrow \mathbb{I}_<$ is $([\delta \longrightarrow \rho_{\mathbb{I}_<}], \delta^P, \rho_{\mathbb{I}_<})$-computable.*

Theorem 22. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space. A measure μ is computable iff $\int d\mu : \mathcal{C}(X, \mathbb{I}_<) \longrightarrow \mathbb{I}_<$ is $([\delta \longrightarrow \rho_{\mathbb{I}_<}], \rho_{\mathbb{I}_<})$-computable.*

Generalization

Definition by tests

Definition 23 (essentially Hertling and Weihrauch 1998). *A measure test over $\mathbf{X}$ is a uniformly θ-computable sequence $\{U_n\}$ of open sets with $\mu(U_n) \leq 2^{-n}$ for all n.*
A point x is measure μ-random over $\mathbf{X}$ if $x \notin \bigcap_n U_n$ for each measure test $\{U_n\}$.

Inspired by a uniform test by Gács and Levin, we give characterization by a function test.
Let $\overline{\rho}_<$ be the representation of lower real line with infinity.

Definition 24. *A function test over $\mathbf{X}$ is a $(\delta, \overline{\rho}_<)$-computable function $f : X \to \overline{\mathbb{R}}$ such that $\mu f = \int_X f d\mu \leq 1$.*

Theorem 25. *A point x is measure μ-random iff $f(x) < \infty$ for each function test f.*

Definition by martingales

We use some terminologies from measure theory.
Let $(X, \mathcal{A}, \mu)$ be a measure space.
A *filtration* is a sequence of sub-σ-algebra $\{\mathcal{A}_n\}$ such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for each n.
A sequence of $\mathcal{A}$-measurable functions $\{f_n\}$ is called *a martingale* if $\int f_n d\mu < \infty$ and $\int_A f_n d\mu = \int_A f_{n+1} d\mu$ for all $A \in \mathcal{A}_n$.

Theorem 26. *A point x is measure μ-random iff $\sup_n f_n(x) < \infty$ for each $([\nu_{\mathbb{N}}, \delta], \overline{\rho}_<)$-computable martingale $\{f_n\}$*

Proof idea. Let $U_{k,m} = \{y \ : \ \sup_{n \le m} \sup f_n(y) > 2^k\}$ and use Doob's maximal inequality. □

Universal complexity of sequences

Let $f :\subseteq 2^* \to \Sigma^\omega$ be a prefix-free computable function.

$$K_f(p) = \min\{\sigma \ : \ f(\sigma) = p\}.$$

If p is not computable, $K_f(p) = \infty$ for all f.

Theorem 27. *There exists a prefix-free computable function $U :\subseteq 2^* \to \Sigma^\omega$ such that*

$$(\forall f)(\exists c)(\forall p)(\exists q)\theta(p) = \theta(q) \text{ and } K_U(q) \leq K_f(p) + c.$$

In the following we write K to mean K_U.

Definition by complexity

Let $\psi^-(p) = X \setminus \theta(p)$.

Definition 28. *A point x is complexity μ-random if*

$$x \in \psi^-(p) \Rightarrow K(p) \geq -\log \mu\psi^-(p) - O(1).$$

This definition is not a straightforward generalization but coincides with the definition on a Cantor space.

Definition 29. *The base $\beta = \{\nu(i)\}$ is complete if all equivalent bases are computably reducible to ν.*

There exists such a complete base.

Theorem 30. *A point x is complexity μ-random iff*

$$x \in \xi(u) \Rightarrow K(u) \geq -\log \mu\xi(u) - O(1)$$

where $\xi(u) = \nu(u)^c$.

When they coincide.

Do measure randomness and complexity randomness coincide?
In general they are different.

Example 31. *For lower unit interval* $\mathbf{I}_<$ *and Lebesgue measure* μ*,*

- *the set of measure* μ*-random points is* $\{1\}$ *and*
- *the set of complexity* μ*-random points is* $\{0\}$*.*

However they coincide on a computable metric space with a computable measure, so on a Cantor space too.
We shall see the coditions on which they coincide.

Almost properties

Definition 32 (almost decidability, Hoyrup and Rojas 2009)**.** *A set A is said to be almost decidable if there are two θ-computable open sets U and V such that:*

$$U \subset A,\ V \subseteq A^c,\ U \cup V \text{ is dense and has measure one.}$$

In a computable metric space with a computable measure, there exists an equivalent basis that is uniformly almost decidable.

Definition 33 (almost disjointness)**.** *Suppose that the base β is complete. $(\mathbf{X}, \mu)$ has the property of almost disjointness if there exists a computable function $d :\subseteq \Sigma^* \times \mathbb{Q} \to \Sigma^*$ such that $\mu(\nu(u)) > 1 - q_1 \geq 1 - q_2$ implies $\nu(u) \cup \nu(d(u, q_i)) = X$, $\nu(d(u, q_1)) \subseteq \nu(d(u, q_2))$ and $\mu(\nu(d(u, q_i))) \leq q_i$ for $i = 1, 2$.*

This property also holds in a computable metric space with a computable measure.

Measure randomness and complexity randomness

Suppose μ is computable.This is an essential hypothesis.

Theorem 34. *With almost decidability, complexity μ-randomness implies measure μ-randomness.*

Each measure test is as a computably countable union of almost decidable base elements.
Then i-th element is covered by a complement of a base element.
By getting rid of the union of base elements until $i - 1$, we get a sequence of closed sets containing all non-random points.
The sum of measures of such sets is at most 1.

Theorem 35. *With almost disjointness, measure μ-randomness implies complexity μ-randomness.*

Each closed set can be covered by a base element with as little loss as you want.

Some natural properties

Theorem 36. *The set of complexity random points has measure 1.*

It is obvious for measure randomness, but it needs another proof for complexity randomness.
Recall that computable permutations preserves randomness.

Theorem 37. *Let $f : X_1 \to X_2$ s.t. $\mu_1(f^{-1}(V)) \leq C\mu_2(V)$ for all open $V \subseteq X_2$.*
If $x \in dom(f)$ is complexity μ_1-random then $f(x)$ is complexity μ_2-random.

Relative randomness

Definition

For $i = 1, 2$ let $\mathbf{X}_i = (X_i, \tau_i, \beta_i, \nu_i)$ be computable topological spaces with complete bases and μ_i be a measure on X_i.
Let $x_1 \in X_1$.

Definition 38. *A x_1-measure μ-test over $\mathbf{X_2}$ is a sequence $\{t_n\}$ of uniformly (δ, θ)-computable functions with $\mu(t_n(x_1)) \leq 2^{-n}$. $y \in X_2$ is x_1-measure μ-random if $x \notin \bigcap_n t_n(x_1)$ for each measure test.*

Definition 39. *A x is x_1-complexity μ-random if*

$$x \in \xi(u) \Rightarrow K_{f(x_1)}(u) \geq -\log(\mu(\xi(u))) - O(1)$$

for all (δ, η^{**})-computable functions $f :\subseteq X \to F^{**}$ such that dom$(f(x_1)) \subseteq 2^*$ and $f(x_1)$ is prefix-free.

Coincide

Note that if x_1 is computable, then x_1-randomness coincides with non-relativized randomness.
Almost all properties can be relativized.

Theorem 40. *With almost disjointness and almost decidability, measure randomness and complexity randomness coincide.*

Remark 41. *Universality of complexity randomness many not hold because of multi-functions.*

Van Lambalgen's Theorem

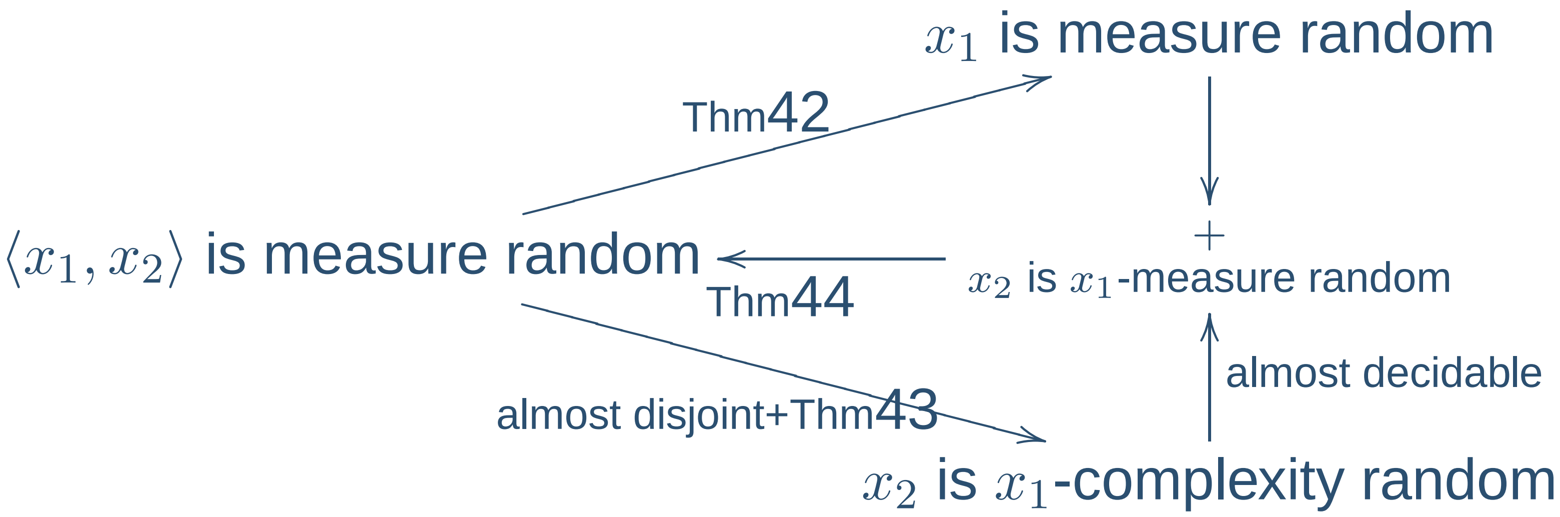
Theorem 42. *If $\langle x_1, x_2 \rangle \in \overline{X}$ is measure $\overline{\mu}$-random, then x_1 is measure μ_1-random.*

Theorem 43. *If X_2 with μ_2 has almost disjointness and $\langle x_1, x_2 \rangle \in \overline{X}$ is measure $\overline{\mu}$-random, then x_2 is x_1-complexity μ_2-random.*

Theorem 44. *If x_1 is measure μ_1-random and x_2 is x_1-measure μ_2-random, then $\langle x_1, x_2 \rangle \in \overline{X}$ is measure $\overline{\mu}$-random.*

Theorem 45. *With almost disjointness and almost decidability, van Lambalgen's Theorem holds.*

Situation



For some reason, it is a difficult problem to find whether van Lambalgen's Theorem holds with no conditions.
Recall that van Lambalgen's theorem is a criterion of natural randomness.
So the pair of the conditions is a sufficient condition for natural randomness.

Summary

Discussion

- We gave the characterizations of measure randomness by martingales.
- We defined complexity randomness and proved it has some natural properties.
- We proposed two conditions which hold on a computable metric space with a computable measure.
- With the conditions, measure randomness and complexity randomness coincide.
- With the conditions, van Lambalgen's Theorem holds.
- The pair of the conditions is a sufficient condition of the space where natural randomness can be defined.

End

Thank you!