A Random Sequence of Reals

Kenshi Miyabe

August 2, 2010

Abstract

We define a random sequence of reals as a random point on a computable topological space. This randomness has three equivalent simple characterizations, namely, by tests, by martingales and by complexity. We prove that members of a random sequence are relatively random. Conversely a relatively random sequence of reals has a random sequence such that each corresponding member is Turing equivalent. Furthermore strong law of large numbers and the law of the iterated logarithm hold for each random sequence.

1 Introduction

1.1 Background

The definition of a random sequence has been studied. A binary sequence is the most simple case. The definition of randomness of binary sequences was first considered by von Mises [34] but his approach was insufficient. The first natural randomness was proposed by Martin-Löf [21], which uses a test concept. Now this randomness is called Martin-Löf randomness. Martin-Löf randomness also has characterizations by complexity and by martingales, which was showed by Schnorr [29]. By the three equivalent characterizations Martin-Löf randomness has been considered a natural randomness of binary sequences.

We would like to consider random sequence of more general values. For example Vovk [36] considered a random sequence of rationals. Clearly we need an appropriate definition of a random sequence of reals.

Martin-Löf randomness is randomness of binary sequences, in other words, randomness on a Cantor space. Many researchers have been studied randomness on general spaces. Hertling and Weihrauch [10] succeeded in defining random elements in computable topological spaces by tests (via Martin-Löf's approach). To give a characterization by complexity, the restricted space was considered, namely, a computable metric space. Gács [7] characterized randomness by complexity on a computable metric space but it was restricted to be compact. This weakness was overcome by Hoyrup and Rojas [12]. It seems to be difficult to give an equivalent characterization by complexity on a computable topological space but Miyabe [23] gave an equivalent characterization by complexity under two assumptions. Furthermore Miyabe [23] also gave a characterization by martingales.

In this paper we define a random sequence of reals as a random point on the space of sequence of reals. The space should be a computable metric space but it is easier to understand the space as a computable topological space. In fact all we have to do is to confirm that the space of sequence of reals is a computable topological space. We also give simple characterization by complexity. Furthermore we prove some properties of a random sequence of reals.

There are some criteria which should hold for natural random sequences, that is, SLLN (Strong Law of Large Numbers) and LIL (the Law of the Iterated Logarithm). SLLN says that the average of a sequence of i.i.d. (independent and identically distributed random variables) converges to the expectation. Kolmogorov [17] in 1930 proved SLLN for non identically ones under variance constraints and Kolmogorov [15] in 1933 provides a necessary and sufficient condition for the validity of SLLN in i.i.d. case. LIL tells us the asymptotic behaviour of sums of random sequences. Khinchin [14] in 1924 proved LIL for i.i.d. coin tossing. Kolmogorov [16] in 1929 generalized to non identically case and Hartman and Wintner [9] in 1941 obtained for i.i.d. case. It is well known that one can examine and prove SLLN and LIL for Martin-Löf randomness. For example Vovk [35] examined LIL for sequences of which the Kolmogorov complexity of each initial segment is high. Li and Vitányi [20] proved a general theorem which implies SLLN and the first part of LIL for Martin-Löf random sequences. Each Martin-Löf randomness is a binary sequence and so this is the most simple case in probability theoretic view. Vovk [36] proved LIL for a random sequence of rationals. Along these lines we prove SLLN and LIL for a random sequence of reals. In our case we can prove randomness version of all theorems above.

1.2 Overview of this paper

In section 3 we study general computable topological spaces. We give some definitions and prove some results, which we will use in the next section. In subsection 3.1 we extend Turing degrees and continuous degrees on computable metric spaces to those on computable topological spaces. In subsection 3.2 we prove that the product of infinite computable topological spaces is also a computable topological space. In subsection 3.3 we give a characterization of randomness by supermartingales. In subsection 3.4 we give a characterization of randomness by restricted martingales and by restricted supermartingales.

In section 4 we study the space of sequences of reals. In subsection 4.1 we give another characterization of randomness by complexity. Theoretically [7], [12] and [23] also gave different characterizations of this randomness by complexity but they are hard to use. We give a useful and simple characterization on this special space. In subsection 4.2 we study the relation between randomness of a sequence and relative randomness of each member. We prove that members of a random sequence are relatively random. Conversely a relatively random sequence of reals has a random sequence such that each corresponding member is Turing equivalent.

In section 5 we study limit theorems of a random sequence. There are some laws which should hold for random sequences. We prove that “strong law of large numbers” and “the law of the iterated logarithm” hold for each random sequence not almost surely.

2 Preliminaries

2.1 Type-2 Theory of Effectivity

This work is based on a computable topological space defined in TTE (Type-2 Theory of Effectivity) [18, 37, 3, 38, 39], which is a foundation of computable topology by the representation approach. We will use essentially the terminology from Weihrauch and Grubba [39]. We assume that the readers are familiar with computability from Σ^* to Σ^*, which has been well studied in computability theory [32, 25, 26, 5].

A *multi-function* from A to B is a triple $f = (A, B, R_f)$ such that $R_f \subseteq A \times B$ denoted by $f : A \rightrightarrows B$. Its inverse is the multi-function $f^{-1} := (B, A, R_f^{-1})$. For $X \subseteq A$ let $f[X] := \{b \in B : (\exists a \in X)(a, b) \in R_f\}$, $\mathrm{dom}(f) := f^{-1}[B]$, and $\mathrm{range}(f) := f[A]$. For $a \in A$ let $f(a) := f[\{a\}]$. If $f(a)$ contains at most one element for every $a \in A$, f can be treated as a usual partial function denoted by $f :\subseteq A \to B$.

Let Σ be a finite alphabet such that $0, 1 \in \Sigma$. By Σ^* we denote the the set of finite words over Σ and by Σ^ω the set of infinite sequences $p : \mathbb{N} \to \Sigma$ over Σ, $p = (p(0)p(1)\ldots)$. For a word $w \in \Sigma^*$ let $|w|$ be its length. Let Σ^n be the set of words of length n and let $\epsilon \in \Sigma^0$ be the empty word. For $p \in \Sigma^\omega$ let $p^{<i} \in \Sigma^*$ be the prefix of p of length $i \in \mathbb{N}$. We write $x \sqsubseteq y$ if x is a prefix of y. We use the "wrapping function" $\iota : \Sigma^* \to \Sigma^*$, $\iota(a_1 a_2 \ldots a_k) := 110a_1 0a_2 0 \ldots a_k 011$ for coding words such that $\iota(u)$ and $\iota(v)$ cannot overlap properly. Let $\langle i, j\rangle := (i + j)(i + j + 1)/2 + j$ be the bijective Cantor pairing function on $\mathbb{N}$. We consider standard functions for finite or countable tupling on σ^* and Σ^ω denoted by $\langle\cdot\rangle$. For $u \in \Sigma^*$ and $w \in \Sigma^* \cup \Sigma^\omega$ let $u \ll w$ iff $\iota(u)$ is a subword of w.

Let $Y_0, \ldots, Y_n \in \{\Sigma^*, \Sigma^\omega\}$ and $Y = Y_1 \times \ldots \times Y_n$. A function $f :\subseteq Y \to Y_0$ is computable if for some Type-2 machine M, f is the function f_M computed by M. Informally, a Type-2 machine is a Turing machine, which reads from input tapes with finite or infinite inscription, operates on work tapes and write one-way to an output tape. For $Y_0 = \Sigma^*$, $f_M(y) = w$, if a Turing machine M on input y halts with w on the output tape. For $Y_0 = \Sigma^\omega$, $f_M(y) = q$, if M on input y computes forever and writes writes $q \in \Sigma^\omega$ on the output tape.

A *multi-representation* of a set M is a surjective multi-function $\gamma : Y \rightrightarrows M$ where $Y \in \{\Sigma^*, \Sigma^\omega\}$. Examples are the canonical notations $\nu_\mathbb{N} :\subseteq \Sigma^* \to \mathbb{N}$ and $\nu_\mathbb{Q} :\subseteq \Sigma^* \to \mathbb{Q}$ of the natural numbers and the rational numbers, respectively, and the representation $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$ of the real numbers. For multi-representations $\gamma_i : Y_i \rightrightarrows M_i$ ($i = 0, \ldots, n$), let $Y = Y_1 \times \ldots \times Y_n$, $M = M_1 \times \ldots \times M_n$ and $\gamma : Y \rightrightarrows M$, $\gamma(y_1, \ldots, y_n) = \gamma_1(y_1) \times \ldots \times \gamma_n(y_n)$. A partial function $h :\subseteq Y \to Y_0$ realizes the multi-function $f : M \rightrightarrows M_0$ if $f(x) \cap \gamma_0 \circ h(y) \neq \phi$ whenever $x \in \gamma(y)$ and $f(x) \neq \phi$. This means that $h(y)$ is a name of some $z \in f(x)$ if y is a name of $x \in \mathrm{dom}(f)$. If $f :\subseteq M \to M_0$ is single-valued, then $h(y)$ is a name of $f(x)$ if y is a name of $x \in \mathrm{dom}(f)$. If only the representations are single-valued, $\delta_0 \circ h(y) \in f(x)$ if $\delta(y) = x$. The multi-function f is called $(\gamma_1, \ldots, \gamma_n, \gamma_0)$-computable if it has a computable realization.A point $x \in M_1$ is γ_1-computable iff $x \in \gamma_1(p)$ for some computable $p \in \mathrm{dom}(\gamma_1)$.

Finally, $\gamma_1 \leq \gamma_0$ (γ_1 is reducible to γ_0) if $M_1 \subseteq M_0$ and the identity $\mathrm{id} : M_1 \to M_0$ is (γ_1, γ_0)-computable. This means that some computable function h translates γ_1-names to γ_0-names, that is, $\gamma_1(p) \subseteq \gamma_0 \circ h(p)$. Computable equivalence is defined canonically:

$\gamma_1 \equiv \gamma_0 \iff \gamma_1 \leq \gamma_0 \wedge \gamma_0 \leq \gamma_1$. Two multi-representations induce the same computability iff they are computably equivalent.

From γ_1 and γ_2 a multi-representation $[\gamma_1, \gamma_2]$ of the product $M_1 \times M_2$ is defined by $[\gamma_1, \gamma_2]\langle y_1, y_2\rangle := \gamma_1(y_1) \times \gamma_2(y_2)$.

2.2 Computable topological spaces

The definitions and results are almost from [39].

A topology τ on a set X is a set of subsets of X, the set of open sets, that is closed under union and finite intersection. A base is a subset of $\beta \subseteq \tau$ such that every $U \in \tau$ is a union of base sets. A topological space (X, τ) is a T_0-space if whenever x and y are distinct points in X, there is an open set containing one and not the other, which means that the points can be identified by their neighborhoods. For more about topology, see [33] etc.

Definition 2.1 (computable topological space; [39]). *An effective topological space is a 4-tuple* $\mathbf{X} = (X, \tau, \beta, \nu)$ *such that* (X, τ) *is a topological* T_0*-space and* $\nu :\subseteq \Sigma^* \to \beta$ *is a notation of a base* β *of* τ*.* $\mathbf{X}$ *is a computable topological space if* $dom(\nu)$ *is recursive and*

$$\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \ : \ (u, v, w) \in S\} \text{ for all } u, v \in dom(\nu) \tag{1}$$

for some r.e. set $S \subseteq (dom(\nu))^3$.

Definition 2.2 ([39]). *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be an effective topological space.*

Define a representation $\delta :\subseteq \Sigma^\omega \to X$ *of the points as*

$$x = \delta(p) \iff (\forall w \in \Sigma^*)(w \ll p \iff x \in \nu(w))$$

and a representation $\theta :\subseteq \Sigma^\omega \to \tau$ *of the set of open sets as*

$$W = \theta(p) \iff \begin{cases} w \ll p \Rightarrow w \in dom(\nu) \\ W = \bigcup \{\nu(w) \ : \ w \ll p\} \end{cases} .$$

We also define a representation $\psi^- :\subseteq \Sigma^\omega \to \mathcal{A}$ *of the set of closed sets by*

$$\psi^-(p) = X \backslash \theta(p).$$

In what follows we assume that any effective topological space is equipped with these representations correspondingly. For example, for an effective topological space $\mathbf{X}_1 = (X_1, \tau_1, \beta_1, \nu_1)$, the representation δ_1 denotes the representation of points in X_1 defined here.

Example 2.3 (computable topological spaces).

(i) *(real line) Define* $\mathbf{R} = (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ *such that* $\tau_{\mathbb{R}}$ *is the real line topology and* ν *is a canonical notation of the set of all open intervals with rational endpoints. The representation* δ *for* $\mathbf{R}$ *is denoted by* ρ.

(ii) *(lower real line with infinity) Define* $\overline{\mathbf{R}}_< = (\overline{\mathbb{R}}, \tau_<, \beta_<, \nu_<)$ *such that* $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ *and* $\nu_<(w) = (q, \infty]$ *where* $q \in \mathbb{Q}$*. The representation* δ *for* $\overline{\mathbf{R}}_<$ *is denoted by* $\overline{\rho}_<$*.*

(iii) *(countable discrete space) Define* $\mathbf{D} = (D, \tau, \beta, \nu)$ *such that* $D = \{d_i \ : \ i \in \mathbb{N}\}$ *is a countable set,* τ *is the discrete topology,* β *is the set of all finite subsets and* ν *is an enumeration.*

(iv) *(Cantor space) Define* $\mathbf{C} = (C, \tau, \beta, \nu)$ *such that* C *is a Cantor space and* $\nu(\sigma) = [\sigma]$ *is the cylinder of* σ*.*

The computable topological space $\mathbf{X} = (X, \tau, \beta, \nu)$ and $\mathbf{X}' = (X, \tau, \beta', \nu')$ are equivalent iff $\nu \leq \theta'$ and $\nu' \leq \theta$. Note that $\nu \leq \theta' \iff \delta' \leq \delta \iff \theta \leq \theta'$. Then $\mathbf{X}$ and $\mathbf{X}'$ are equivalent $\iff \delta \equiv \delta' \iff \theta \equiv \theta'$.

Definition 2.4 (almost decidability; Miyabe [23], see also [12])**.** *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be a computable topological space and* μ *be a measure on* X*. A set* A *is said to be almost decidable if there are two* θ*-computable open sets* U *and* V *such that:*

$$U \subset A,\ V \subseteq A^c,\ U \cup V \text{ is dense and has measure one.}$$

We say that the elements of a sequence (A_i) *are uniformly almost decidable if there are two sequences* (U_i) *and* (V_i) *of uniformly* θ*-computable open sets satisfying the conditions above. The computable topological space with the measure has the property of almost decidability if there is an equivalent basis that is uniformly almost decidable.*

2.3 Computable metric spaces

The followings are from [12].

A computable metric space is a triple $\mathcal{X} = (X, d, S)$, where

- (X, d) is a separable complete metric space (polish metric space),
- $S = \{s_i \ : \ i \in \mathbb{N}\}$ is a countable dense subset of X,
- The real numbers $d(s_i, s_j)$ are all computable, uniformly in $\langle i, j\rangle$.

The elements of S are called the ideal points. The numbering ν_S defined by $\nu_S(i) = s_i$ makes S a numbered set. Without loss of generality, ν_S can be supposed to be injective: as $d(s_i, s_j) > 0$ can be semi-decided, ν_S can be effectively transformed into an injective numbering. Then a sequence of ideal points can be uniquely identified with the sequence of their names.

The numbered sets S and $\mathbb{Q}_{>0}$ induce the numbered set of ideal balls $\mathcal{B} = \{B(s_i, q_i) \ : \ s_i \in S,\ q_j \in \mathbb{Q}_{>0}\}$, the numbering being $\nu_{\mathcal{B}}(\langle i, j\rangle) = B(s_i, q_j)$. We write $B\langle i, j\rangle$ for $\nu_{\mathcal{B}}(\langle i, j\rangle)$. Then $\mathbf{X} = (X, \tau, \beta, \nu_{\mathcal{B}})$ is a computable topological space. A sequence (x_n) of points is said to be a fast Cauchy sequence if $d(x_n, x_{n+1}) < 2^{-n}$ for all n. The canonical representation is the Cauchy representation (S, δ_C) defined by $\delta_C(p) = x$ for all fast sequence p of ideal points converging to x.

Let $\mathcal{M}(X)$ be the set of Borel probability measures over X endowed with the weak topology. Let $D \subseteq \mathcal{M}(X)$ be the set of those probability measures that are concentrated in finitely many points of S and assign rational values to them, which is a dense subset [1]. The prokhorov metric ρ on $\mathcal{M}(X)$ is defined by $\rho(\mu, \nu) = \inf\{\epsilon \in \mathbb{R}^+ : \mu(A) \leq \nu(A^\epsilon) + \epsilon$ for every Borel set $A\}$ where $A^\epsilon = \{x : d(x, A) < \epsilon\}$. Then $((X), D, \rho)$ is a computable metric space. We say that a measure μ is computable if it is a computable point of $(\mathcal{M}(X), D, \rho)$. A computable probability space is a pair $(\mathcal{X}, \mu)$ where $\mathcal{X}$ is a computable metric space and μ a computable Borel probability measure on X.

A set A is said to be almost decidable if there are two r.e. open sets U and V such that:

$$U \subset A,\ V \subseteq A^c,\ U \cup V \text{ is dense and has measure one.}$$

We say that the elements of a sequence (A_i) are uniformly almost decidable if there are two sequences (U_i) and (V_i) of uniformly r.e. sets satisfying the conditions above.

Theorem 2.5 ([12]). *There is a sequence (r_n) of uniformly computable reals such that $(B(s_i, r_n))_{\langle i,n \rangle}$ is a basis of uniformly almost decidable balls.*

Definition 2.6 ([12]). *A binary representation of a computable probability space (X, μ) is a pair (δ, μ_δ) where μ_δ is a computable probability measure on 2^ω and $\delta : (2^\omega, \mu_\delta) \to (X, \mu)$ is a surjective morphism such that , calling $\delta^{-1}(x)$ the set of expansions of $x \in X$:*

(i) *there is a dense full-measure Π^0_2-set D of points having a unique expansion,*

(ii) *$\delta^{-1} : D \to \delta^{-1}(D)$ is computable.*

Theorem 2.7 ([12]). *Let δ be a binary representation on a computable probability space (X, μ). Each point having a μ_δ-random expansion is μ-random and each μ-random point has a unique expansion, which is μ_δ-random.*

One of the most important notions in computability theory is Turing reducibility. Intuitively a binary sequence A is Turing reducible to another binary sequence B if there is an algorithm to compute A by using B as an oracle. For the detail see [32]. Miller [22] defined Turing degrees and continuous degrees on computable metric spaces. The following are from [3].

Let X be a computable metric space with Cauchy representation δ_X. It is natural to define the Turing degree $\deg_T(x)$ of an element $x \in X$ by

$$\deg_T(x) = \min\{\deg_T(p) : \delta_X(p) = x\}.$$

For $X = \mathbb{R}$ and any real number $x \in \mathbb{R}$, the Turing degree $\deg_T(x)$ is well defined and coincides with the Turing degree of the fractional part of the binary expansion of x.

Theorem 2.8 ([22]). *Consider $X = C[0, 1]$. There exists some $f \in X$ such that $deg_T(f)$ is not well defined; i.e., there is no Cauchy name for f with minimal Turing degree.*

Definition 2.9. *Let X and Y be computable metric spaces. An element $x \in X$ is representation reducible to an element $y \in Y$ is there is a (δ_X, δ_Y)-computable function $f :\subseteq Y \to X$ with $f(y) = x$.*

This relation is clearly reflexive and transitive. Let us call the equivalence classes of elements of computable metric spaces under this reducibility relation continuous degrees.

Proposition 2.10 ([22])**.** *Every continuous degree contains a real-analytic function.*

Theorem 2.11. *There is a natural, nontrivial embedding from the Turing degrees into the continuous degrees and a natural, nontrivial embedding from the continuous degrees into the enumeration degrees.*

2.4 Computability of Measures

Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space and δ, θ be representations defined in Definition 2.2. Let $\mathcal{M}(X)$ denote the Borel probability measures over X and τ_w the weak topology on $\mathcal{M}(X)$. Let D be the family of open sets of the forms $\{\mu \in \mathcal{M}(X) : \mu(B) > q\}$ where B is the finite union of base elements. Note that D is a countable subbase of τ_w.

Theorem 2.12 ([23])**.** *Let $\beta^{\mathcal{M}}$ be the base generated from D and $\nu^{\mathcal{M}}$ be its enumeration. Then $\mathcal{M}(\mathbb{X}) = (\mathcal{M}(X), \tau_w, \beta^{\mathcal{M}}, \nu^{\mathcal{M}})$ is a computable topological space.*

Definition 2.13 ([23])**.** *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space. A measure $\mu \in \mathcal{M}(X)$ is computable iff it is $\delta^{\mathcal{M}}$-computable.*

Theorem 2.14 ([23])**.** *A measure μ is computable iff its restriction to open sets is $[\theta \to \rho_{\mathbb{I}_<}]$-computable.*

Theorem 2.15 ([23])**.** *The integral operator $\int : C(X, \mathbb{I}_<) \times \mathcal{M}(X) \to \mathbb{I} <$ is $([\delta \to \rho_{\mathbb{I}_<}], \delta^{\mathcal{M}}, \rho_{\mathbb{I}_<})$-computable.*

Theorem 2.16 ([23])**.** *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space. A measure μ is computable iff $\int d\mu : C(X, \mathbb{I}_<) \to \mathbb{I}_<$ is $([\delta \to \rho_{\mathbb{I}_<}], \rho_{\mathbb{I}_<})$-computable.*

3 On Computable Topological Spaces

In this section we define some notions and prove some results on computable topological spaces in general setting. We will use the results on a special computable topological space in the next section.

3.1 Continuous Degrees

Similar to the work by Miller [22], we define continuous degrees on computable topological spaces. Note that representation reducibility compares two points in (maybe) different spaces.

Definition 3.1. *For $i = 1, 2$, let $\mathbf{X}_i = (X_i, \tau_i, \beta_i, \nu_i)$ be computable topological spaces. For $x_1 \in X_1$ and $x_2 \in X_2$, we say that (x_1, X_1) is representation reducible to (x_2, X_2) (denoted by $(x_1, X_1) \leq_r (x_2, X_2)$) iff there exists a (δ_1, δ_2)-computable function f such that $f(x_2) = x_1$ (single-valued at x_2).*

When the considered space is clear, we write only $x_1 \leq_r x_2$. We sometimes say that x_1 is x_2-computable to mean $x_1 \leq_r x_2$. We will also have another reducibility by replacing the condition of f in the definition: i.e., a total computable function and multi-valued.

The relation $\leq_r$ is reflexive and transitive and so $\equiv_r$ is a equivalent relation. Hence we can define degrees.

Definition 3.2. *We say $x_1 \in X_1$ is representation equivalent to $x_2 \in X_2$ (denoted by $x_1 \equiv_r x_2$ iff $x_1 \leq_r x_2$ and $x_2 \leq_r x_1$. Continuous degrees are the equivalent classes of representation equivalent points. The continuous degree of a point x in a computable topological space $\mathbf{X}$ is denoted by $deg_{\mathbf{X}}(x)$.*

For a Cantor space $\mathbf{C}$ and points $x, y \in \mathbf{C}$,

$$x \leq_r y \iff x \leq_T y.$$

Hence there is a natural embedding from the Turing degrees into the continuous degrees. For real line $\mathbf{R}$ and a point $r \in \mathbf{R}$, the continuous degree $\deg_{\mathbf{R}}(x)$ coincides with the Turing degree of the fractional part of the binary expansion of r. In these spaces we often use the word "Turing degree" rather than "continuous degree" for simplicity.

The continuous degree of a point may depend on the basis and its notation. In other words a point $x \in X$ may have different continuous degrees when considered computable topological spaces are different. Recall that $\mathbf{0} = \deg_{\mathbf{C}}(\phi)$ is the degree consisting of the computable sets and that $\mathbf{0}'$ is the jump of it. The halting probability Ω has the Turing degree $\mathbf{0}'$ on a Cantor space, that is, $\deg_{\mathbf{C}}(\Omega) = \mathbf{0}'$.

Example 3.3.

The degree $deg_{\mathbf{I}_<}(\Omega) = \mathbf{0}$ because we can enumerate all rationals below Ω.

The degree $deg_{\mathbf{I}_<}(1 - \Omega) = \mathbf{0}'$ because we can enumerate all rationals above Ω from the representation of $1 - \Omega$.

However if they are equivalent, then the point has the same continuous degree.

Proposition 3.4. *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be equivalent computable topological spaces. Then $deg_{\mathbf{X}_1}(x) = deg_{\mathbf{X}_2}(x)$.*

Proof. This is because $\delta_1 \equiv \delta_2$. □

By Theorem 2.8 there exists a point that does not have any point on a Cantor space such that they are representation equivalent. However each random point on a computable probability space does have.

Proposition 3.5. *Let (X, μ) be a computable probability space. For each μ-random point, there exists a point on a Cantor space such that they are representation equivalent.*

Proof. Each μ-random point has a unique expansion by Theorem 2.7. Hence it is representation equivalent to the representation. □

3.2 Products of Infinite Spaces

The products of computable topological spaces are important spaces. Weihrauch and Grubba [39] defined the product of only finite spaces. Here we define products of infinite spaces. The definitions and proofs of the results are standard in general topology.

The following definitions are from [33]. The *Cartesian product* of the sets X_i is the set

$$\prod_{i\in\mathbb{N}} X_i = \left\{ x \; : \; \mathbb{N} \to \bigcup_{i\in\mathbb{N}} X_i \mid x(i) \in X_i \text{ for each } i \in \mathbb{N} \right\}.$$

The value of $x \in \prod X_i$ at i is usually denoted x_i, rather than $x(i)$, and x_i is referred to as the i-th coordinate of x. The space X_i is the i-th factor space.

The *Tychonoff topology* (or *product topology*) on $\prod X_i$ is obtained by taking as a base for the open sets, sets of the form $\prod U_i$, where

(i) U_i is open in X_i for each $i \in \mathbb{N}$,

(ii) For all but finitely many coordinates, $U_i = X_i$.

Note that "U_i is open" can be replaced by "U_i is a base for the topology of X_i".

In order to make the base effective, we define an index of a finite set. Let $D_0 = \phi$. If $n > 0$ has the form $2^{x_1} + 2^{x_2} + \cdots + 2^{x_r}$, where $x_1 < \ldots x_r$, then let $D_n = \{x_1, \ldots x_r\}$. We say that n is a *strong index* for D_n.

Definition 3.6. *For $i \in \mathbb{N}$, let $\mathbf{X}_i = (X_i, \tau_i, \beta_i, \nu_i)$ be effective topological spaces with representations δ_i, θ_i, ... Let $\overline{\nu}$ be a notation of the base for the product topology such that*

$$\overline{\nu}\langle s, \langle u_1, \ldots, u_{l(s)}\rangle\rangle = \prod U_i$$

where

(i) $D_s = \{x_1, \ldots, x_{l(s)}\}$ *and* $x_1 < \ldots < x_{l(s)}$,

(ii) $U_i = \nu_i(u_j)$ *if* $i = x_j$,

(iii) $U_i = X_i$ *if* $i \notin D_s$.

Define the infinite product

$$\prod \mathbf{X}_i = (\prod X_i, \overline{\tau}, \overline{\beta}, \overline{\nu})$$

of effective topological spaces such that $\overline{\nu}$ is the above, $\overline{\beta} = range(\overline{\nu})$ and $\overline{\tau}$ is the product topology generated by $\overline{\beta}$.

Compare with the definition of the Tychonoff topology. The finite set D_s tells us the coordinates which $U_i = X_i$ does not hold.

Under some natural conditions $\prod \mathbf{X}_i$ comes to be a computable topological space. We say that an r.e. set S is an *intersection set* of a computable topological space $\mathbf{X}$ if S satisfies the formula (1) in Definition 2.1.

Proposition 3.7. *For $i \in \mathbb{N}$, let $\mathbf{X}_i$ be computable topological spaces with the intersection set S_i such that $dom(\nu_i)$ is recursive uniformly in n and the sequence $\{S_i\}$ is uniformly r.e. Then the infinite product $\prod \mathbf{X}_i$ is a computable topological space.*

Proof. First dom($\overline{\nu}$) is recursive because dom(ν_i) is uniformly recursive.

Next we give the intersection set S of $\prod \mathbf{X}_i$. The intersection of two bases can be written for each coordinate. Let

$$\begin{aligned}\overline{S} = \{(\langle s, \langle u_1, \ldots, u_{l(s)}\rangle\rangle, \langle t, \langle v_1, \ldots, v_{l(t)}\rangle\rangle, \langle r, \langle w_1, \ldots, w_{l(r)}\rangle\rangle) \ :\\ D_s = \{x_1, \ldots, x_{l(s)}\}, D_t = \{y_1, \ldots, y_{l(t)}\}, D_r = D_s \cup D_t = \{z_1, \ldots, z_{l(r)}\},\\ (z_i = x_k \notin D_t \Rightarrow w_i = u_k) \wedge (z_i = y_j \notin D_s \Rightarrow w_i = v_j)\\ \wedge (z_i = x_k = y_j \Rightarrow (u_k, v_j, w_i) \in S_{z_i})\}.\end{aligned}$$

Then $\overline{S}$ satisfies (1) and it is r.e. because $\{S_i\}$ is uniformly r.e. □

3.3 Martingales

Miyabe [23] gave a characterization of randomness via martingales. Here we give a characterization by supermartingales.

We use some terminologies from measure theory referring to [28]. For a measure space $(X, \mathcal{A}, \mu)$, *a filtration* is a sequence of sub-σ-algebra $\{\mathcal{A}_n\}$ such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for each n. For a filtered measure space $(X, \mathcal{A}, \mathcal{A}_j, \mu)$, a sequence $\{f_n\}$ of $\mathcal{A}$-measurable functions is called a *martingale* if $\int_X f_n d\mu < \infty$ and $\int_A f_n d\mu = \int_A f_{n+1} d\mu$ for all $A \in \mathcal{A}_n$. We say that $\{f_n\}$ is a *submartingale* if $\int f_n d\mu < \infty$ and $\int_A f_n d\mu \leq \int_A f_{n+1} d\mu$ for all $A \in \mathcal{A}_n$ and a *supermartingale* if $\int f_n d\mu < \infty$ and $\int_A f_n d\mu \geq \int_A f_{n+1} d\mu$ for all $A \in \mathcal{A}_n$.

For two measures μ and ν, we call ν absolutely continuous w.r.t. μ and write $\nu \ll \mu$ if

$$N \in \mathcal{A}, \ \mu(N) = 0 \ \Rightarrow \nu(N) = 0.$$

Theorem 3.8 (Radon-Nikodým)**.** *If μ is σ-finite, then the following assertions are equivalent.*

(i) $\nu(A) = \int_A f(x)\mu(dx)$ *for some a.e. unique positive measurable function* f*.*

(ii) $\nu \ll \mu$*.*

Theorem 3.9 (Doob's maximal inequality)**.** *Let $\{f_n\}$ be a non-negative submartingale. Then we have for all $s > 0$*

$$\mu\left(\left\{\max_{i \leq j \leq N} f_n \geq s\right\}\right) \leq \frac{1}{s} \int f_N d\mu.$$

We start from the following characterization. Let $\mathbf{X}$ be a computable topological space and μ be a computable measure on $\mathbf{X}$.

Definition 3.10 (Hertling and Weihrauch [10], Miyabe [23])**.** *A measure test over $\mathbf{X}$ is a sequence $\{U_n\}$ of uniformly θ-computable open sets with $\mu(U_n) \leq 2^{-n}$ for all n. A point x is measure μ-random if $x \notin \bigcap_n U_n$ for each measure test $\{U_n\}$.*

Theorem 3.11 (Miyabe [23])**.** *A point x is measure μ-random iff $\sup_n f_n(x) < \infty$ for each non-negative $([\nu_{\mathbb{N}}, \delta], \overline{\rho}_<)$-computable martingale $\{f_n\}$.*

Similar to the case on a Cantor space, we can replace a martingale with a supermartingale. To do that, we prepare a lemma.

Lemma 3.12. *For a supermartingale* $\{f_n\}$*, let* $U_m = \{y : \sup_{n\le m} f_n(y) > k\}$*. Then*

$$k\mu(U_m) \le \int_X f_0 d\mu.$$

Proof. For each n, let ν_n be the measure such that

$$\nu_n(A) = \int_A (f_n - f_{n+1}) d\mu$$

for all $A \in \mathcal{A}_n$. Then ν_n is non-negative and absolutely continuous with respect to μ. By Radon-Nikodým theorem (Theorem 3.8), there exists non-negative g_n such that

$$\nu(A) = \int_A g_n d\mu.$$

Let $h_n = f_n + \sum_{i<n} g_n$ for each n. Then $h_n \ge f_n$ and $\{h_n\}$ is a martingale because, for each $A \in \mathcal{A}_n$,

$$\int_A h_{n+1} d\mu = \int_A (h_n + f_{n+1} - f_n + g_n) d\mu = \int_A h_n d\mu.$$

Let $V_m = \{y : \sup_{n\le m} h_n(y) > k\}$. Note that $\mu(V_m) \ge \mu(U_m)$ because $g_n \ge 0$ for each n. By Doob's maximal inequality (Theorem 3.9), we get

$$k\mu(U_m) \le k\mu(V_m) \le \int_X h_m d\mu = \int_X h_0 d\mu = \int_X f_0 d\mu.$$

□

Theorem 3.13. *A point* $x \in X$ *is* μ*-random iff* $\sup_n f_n(x) < \infty$ *for each non-negative* $([\nu_{\mathbb{N}}, \delta], \overline{\rho}_<)$*-computable supermartingale* $\{f_n\}$*.*

Proof. It suffices to show "only if" direction. Suppose that there exists a martingale $\{f_n\}$ such that $\sup_n f_n(x) = \infty$. Let $U_{k,m} = \{y : \sup_{n\le m} f_n(y) > 2^k\}$. By Lemma 3.12,

$$\mu(U_{k,m}) \le 2^{-k} \int_X f_m d\mu \le 2^{-k}.$$

Let $U_k = \{y : \sup_n f_n(x) > 2^k\}$ for each k. Note that $U_{k,m} \subseteq U_{k,m+1}$ and $U_{k,m}$ is θ-computable uniformly in k and m. Since $U_k = \bigcup_m U_{k,m}$,

$$\mu(U_k) = \mu(\bigcup_m U_{k,m}) = \sup_m \mu(U_{k,m}) \le 2^{-n}.$$

Then $\{U_k\}$ is a measure test and $x \in \bigcap_n U_n$. Hence x is not μ-random. □

3.4 Martingales on the Product of Infinite Spaces

In this subsection we consider an infinite product computable topological space $\overline{X} = \prod \mathbf{X_i}$ defined in Definition and Proposition 3.7. Let $\mu = \prod \mu_i$ be the infinite product measure (see Bogachev [2] etc) on $\overline{X}$ where μ_i is a measure on $\mathbf{X}_i$ for each i. In the following we study restricted martingales on this measure space. We say that a martingale $\{f_n\}$ is restricted if $f_n(x)$ depends only on $x_1, \cdots, x_n$. Hence for $x, y \in \prod X_i$, if $x_j = y_j$ for each $j \le n$ then $f_n(x) = f_n(y)$. Intuitively the martingale $\{f_n\}$ is a betting strategy and f_n forecasts the n-th value x_n given the values $x_1, \cdots, x_{n-1}$. So the martingale only forecasts the next value. The goal is to prove that by such a martingale we can get infinite money against a non-random sequence of reals under a condition.

We would like to assume that μ is computable. It is clear that if μ is computable then μ_i is uniformly computable. The converse needs the following condition.

Lemma 3.14. *Let μ_i be uniformly computable measure on $\mathbf{X}_i$. If $\mathbf{X}_i$ with μ_i has the property of almost decidability uniformly, then $\prod \mathbf{X}_i$ with μ has also the property of almost decidability.*

Proof. Let $\{U_{i,j}\}_j$ be an equivalent base such that the pair of sequences $\{U_{i,j}\}_j$ and $\{V_{i,j}\}_j$ is the witness of almost decidability. Note that the space with this base is equivalent to the original space. Then there exists an equivalent base of $\prod \mathbf{X}_i$ such that each base B has the form $\prod W_i$ where $W_i = U_{i,j}$ for some j for finitely many i and $W_i = X_i$ otherwise. For this B, let $B' = \prod W'_i$ where $W'_i = V_{i,j}$ if $W_i = U_{i,j}$ and $W'_i = X_i$ otherwise. Then B and B' satisfy the conditions. Since the uniformity is clear, $\prod \mathbf{X}_i$ has also the property of almost decidability. □

Lemma 3.15. *If $\mathbf{X}_i$ with μ_i has the property of almost decidability uniformly, then μ is computable.*

Proof. By Lemma 3.14, $\overline{\mathbf{X}}$ has the property of almost decidability. Hence $\mu(\overline{\nu}(u))$ is computable uniformly in u. It follows that μ with the restriction to open sets is $[\theta \to \rho_{\mathbb{I}_<}$-computable. Then μ is computable. □

In what follows, we assume that μ_i are uniformly computable so that μ is computable.

Theorem 3.16. *Let $\mathbf{X}_i$ have the property of almost decidability uniformly in i. Then a point $x \in \prod X_i$ is μ-random iff $\sup_n f_n(x) < \infty$ for each restricted non-negative $([\nu_{\mathbb{N}}, \delta], \overline{\rho}_<)$-computable martingale $\{f_n\}$.*

Proof. The direction "only if" is followed by Theorem 3.11 but the direction "if" is not followed by Theorem 3.11 because we have restricted martingales.

Let (U_j) and (V_j) be uniformly θ-computable open sets satisfying the conditions in Definition of almost decidability generated by Lemma 3.14. Inspired from the work of Hoyrup and Rojas [12], we defined the cells $\Gamma(w)$ for $w \in 2^*$ by induction:

$$\Gamma(\epsilon) = X,\ \Gamma(w0) = \Gamma(w) \cap U_i,\ \text{and}\ \Gamma(w1) = \Gamma(w) \cap V_i$$

where ϵ is the empty word and $i = |w|$. These are almost decidable sets uniformly in w. Since each U_i and V_i has the form $\prod W_j$ where $W_j = X_j$ for all but finitely many j, each $\Gamma(w)$ has also the same form. Let

$$\Gamma(w) = \prod W_i^w \text{ and } \Gamma_n(w) = \prod_{i \le n} W_i^w \times \prod_{i > n} X_i.$$

Note that $\mu(U_j)$ is computable uniformly in j by Lemma 3.15 and almost decidability of U_j. Hence $\mu(\Gamma(w))$ and $\mu(\Gamma_n(w))$ are computable uniformly in w and n.

Now suppose that y is not measure μ-random. Then there exists a measure test $\{U_m\}$ such that $y \in \bigcap_m U_m$. Since $\Gamma(w)$ is a base and $\Gamma(w0) \cup \Gamma(w1)$ has measure 1 for each w, there exists prefix-free r.e. sets $\{S_m\}$ of 2^* such that $\mu(U_m) = \mu(\bigcup_{w \in S_m} \Gamma(w))$. Let

$$f_n(x) = \sum_m \sum_{w \in S_m,\ x \in \Gamma_n(w)} \mu\Gamma(w)/\mu\Gamma_n(w).$$

We shall prove that $\{f_n\}$ is a martingale. First $\int f_0 d\mu = \sum_m \mu(U_m) < \infty$. So it suffices to show for each $w \in S_m$. For each $A \in \mathcal{A}_n$,

$$\begin{aligned}
\int_A f_{n+1} d\mu =& \mu(A \cap \Gamma_{n+1}(w))\mu\Gamma(w)/\mu\Gamma_{n+1}(w) \\
=& \mu(A \cap \Gamma_n(w)) \cdot \mu_{n+1}(W_{n+1}^w)\mu\Gamma(w)/(\mu(\Gamma_n(w)) \cdot \mu_{n+1}(W_{n+1}^w)) \\
=& \mu(A \cap \Gamma_n(w))\mu\Gamma(w)/\mu(\Gamma_n(w)) \qquad = \int_A f_n d\mu.
\end{aligned}$$

□

Of course we can replace a martingale with a supermartingale.

Corollary 3.17. *Let* $\mathbf{X}_i$ *have the property of almost decidability uniformly in i. A point* $x \in \prod X_i$ *is* μ*-random iff* $\sup_n f_n(x) < \infty$ *for each restricted non-negative* $([\nu_{\mathbb{N}}, \delta], \overline{\rho}_<)$*-computable supermartingale* $\{f_n\}$.

Proof. The direction "if" is followed by Theorem 3.16 and the direction "only if" is followed by Theorem 3.13. □

Especially we will come across many martingales with the following forms. Hence the theorem will be useful, although the converse may not hold.

Theorem 3.18. *Let* $\mathbf{X}_i$ *have the property of almost decidability uniformly in i. If a point* $x \in \prod X_i$ *is* μ*-random, then* $\sup_n f_n(x) < \infty$ *for each* $\{f_n\}$ *such that*

$$f_{n+1}(x) = g_{n+1}(f_n(x), x_{n+1})$$

where g is a non-negative $([\nu_{\mathbb{N}}, \overline{\rho}_<, \delta_{n+1}], \overline{\rho}_<)$*-computable function and*

$$y \ge \int_{X_{n+1}} g_{n+1}(y, z)\mu_{n+1}(dz)$$

for each $y \in \overline{\mathbb{R}}$.

Proof. It suffices to show that $\{f_n\}$ is a $([\nu_{\mathbb{N}}, \delta], \overline{\rho}_<)$-computable supermartingale. For each $A \in \mathcal{A}_n$,

$$\begin{aligned}\int_A f_{n+1} d\mu &= \int_A g_{n+1}(f_n(x), x_{n+1}) d\mu \\ &= \int_A d\mu \int_{X_{n+1}} g_{n+1}(f_n(x), x_{n+1}) \mu(dx_{n+1}) \qquad \text{(by Fubini)} \\ &\leq \int_A f_n(x) d\mu.\end{aligned}$$

□

We can replace a $([\nu_{\mathbb{N}}, \overline{\rho}_<, \delta_{n+1}], \overline{\rho}_<)$-computable function with a $([\nu_{\mathbb{N}}, \overline{\rho}, \delta_{n+1}], \overline{\rho})$-computable function.

Corollary 3.19. *Let $\mathbf{X}_i$ have the property of almost decidability uniformly in i. If a point $x \in \prod X_i$ is μ-random, $\sup_n f_n(x) < \infty$ for each $\{f_n\}$ such that $f_{n+1}(x) = g_{n+1}(f_n(x), x_{n+1})$ where g is non-negative and $([\nu_{\mathbb{N}}, \overline{\rho}, \delta_{n+1}], \overline{\rho})$-computable and $y \geq \int_{X_{n+1}} g_{n+1}(y, x) \mu_{n+1}(dx)$ for each $y \in \overline{\mathbb{R}}$.*

3.5 Borel-Cantelli Lemmas

Borel-Cantelli Lemmas are very useful lemmas in probability theory. This is also true for our purpose. We prove two effective version of Borel Cantelli Lemmas. The first one was proved by [8] on computable probability space. This also holds on a computable topological space. The proof is a straightforward modification so we omit the proof.

Definition 3.20 (see [8])**.** *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space. A Borel-Cantelli test (BC-test) is a sequence $\{E_n\}$ of uniformly θ-computable open sets such that $\sum_n \mu(E_n) < \infty$. We say that x fails the BC-test if $x \in E_n$ infinitely often and that x passes it otherwise.*

Lemma 3.21. *A point x is μ-random iff x passes each BC-test.*

Lemma 3.22 (Effective BC1)**.** *Let $\{E_n\}$ be a BC-test. If x is μ-random then $x \notin E_n$ for all but finite n.*

The second one needs a little more condition.

Lemma 3.23 (Effective BC2)**.** *Let F_n be a sequence of uniformly ψ^--computable closed sets such that*

$$\mu\left(\bigcap_{n=k}^{l} F_n^c\right) = \prod_{n=k}^{l} (1 - \mu(F_n))$$

and $\mu(F_n)$ has a non-decreasing computable lower-bound whose sums tend to infinity, that is,

$$\mu(F_n) \geq g(n) \text{ and } \sum_{n=1}^{\infty} g(n) = \infty$$

for some computable function $g : \mathbb{N} \to \mathbb{N}$. If x is μ-random then $x \in F_n$ for infinitely many n.

Proof. Fix temporary $m \in \mathbb{N}$. Let

$$H_k = \bigcap_{n=m}^{b_k} F_n^c$$

where $\{b_k\}$ is a computable sequence of $\mathbb{N}$ such that

$$\exp(-\sum_{n=m}^{b_k} g(n)) \leq 2^{-k}$$

for all k. Recall that $\psi^-(p) = X \backslash \theta(p)$ for all $p \in \Sigma^\omega$. Then $\{H_k\}$ is a sequence of uniformly θ-computable open sets. Furthermore, by the inequality $1 - x \leq \exp(-x)$ for $x \geq 0$,

$$\mu(H_k) = \prod_{n=m}^{b_k}(1 - \mu(F_n)) \leq \exp(-\sum_{n=m}^{b_k} \mu(F_n)) \leq \exp(-\sum_{n=m}^{b_k} g(k)) \leq 2^{-k}.$$

Since x is μ-random, there exists K such that $x \notin H_K$. It follows that $x \in F_n$ for some $n \geq m$. Since m is arbitrary, $x \in F_n$ for infinitely many n. □

4 Random Sequences of Reals

Randomness of a binary sequence has been well studied. For example a Martin-Löf random sequence is a random sequence of $\{0, 1\}$. Clearly we are also interested in a random sequence of reals. Since the theory of randomness on a computable topological space has been developed, we can naturally define a random sequence of reals as a random point on the computable topological space $\mathbf{R}^{\mathbb{N}}$.

Recall that $\mathbf{R} = (\mathbb{R}, \tau_{\mathbb{R}}, \beta_{\mathbb{R}}, \nu_{\mathbb{R}})$ is a computable topological space whose base elements are open intervals with rational endpoints. Let $\mathbf{R}^{\mathbb{N}} = (\mathbb{R}^{\mathbb{N}}, \tau, \beta, \nu)$ be the computable topological space generated from $\mathbf{R}$ by Definition 3.4 and Proposition 3.7. Let μ_i be a uniformly computable measure on $\mathbb{R}$ for each i. We consider a measure $\mu = \prod \mu_i$ on $\mathbb{R}^{\mathbb{N}}$. Note that μ is computable by Lemma 3.15. Furthermore if μ_i are continuous, then $\mu(\nu(u))$ is computable uniformly in u. Recall that $\nu(u)$ has the form $\nu(u) = \prod W_i$ where $W_i = \nu_{\mathbb{R}}(v_i)$ for finitely many i and $W_i = \mathbb{R}$ otherwise. Then $\mu(\nu(u)) = \prod \nu_{\mathbb{R}}(v_i)$. In this section we study randomness on this computable topological space with this measure, in other words, a random sequence of reals on this measure.

4.1 Another Characterization by Complexity

Randomness on a computable topological space has three equivalent characterization, that is, by tests (Definition 3.10), by martingales (Theorem 3.11) and by complexity. Theoretically [7], [12] and [23] also gave different characterizations of this randomness by complexity but they are hard to use. Tests and martingales are easy to handle but in contrast complexity is difficult to use. However randomness on $\mathbf{R}^{\mathbb{N}}$ has a natural characterization.

Now we recall the following useful theorem. See Downey [6] for the detail.

Theorem 4.1 (Kraft-Chaitin Thorem; Levin [19], Schnorr [30], Chaitin [4]). *Let $\langle d_i, \tau_i \rangle$ be a computable sequence of pairs (which we call requests), with $d_i \in \mathbb{N}$ and $\tau_i \in \Sigma^*$ such that $\sum_i 2^{-d_i} \leq 1$. Then there is a prefix-free machine M and string σ_i of length d_i such that $M(\sigma_i) = \tau_i$ for all i. Furthermore, and index for M can be obtained effectively from an index for our sequence of requests.*

We call an effectively enumerated set of requests $\langle d_i, \tau_i \rangle$ such that $\sum_i 2^{-d_i} \leq 1$ a *KC set*. The *weight* of this set is $\sum_i 2^{-d_i}$.

Theorem 4.2. *Let $\mu = \prod \mu_i$ be a computable measure on $\mathbf{R}^{\mathbb{N}}$ such that μ_i are continuous for all i. A point $x \in \mathbf{R}^{\mathbb{N}}$ is μ-random iff*

$$x \in \nu(u) \Rightarrow K(u) \geq -\log \mu(\nu(u)) - O(1).$$

Proof. (only if direction)

Let

$$W_k = \{u \ : \ K(u) < -\log \mu(\nu(u)) - k\} = \{u_1^k, u_2^k, \cdots, \}$$

and

$$V_k = \bigcup_{u \in W_k} \nu(u).$$

Recall that $\mu(\nu(u))$ is computable uniformly in u. Hence $\{V_k\}$ is a sequence of θ-computable open sets uniformly in k. Furthermore

$$\mu(V_k) \leq \sum \mu(\nu(u)) \leq 2^{-k} \sum 2^{-K(u)} \leq 2^{-k}.$$

Hence $\{V_k\}$ is a measure test. If x is μ-random, there exists K such that $x \notin V_K$. It follows that $x \in \nu(u) \Rightarrow K(u) \geq -\log \mu(\nu(u)) - K$.

(if direction)

Suppose that $\{U_k\}$ is a measure test such that $x \in \bigcap_k U_k$. Then there exists a computable sequence $\{u_i^k\}$ such that $U_k = \bigcup_i \nu(u_i^k)$. Recall that $\nu(u)$ has the form $\prod W_n$ where W_n is an open interval with rational endpoints for finitely many n and $W_n = \mathbb{R}$ otherwise. Hence $\nu(u) \backslash \nu(v)$ is the disjoint union of $\prod W_n$ where, for finitely many n, W_n is a interval with rational endpoints which is open, closed, left-closed and right-open or left-open and right-closed. (For example, $(1,3) \times (1,3) \times \mathbb{R}^{\mathbb{N}} \backslash (0,2) \times (0,2) \times \mathbb{R}^{\mathbb{N}} = (1,2) \times [2,3) \times \mathbb{R}^{\mathbb{N}} \cup [2,3) \times (1,3) \times \mathbb{R}^{\mathbb{N}}$.) It follows that, for each j, we can compute $\{w_i\}$ from u and v such that $\nu(u) \backslash \nu(v) \subseteq \bigcup_i \nu(w_i)$ and $\mu(\bigcup_i \nu(w_i)) \leq \mu(\nu(u) \backslash \nu(v)) + 2^{-j}$. (In the above example, $\bigcup_i \nu(w_i) = (1,2) \times (1.99, 3) \times \mathbb{R}^{\mathbb{N}} \cup (1.99, 3) \times (1,3) \times \mathbb{R}^{\mathbb{N}}$.) Hence $\nu(i) \backslash \bigcup_{j<i} \nu(u_j^k)$ and $\bigcup_{j \leq i} \nu(u_j^k)$ also have the same form. It follows that there exists a computable sequence $\{w_i^k\}$ such that $x \in \bigcup_i \nu(w_i^k)$ and $\mu(\bigcup_i \nu(w_i^k)) \leq \mu(U_k) + 2^{-k} \leq 2^{-k+1}$.

We define a KC set $L = \{\langle [-\log \mu\nu(w_i^{2k+1})] - k + 2, w_i^{2k+1} \rangle\}$. Note that

$$[-\log \mu\nu(w_i^{2k+1})] \leq -\log \mu\nu(w_i^{2k+1}) + 1.$$

Then we obtain

$$\sum_{k,i} 2^{-[-\log \mu\nu(w_i^{2k+1})]+k-2} \le \sum_{k,i} 2^{\log \mu\nu(w_i^{2k+1})+k-1} \le \sum_{k,i} \mu(\nu(w_i^{2k+1}))2^{k-1}$$
$$\le \sum_{k,i} 2^{-(2k+1)+1}2^{k-1} \le \sum_{k} 2^{-k-1} \le 1.$$

Then by Kraft-Chaitin theorem $K(w_i^{2k+1}) \le -\log \mu\nu(w_i^{2k+1}) - k + 1$. Furthermore for each k there exists i such that $x \in \nu(w_i^{2k+1})$. □

4.2 Relative Randomness

We have studied a sequence of reals. Each member of the sequence is a real, so it can be random. Furthermore each member can be random relative to any finite other members. In this section we study the relation between randomness of a sequence and relative randomness of each member.

Randomness of a binary sequence has been well studied. So we would like to interpret a sequence of reals as a binary sequence. Note that the class of sequences of reals and the class of binary sequences have the same cardinality, namely, the cardinality of continuum.

Recall that a real in the unit interval can be identified with a binary sequence. By using a pairing function a sequence of reals in the unit interval also can be identified with a binary sequence. If we can interpret a real as a real in the unit interval, we can naturally get the goal. To do that we follow the method in Kautz [13] and generalize to real line. See also Downey and Hirschfeldt's book [6].

Let $\mathbf{R} = (\mathbb{R}, \tau, \beta, \nu)$ and μ be a computable measure on $\mathbf{R}$. We say that a measure μ is *atomic* if $\mu(\{r\}) > 0$ for some $r \in \mathbb{R}$. We call such an r an *atom* of μ. A nonatomic measure is called a *continuous* measure.

Definition 4.3. *Let p, q be rationals. For each interval (p, q), we define the interval $I_\mu(p, q) = (\mu(-\infty, p], \mu(-\infty, q))$. For $\alpha \in [0, 1]$, we define $rs_\mu(\alpha) \in \mathbb{R}$ as the unique r such that $\alpha \in I_\mu(p, q) \Rightarrow r \in (p, q)$.*

Notice that $\lambda(I_\mu(p, q)) = \mu(p, q)$ where λ is the uniform measure on $[0, 1]$. Furthermore $I_\mu(p, q)$ is an r.e. open set computable from p and q because $I_\mu(p, q) = (1 - \mu(p, \infty), \mu(-\infty, q))$.

Remark 4.4. *Note that $rs_\mu(\alpha)$ may not be defined. If α is not computable then $rs_\mu(\alpha)$ is defined.*

Note that rs is a partial function from $[0, 1]$ to $\mathbb{R}$. By this function we would like to identified $r \in \mathbb{R}$ with $\alpha \in [0, 1]$. Since $\mathrm{rs}_\mu(\alpha)$ may not be defined, it is impossible to do that for all $r \in \mathbb{R}$. However we finally prove that this function preserves randomness. To see this we prove some properties of this function. The following theorems are generalization from [13] and [6] but the proof needs a little modification.

Theorem 4.5. *If $r = rs_\mu(\alpha)$ is defined, then*

(i) $r \le_T \alpha$*, and*

(ii) *if r is not an atom of μ then $\alpha \leq_r r$.*

Proof. We would like to search p, q from α such that

$$\alpha \in I_\mu(p,q) \iff \mu(-\infty, p] = 1 - \mu(p, +\infty) < \alpha < \mu(-\infty, q).$$

The measure μ is computable and intervals $(p, +\infty)$ and $(-\infty, q)$ are open. Then the measure of them are r.e. reals. Hence we can enumerate such p and q. For each n, there exist q and q such that $q - p < 2^{-n}$ because, otherwise, $\mathrm{rs}_\mu(\alpha)$ would not be defined. Hence $\mathrm{rs}_\mu(\alpha) \leq_T \alpha$.

For the converse, let $\beta = \mu(-\infty, r)$. Then $\beta \leq_T r$ because r is not an atom of μ and μ is computable. Furthermore $\beta \in I_\mu(p,q) \Rightarrow r \in (p,q)$. For each n there exist $p_n, q_n \in \mathbb{Q}$ such that $\alpha \in I_\mu(p_n, q_n)$ and $|p_n - q_n| < 2^{-n}$. Since $r \in (p_q, q_n)$, we obtain

$$\mu(-\infty, p_n] \leq \mu(-\infty, r) \leq \mu(-\infty, q_n).$$

If $\mu(-\infty, p_n) = \mu(-\infty, r)$ or $\mu(-\infty, r) = \mu(-\infty, q_n)$ then r would not be defined. Hence $\beta \in I_\mu(p_n, q_n)$. Since r is not an atom, there is only one real such that $\gamma \in \bigcap_n I_\mu(p_n, q_n)$ Hence $\alpha = \beta = \gamma$. It follows that $\alpha \leq_T \mathrm{rs}_\mu(\alpha)$. □

Theorem 4.6. *All atoms of a computable measure are computable.*

Proof. Suppose that $2\epsilon > \mu(\{r\}) > \epsilon > 0$. By countable additivity of μ, there exists $p_0, q_0 \in \mathbb{Q}$ such that

$$\mu(p_0, q_0) \leq \mu([p_0, q_0]) < 2\epsilon.$$

Then r is only one atom in (p_0, q_0) whose measure is larger than ϵ. Given n, we can enumerate $p, q \in (p_0, q_0) \cap \mathbb{Q}$ such that $|q - p| < 2^{-n}$ and $\mu(p,q) \geq \epsilon$. If $r \notin (p,q)$, then

$$\mu(p,q) \leq \mu(p_0, q_0) - \mu(\{r\}) < 2\epsilon - \epsilon = \epsilon.$$

This is a contradiction. It follows that $r \in (p,q)$. Hence r is computable. □

Corollary 4.7. *If μ is continuous and $rs_\mu(\alpha)$ is defined then $rs_\mu(\alpha) \equiv_T \alpha$.*

Theorem 4.8. (i) *If $\alpha \in [0,1]$ is Martin-Löf random then $rs_\mu(\alpha)$ is μ-random.*

(ii) *If $rs_\mu(\alpha)$ is μ-random and is non an atom of μ, then α is Martin-Löf random.*

Proof. (i) Suppose that $r = \mathrm{rs}_\mu(\alpha)$ is not μ-random. Let $\{U_n\}$ be a sequence of uniformly θ-computable open sets such that $r \in U_n$ and $\mu(U_n) \leq 2^{-n}$ for all n. Now we fix n. Let $\{p_i\}, \{q_i\}, \{s_i\}$ be uniformly computable sequences of rationals such that (p_i, q_i) is pairwise disjoint and $U_n = \bigcup_i (p_i, q_i) \cup \bigcup_i \{s_i\}$. Suppose that r is rational. Since r is not μ-random, r is not an atom of μ. Then $\alpha \leq r$ by Theorem 4.5. It follows that α is computable and not Martin-Löf random. Hence we can assume that $r \in \bigcup (p_i, q_i)$. Let $V_n = \bigcup_i I_\mu(p_i, q_i)$. Then $\alpha = \mu(-\infty, r) \in V_n$ because $\mu(-\infty, p_i) \leq \mu(-\infty, r) \leq \mu(-\infty, q_i)$ for some i. Furthermore

$$\lambda(V_n) \leq \sum \mu(p_i, q_i) \leq \mu(U_n) = 2^{-n}.$$

It follows that α is not Martin-Löf random.

(ii)Suppose α is not Martin-Löf random and $r = \mathrm{rs}_\mu(\alpha)$ is not computable. Let $\{U_n\}$ be a Martin-Löf test covering α and $\{V_n\}$ be uniformly c.e. sets of generators for the $\{U_n\}$. For each n and k, proceed as follows. Let σ be the k-th string in the enumeration of V_{n+1}. Let $a, b \in \mathbb{Q}$ such that $[a, b] = [\sigma]$, and let $I_k = (a - 2^{-(n+k+3)}, b + 2^{-(n+k+3)})$. Let

$$W_n = \bigcup_k \{(p, q) \ : \ [\mu(-\infty, p), \mu(-\infty, q]] \subseteq I_k\}.$$

Here

$$\begin{aligned}&[\mu(-\infty, p), \mu(-\infty, q]] \subseteq I_k \\ \iff\ & a - 2^{-(n+k+3)} < \mu(-\infty, p) \text{ and } 1 - \mu(q, +\infty) < b + 2^{-(n+k+3)}.\end{aligned}$$

Hence W_n are uniformly r.e. open sets. Furthermore

$$\begin{aligned}\mu(W_n) &\le \sum \mu(p, q) \le \sum \mu[p, q] \le \sum \lambda(I_k) \\ &\le \mu(U_{n+1}) + \sum_k 2^{-(n+k+2)} \le 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}.\end{aligned}$$

Hence $\{W_n\}$ is a measure μ-test.

Now it suffices to show that there exist p and q such that

$$[\mu(-\infty, p), \mu(-\infty, q]] \subseteq I_k \text{ and } \alpha \in I_\mu(p, q)$$

for $\alpha \in [a, b]$ and $I_k = (a - 2^{-(n+k+3)}, b + 2^{-(n+k+3)})$ because then $r \in (p, q) \subseteq W_n$. Since r is not an atom of μ, there exist p, q such that $r \in (p, q)$ and

$$\lambda(I_\mu(p, q)) + \mu(\{p, q\}) = \mu[p, q] < 2^{-(n+k+3)}.$$

Since $\alpha \in I_\mu(p, q) \subseteq [\mu(-\infty, p), \mu(-\infty, q]]$, we obtain

$$\mu(-\infty, q] - 2^{-(n+k+3)} < \alpha < \mu(-\infty, p) + 2^{-(n+k+3)}.$$

Then

$$a - 2^{-(n+k+3)} \le \alpha - 2^{-(n+k+3)} < \mu(-\infty, p)$$

and

$$\mu(-\infty, q] < \alpha + 2^{-(n+k+3)} < b + 2^{-(n+k+3)}.$$

It follows that $[\mu(-\infty, p), \mu(-\infty, q]] \subseteq I_k$. ◻

The above theorem says that rs_μ preserves randomness if rs_μ is defined. Conversely the inverse function also preserves randomness.

Proposition 4.9. *If $r \in \mathbb{R}$ is μ-random then there is a Martin-Löf random real α such that $rs_\mu(\alpha) = r$.*

Proof. If r is an atom of μ, there exist an interval (a, b) such that $\beta \in (a, b) \Rightarrow \mathrm{rs}_\mu(\beta) = r$. Then there exists a Martin-Löf random real $\alpha \in (a, b)$.

Next we assume that r is not an atom of μ. Let $\alpha = \mu(-\infty, r)$. Then for each n let

$$R_n = \{p \ : \ |\alpha - \mu(-\infty, p)| < 2^{-(n+1)}\}.$$

Since $\lim_{p\to r} \mu(-\infty, p) = \mu(-\infty, r)$, there exist $p_0, q_0 \in R_n$ such that $p_0 < r < q_0$.

Suppose that $p, q \in R_n$ such that $p < r < q$. If $\mu(-\infty, p) = \mu(-\infty, q)$ then $\mu(-\infty, p) = \mu(-\infty, q) = \mu(-\infty, r) = \alpha$ and $\mu(p, q) = 0$. It follows that $r \in (p, q)$ is not μ-random, which is a contradiction. Hence $p, q \in R_n$ and $p < r < q$ imply $\mu(-\infty, p) < \mu(-\infty, q)$.

Let

$$P_n = \{p \in \mathbb{Q} \ : \ \alpha - 2^{-(n+1)} < \mu(-\infty, p) \le \mu(-\infty, p] < \alpha + 2^{-(n+1)}\}.$$

Then $p_1 = p_0 \in P_n \cap R_n$ because

$$\mu(-\infty, p_0] \le \mu(-\infty, q_0) < \alpha + 2^{-(n+1)}.$$

Let $q_1 \in (r, q_0)$. Then $q_1 \in P_n \cap R_n$ because

$$\alpha - 2^{-(n+1)} < \mu(-\infty, r) \le \mu(-\infty, q_1)$$

and

$$\mu(-\infty, q_1] \le \mu(-\infty, q_0) < \alpha + 2^{-(n+1)}.$$

Hence there exist $p_1, q_1 \in P_n \cap R_n$ such that $p_1 < r < q_1$.

We shall prove that $\mathrm{rs}_\mu(\alpha)$ is defined. Suppose, for a contradiction, that $\mathrm{rs}_\mu(\alpha)$ is not defined. Then α is computable. Then P_n is an r.e. set. Let $V_n = \bigcup_{p,q\in P_n}(p, q)$. Note that $\{V_n\}$ is a sequence of θ-computable open sets. Then $r \in V_n$ and

$$\mu(V_n) = \mu(\inf P_n, \sup P_n) \le 2^{-n}.$$

Hence r is not μ-random, which is a contradiction. It follows that $\mathrm{rs}_\mu(\alpha)$ is defined.

Furthermore $\mathrm{rs}_\mu(\alpha) = r$ by the proof of Theorem 4.5 (ii). Note that r is μ-random and is not an atom of μ. Then α is Martin-Löf random by Theorem 4.8. □

In the following we characterize randomness of a sequence of reals by randomness of a binary sequence. Let $\langle n, m\rangle$ be a pair function defined as $n+(m+n)(m+n+1)/2$. Let A_n be a binary sequence for each n. We define $\oplus_{i=0}^n A_n = (\cdots((A_0 \oplus A_1) \oplus A_2 \cdots) \oplus A_n)$. We write $A_n(m)$ to mean the m-th bit of A_n. We also define $\oplus_n A_n = A$ as $A(\langle n, m\rangle) = A_n(m)$ for all n and m.

We use a result from [24].

Definition 4.10 ([24])**.** *A sequence $\{A_n\}$ of binary sequences is relative 1-random if A_n is $\oplus_{i=0}^{n-1} A_i$-random for each n.*

Theorem 4.11 ([24])**.** *For a relative 1-random sequence $\{A_n\}$ there exists $\{B_n\}$ such that $B_n =^* A_n$ for each n and $\oplus B_n$ is 1-random.*

This theorem follows a more natural result. If a sequence of reals is random, then members of a random sequence are relatively random. Even if each member is random relative to previous members, a sequence many not be random. However a relatively random sequence of reals has a random sequence such that each corresponding member is representation equivalent.

Theorem 4.12. *Let $\{x_n\} \in \mathbf{R}^{\mathbb{N}}$ be a sequence of reals and $\mu = \prod \mu_n$ be a computable measure on $\mathbf{R}^{\mathbb{N}}$ such that μ_n is continuous for each n. A sequence $\{rs_{\mu_n}(x_n)\} \in \mathbf{R}^{\mathbb{N}}$ is μ-random iff $\oplus x_n \in 2^\omega$ is Martin-Löf random.*

Proof. Let $\{U_m\}$ be a Martin-Löf test on a Cantor space covering $\oplus x_n$. Then there exists a prefix-free set W_m such that $U_m = W_m^{[\preceq]}$. For each $\sigma \in W_m$, there exists a sequence of classes $\{A_n^\sigma\}$ such that

$$\oplus x_n \in [\sigma] \iff x_n \in A_n^\sigma \text{ for all } n \text{ and } \prod \lambda(A_n^\sigma) = 2^{-|\sigma|}.$$

For each n and σ we can construct θ-computable open sets such that

$$x_n \in A_n^\sigma \Rightarrow \mathrm{rs}_{\mu_n}(x_n) \in B_n^\sigma \text{ and } \mu_n(B_n^\sigma) \leq \lambda(A_n^\sigma)$$

similar to the proof of Theorem 4.8. Let

$$V_n = \bigcup_{\sigma \in W_n} \{z \ : \ z_n \in B_n^\sigma \text{ for all } n\}.$$

Then

$$\prod \mu_n(V_n) = \sum_\sigma \prod \mu_n(B_n^\sigma) \leq \sum_\sigma \prod \lambda(A_n^\sigma) = \lambda(W_n) \leq 2^{-n}.$$

Furthermore

$$\begin{aligned} \oplus x_n \in U_m \Rightarrow & \oplus x_n \in [\sigma] \text{ for some } \sigma \in W_m \\ \Rightarrow & x_n \in A_n^\sigma \text{ for all } n \text{ for some } \sigma \in W_m \\ \Rightarrow & \mathrm{rs}_{\mu_n}(x_n) \in B_n^\sigma \text{ for all } n \text{ for some } \sigma \in W_m \\ \Rightarrow & \{\mathrm{rs}_{\mu_n}(x_n)\} \in V_n. \end{aligned}$$

The converse also holds by the same way. □

Corollary 4.13. *Let μ_n be a continuous measure on* **R** *for each n. If x_n is $\oplus_{i<n} x_i$-μ_n-random for each n, then there exists $\{y_n\}$ such that $\{y_n\}$ is $\prod \mu_n$-random and $y_n \equiv_T x_n$ for all n.*

Proof. Suppose that x_n is $\oplus_{i<n} x_i$-μ-random for each n. Then $\mathrm{rs}_{\mu_n}^{-1}(x_n)$ is $\oplus_{i<n} \mathrm{rs}_{\mu_i}^{-1}(x_i)$-random for each n. By Theorem 4.11 there exists $\{A_n\}$ such that

$$\mathrm{rs}_{\mu_n}^{-1}(x_n) =^* A_n \tag{2}$$

and

$$\oplus A_n \text{ is Martin-Löf random} \tag{3}$$

Let $y_n = \mathrm{rs}_\mu(A_n)$ for each n. Here y_n is defined for each n because x_n is μ_n-random and so A_n is Martin-Löf random. By Theorem 4.5 and (2) we obtain

$$y_n \equiv_T A_n =^* \mathrm{rs}_\mu^{-1}(x_n) \equiv_T x_n.$$

By Theorem 4.12 and (3), $\{y_n\}$ is $\prod \mu_n$-random. □

Especially we consider the case that each μ_n is the uniform measure. Let μ_I be the uniform measure on $[0, 1] \subseteq \mathbb{R}$. Let $\mathbb{I} = [0, 1]$ and $\mathbf{I} = (\mathbb{I}, \tau, \beta, \nu)$ be the computable topological space where τ, β, ν is the usual one.

Corollary 4.14. *A sequence $\{x_n\} \in \mathbf{I}^\mathbb{N}$ is $\prod \mu_I$-random on $\mathbf{I}^\mathbb{N}$ iff $\oplus x_n$ is Martin-Löf-random.*

Corollary 4.15. *If x_n is $\oplus_{i<n} x_i$-random for all n then there exists $\{y_n\}$ such that $\{y_n\}$ is random and $y_n =^* x_n$ for all n.*

Proof. We obtain $y_n =^* x_n$ by the same proof of Theorem 4.12. □

5 Limit Theorems

In probability theory many limit theorems are proved. Especially strong law of large numbers and the law of the iterated logarithm are considered as laws which should hold for a random sequence. In this section we prove these theorem for a random sequence of reals defined in the previous section. The proofs are more or less same as those in probability theory except requirements of computability. Our proofs use characterizations by tests and by martingales because the proof in probability theory also use measures and martingales. However we have another characterization of randomness by complexity. Hence it also means that the complexity requirement implies these laws.

Subsequently we consider the computable topological space $\mathbf{R}^\mathbb{N}$ and a computable measure $\mu = \prod \mu_n$.

5.1 Convergence Theorem

In this subsection we prove convergence theorem which we will use later. Recall randomness is characterized by $([\nu_\mathbb{N}, \delta], \overline{\rho}_<)$-computable martingales. In contrast we shall only prove that the convergence theorem hold for $([\nu_\mathbb{N}, \delta], \overline{\rho})$-computable martingales. This additional requirement is needed in the proof.

Theorem 5.1 (non-negative convergence theorem)**.** *Let $\mu = \prod \mu_n$ be a computable measure. If $y = \{y_i\}$ is μ-random, then $f_n(y)$ converges for each non-negative $([\nu_\mathbb{N}, \delta], \overline{\rho})$-computable martingale.*

Proof. Suppose that y is μ-random. Then $\sup_n f_n(y) < \infty$ by Theorem 3.11. Hence, if $f_n(y)$ does not converge, there exist $a, b \in \mathbb{Q}^+$ such that

$$b < a \text{ and } \limsup_n f_n(y) > a \text{ and } \liminf_n f_n(y) < b.$$

For these a and b, we shall construct a new supermartingale $\{g_n\}$ such that $\sup_n g_n(y) = \infty$ to follow a contradiction.

We take $(\delta, \nu_{\mathbb{N}})$-computable functions σ_i, τ_i such that $\sigma_0(x) = \tau_0(x) = 0$ and 2

$$\sigma_i(x) \in \{n \ : \ f_n(x) < b \text{ and } n \geq \tau_{i-1}\} \text{ and } \tau_i(x) \in \{n \ : \ f_n(x) > a \text{ and } n \geq \sigma_i\}.$$

Note that σ_i, τ_i may not be defined for some x then they are ∞. We also assume that the relations $\sigma_i(x) < n$ and $\tau_i(x) < n$ are computable uniformly in n. For each n and i, let

$$g_n^i(x) = \begin{cases} 1 & \text{if } n < \sigma_i(x) \\ f_n(x)/b & \text{if } \sigma_i(x) \leq n < \tau_i(x) \\ a/b & \text{if } \tau_i(x) \leq n \end{cases}$$

and let $g_n = \prod_i g_n^i$. Then g_n is $([\nu_{\mathbb{N}}, \delta], \overline{\rho})$-computable.

We shall prove that $\{g_n\}$ is a supermartingale. We divide X into the following parts.

(i) $A_n^i = \{x \ : \ \tau_{i-1}(x) \leq n < \sigma_i(x) \leq n+1\}$.

(ii) $B_n^i = \{x \ : \ \sigma_i(x) \leq n < n+1 < \tau_i(x)\}$.

(iii) $C_n^i = \{x \ : \ \sigma_i(x) \leq n < \tau_i(x) \leq n+1\}$.

(iv) $D_n^i = \{x \ : \ \tau_i(x) \leq n < n+1 < \sigma_i(x)\}$.

For each n, these are pairwise disjoint. Furthermore the union of them is X. We shall prove that $\int_A g_n d\mu \geq \int_A g_{n+1} d\mu$ for all $A \in \mathcal{A}_n$ for each part.

(i) Suppose $A \subseteq A_n^i$. Then $g_n^i(x) = 1$, $g_{n+1}^i(x) = f_n(x)/b$ and $\sigma_i(x) = n+1$. It follows that

$$\int_A g_{n+1} d\mu = \int_A \prod_{j \neq i} g_{n+1}^j \cdot f_{n+1}/b d\mu < \int_A \prod_{j \neq i} g_{n+1}^j \cdot b/b d\mu = \int_A \prod_{j \neq i} g_n \cdot g_n^i d\mu = \int_A g_n d\mu.$$

Note that $g_{n+1}^j = g_n^j$ if $j \neq i$.

(ii) In this case $g_n^i(x) = f_n(x)/b$ and $g_{n+1}^i(x) = f_{n+1}(x)/b$. Then

$$\int_A g_{n+1} d\mu = \int_A \prod_{j \neq i} g_{n+1}^j f_{n+1}(x)/b d\mu = \int_A \prod_{j \neq i} g_n^j f_n(x)/b d\mu = \int_A g_n d\mu.$$

(iii) Now $g_n^i(x) = f_n(x)/b$, $g_{n+1}^i = a/b$ and $\tau_i(x) = n+1$. Hence

$$\begin{aligned} \int_A g_{n+1} d\mu = \int_A \prod_{j \neq i} g_{n+1}^j a/b d\mu < \int_A \prod_{j \neq i} g_{n+1}^j f_{n+1}(x)/b d\mu &= \int_A \prod_{j \neq i} g_n^j f_n(x)/b d\mu \\ &= \int_A g_n d\mu. \end{aligned}$$

(iv) This is because $g_n^i(x) = g_{n+1}^i(x) = 1$.

Then $\{g_n\}$ is a supermartingale.

Recall that $\limsup_n f_n(y) > a$ and $\liminf_n f_n(y) < b$. Then for each i, there exist n such that $\tau_i(y) \leq n$. It implies $g_n^i(y) = a/b$ and $g_n(y) \geq (a/b)^i$. This contradicts to the fact that y is μ-random. □

Corollary 5.2. *Let $\mu = \prod \mu_n$ be a computable measure and $m_n = \int |z| d\mu_n$. Suppose that*

(i) *m_n are uniformly computable and*

(ii) *$\sum m_n < \infty$.*

If $y = \{y_n\}$ is μ-random, then $\sum |y_n| < \infty$. Hence $\sum y_n$ converges.

Proof. Let $f_n(x) = Q + \sum_{i=1}^{n}(|x_i| - m_i)$ for each n where Q is rational such that $\sum m_n < Q$. Then $\{f_n\}$ is a non-negative $([\nu_{\mathbb{N}}, \delta], \overline{\rho})$-computable martingale. Hence $Q + \sum(|y_n| - m_n)$ converges by Theorem 5.1. Since $\sum m_n < \infty$, we obtain $\sum |y_n| < \infty$. □

5.2 The Strong Law of Large Numbers

Now we shall prove SLLN for a random sequence of reals. We recall the statements in probability theory.

Theorem 5.3 (SLLN under variance constraints)**.** *Let (W_n) be a sequence of independent random variables such that*

$$E(W_n) = 0 \text{ and } \frac{Var(W_n)}{n^2} < \infty.$$

Then $n^{-1} \sum_{k \le n} W_k \to 0$, a.s.

Theorem 5.4 (SLLN for i.i.d.)**.** *Let $\{X_i\}$ be independent and identically distributed random variables. Define $S_n = \sum_{k \le n} X_k$. Then,*

$$E(|X_1|) < \infty \Rightarrow \frac{S_n}{n} \to E(X_1) \text{ a.s.},$$
$$E(|X_1|) = \infty \Rightarrow \limsup_{n \to \infty} \frac{|S_n|}{n} = +\infty \text{ a.s.}.$$

First we shall prove the counterpart of Theorem 5.3 for a random sequence. Randomness version of SLLN with the existence of forth moment also can be seen in [11]. We prepare two famous lemmas.

Lemma 5.5 (Cesàro's Lemma)**.** *Suppose that (b_n) is a sequence of strictly positive real numbers with $b_n \to \infty$, and that (v_n) is a convergent sequence of real numbers: $v_n \to v_\infty \in \mathbb{R}$. Let $b_0 = 0$. Then*

$$\frac{1}{b_n} \sum_{k=1}^{n} (b_k - b_{k-1}) v_k \to v_\infty \ (n \to \infty).$$

Lemma 5.6 (Kronecker's Lemma)**.** *Let (v_n) denote a sequence of strictly positive real numbers with $b_n \to \infty$. Let (x_n) be a sequence of real numbers, and define $s_n = \sum_{k \le n} x_k$. Then*

$$\left(\sum \frac{x_n}{b_n} \text{ converges} \right) \Rightarrow \left(\frac{s_n}{b_n} \to 0 \right).$$

Theorem 5.7 (Randomness version of SLLN)**.** *Let $\mu = \prod \mu_i$ be a measure on $\mathbf{R}^{\mathbb{N}}$ such that $m_i = \int z\mu_i(dz)$ and $v_i = \int (z - m_i)^2 \mu_i(dz)$ are uniformly computable. Then*

$$\{y_i\} \text{ is } \mu\text{-random and } \sum_i \frac{v_i}{i^2} < \infty \Rightarrow \frac{\sum_{i=1}^n (y_i - m_i)}{n} \to 0.$$

Proof. The idea of the proof is based on Vovk [31]. Without loss of generality, we can assume that $m_i = 0$ for all i. For each n, let $\{f_n\}$ such that

$$f_n(x) = \sum_{i=1}^n \frac{x_i}{i}$$

where $x = \{x_i\} \in \mathbb{R}^{\mathbb{N}}$. Let

$$f_n^*(r, x_n) = r + \frac{x_n}{n}.$$

. Note that $f_{n+1}(x) = f_{n+1}^*(f_n(x), x_{n+1})$. Furthermore $\int_{X_n} x_n \mu_n(dx_n) = m_n = 0$. Then $r \geq \int_{X_n} (r + \frac{x_n}{n}) \mu_n(dx_n)$. Hence $\{f_n\}$ is a $([\nu_{\mathbb{N}}, \delta], \overline{\rho})$-computable martingale by Theorem 3.18. Note that $f_n(x)$ may be negative, hence convergence theorem cannot be adapted.

We also let

$$V_n = \sum_{i=1}^n \frac{v_i}{i^2}.$$

Note that V_n is uniformly computable. Since V_n is increasing and $\lim_n V_n < \infty$, this converges.

Let

$$g_n = (f_n)^2 - V_n.$$

Then $\{g_n\}$ is a $([\nu_{\mathbb{N}}, \delta], \overline{\rho})$-computable martingale because

$$g_n(x) = \sum_{i=1}^n 2\left(\sum_{j=1}^{i-1} \frac{x_j}{j}\right)\frac{x_i}{i} + \sum_{i=1}^n \frac{x_i^2 - v_i}{i^2}.$$

Let $h_n = (f_n + 1)^2 - V_n$. Then $\{h_n\}$ is a $([\nu_{\mathbb{N}}, \delta], \overline{\rho})$-computable martingale because $h_n = g_n + 2f_n + 1$. Then by convergence theorem, $g_n(y)$ and $h_n(y)$ converge. Hence $f_n(y) = (h_n(y) - g_n(y) - 1)/2 = \sum_{i=1}^n \frac{y_i}{i}$ also converges. By Kronecker's lemma,

$$\frac{\sum_{i=1}^n y_i}{n} \to 0.$$

□

Next we prove SLLN for the case where all measures are the same. We prepare a lemma.

Lemma 5.8 (Randomness version of Kolmogorov's Truncation Lemma)**.** *Let μ_1 be a computable measure on $\mathbf{R}$ such that $m = \int |z| d\mu_1$ is computable. Let*

$$g_n(x) = \begin{cases} x & if\ |x| \leq n \\ 0 & if\ |x| > n \end{cases}.$$

Then

(i) $m_n = \int g_n(x)d\mu \to \int xd\mu$.

(ii) *If $x \in \mathbb{R}^{\mathbb{N}}$ is $\prod \mu_1$-random, then there exists a computable sequence $\{a_n\}$ such that $g_{a_n}(x_n) = x_n$ for all but finitely many n.*

(iii) $\sum_n n^{-2} \int (g_n(x) - m_n)^2 d\mu < \infty$.

Proof. (i) and (iii) are straightforward by Kolmogorov's Truncation Lemma in probability theory. See Williams [40] etc.
(ii) Since $\mu = \prod \mu_1$ is computable, there exists a computable sequence $\{a_n\}$ such that $\mu(\{x : |x| > a_n\}) < 2^{-n}$. Let $E_n = \{x : g_{a_n}(x_n) \neq x_n\} = \{x : |x_n| > a_n\}$. Then $\mu(E_n) \leq \mu(\{x : |x| > a_n\}) \leq 2^{-n}$. Let $V_n = \bigcup_k E_{n+k+1}$. Then V_n is uniformly θ-computable and

$$\mu(V_n) \leq \sum_k 2^{-n-k-1} \leq 2^{-n}.$$

Hence $\{V_n\}$ is a Martin-Löf test. Since x is μ-random, there exists m such that $x \notin V_m$. Hence $x \notin E_{m+k+1}$ for all k. It follows that $g_{a_n}(x_n) = x_n$ for all $n \geq m + 1$. □

The following theorem is randomness version of the first part of Theorem 5.4.

Theorem 5.9. *Let μ_1 be a computable measure on* **R** *such that $m = \int |z|d\mu_1 < \infty$ is computable. Suppose that $\mu = \prod \mu_1$ is the products of infinite same measures. Then*

$$\{y_i\} \text{ is } \mu\text{-random} \Rightarrow \frac{\sum_{i=1}^n y_i}{n} \to \int xd\mu.$$

Proof. By (ii) and (iii) of Lemma 5.8 , we need only show that

$$n^{-1} \sum_{i=1}^n g_i(y_i) \to \int xd\mu.$$

But by Theorem 5.7,

$$n^{-1} \sum_{k \leq n} g_k(y_k) = n^{-1} \sum m_k \to \lim_n m_n = \int xd\mu.$$

The last implication is by Cesàro's Lemma and (i) of lemma 5.8. □

The next theorem is randomness version of the second part of Theorem 5.4.

Theorem 5.10. *Let μ_1 be a computable measure on* **R** *such that $m = \int |z|d\mu_1 = \infty$ is computable. Suppose that $\mu = \prod \mu_1$ is the product of infinite same measures. Then*

$$\{y_i\} \text{ is } \mu\text{-random} \Rightarrow \limsup_n \frac{|\sum_{i=1}^n y_i|}{n} = \infty.$$

Proof. Suppose that $\limsup_n |\sum_{i=1}^n y_i|/n < b$. Then $|\sum_{i=1}^n y_i| < nb$ for all but finite n. It follows that $|y_i| < 2nb$ for all but finite n. For this b,

$$\int x d\mu \leq \sum_{m=0}^{\infty} 2(m+1)b\mu(\{x \ : \ 2mb \leq |x| < 2(m+1)b) \tag{4}$$

$$= \sum_{n=0}^{\infty} 2b\mu(\{x \ : \ |x| \geq 2nb). \tag{5}$$

Let $F_n = \{x \in \mathbb{R}^{\mathbb{N}} \ : \ |x_n| \geq 2nb\}$. Then $\mu(F_n)$ is computable and $\sum \mu(F_n) \geq \int_{\mathbb{R}} |x| d\mu = \infty$. By BC2, $y \in F_n$ for infinitely many n. This is a contradiction. □

Theorem 5.9 and Theorem 5.10 says that, if a sequence of reals is random with respect tot the product measure of infinite same measure μ_1, then the arithmetic mean converges to $\int z d\mu_1$. This is also true for each moment.

We use the following definitions and theorem. They are proved by Hertling and Weihrauch [10]. Here we restate them in our terminology.

Definition 5.11 ([10])**.** *Let* $\mathbf{X} = (X, \beta, \tau, \nu)$ *be a computable topological space and* μ *be a measure on it. A set* $D \subseteq X$ *is called fast enclosable iff it is measurable and if there is a uniformly* θ*-computable sequence* $(U_n)_n$ *of open sets with* $D \subseteq U_n$ *and* $\mu(U_n \setminus D) \leq 2^{-n}$ *for all n.*

Definition 5.12 ([10])**.** *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be computable topological spaces and* μ_X *and* μ_Y *be measures on them respectively. A function* $f :\subseteq X \to Y$ *is called recursively measure bounded if* $\mathrm{dom} f$ *is measurable and there is a total recursive function r such that for all open sets* $V \subseteq Y$*:*

$$\mu_Y(V) \leq 2^{-r(n)} \Rightarrow \mu_X(f^{-1}(V)) \leq 2^{-n}.$$

We also call it measure invariant if, for all open sets $V \subseteq Y$*,*

$$\mu_Y(V) = \mu_X(f^{-1}(V)).$$

Theorem 5.13 ([10])**.** *Let* $f :\subseteq X \to Y$ *be a* (δ_X, δ_Y)*-computable, recursively measure-bounded function with a fast enclosable domain. If* $x \in \mathrm{dom} f$ *is a random element of* X*, then* $f(x)$ *is a random element of* Y*.*

Now the following theorem naturally follows.

Theorem 5.14. *Let* μ_1 *be a computable measure on* $\mathbf{R}$ *such that* $m = \int |z| d\mu_1 < \infty$ *and* $m_p = \int (z-m)^p d\mu_1 < \infty$ *are computable. Suppose that* $\mu = \prod \mu_1$ *is the product of infinite same measures. Then*

$$\{y_i\} \text{ is } \mu\text{-random} \ \Rightarrow \ \frac{\sum_{i=1}^n (y_i - m)^p}{n} \to \int (z-m)^p d\mu.$$

Proof. Let $f(z) = (z - m)^p$. Clearly $\mathrm{dom}(f) = \mathbb{R}$ is fast enclosable. Let μ_p be the measure such that $\mu_p(V) = \mu(f^{-1}(V))$ for all open sets $V \subseteq \mathbb{R}$. Then f is measure invariant. Suppose that $\{y_i\}$ is μ-random. By Theorem 5.13, $\{f(y_i)\}$ is μ_p-random. Then by Theorem 5.9,

$$\frac{\sum_{i=1}^{n}(y_i - m)^p}{n} \to \int z d\mu_p.$$

Now it suffices to show

$$\int z d\mu_p = \int f(z) d\mu$$

if $\mu_p(V) = \mu(f^{-1}(V))$ for all open sets $V \subseteq \mathbb{R}$. We only show this on the hypothesis that f is increasing. This is because

$$r \times \mu_p(\{z \ : \ z > r\}) = r \times \mu(\{z \ : \ z > f^{-1}(r)\}) = r \times \mu(\{z \ : \ f(z) > r\})$$

for each $r \in \mathbb{R}$. □

5.3 The Law of the Iterated Logarithm

In this subsection we prove LIL for a random sequence. First we recall the statements in probability theory from [27].

Let $\{X_n\}$ be a sequence of independent random variables with zero means and finite variances. Put $\sigma_n^2 = \mathrm{Var} X_n$ and $B_n = \sum_{k=1}^{n} \sigma_k^2$.

Theorem 5.15. *Suppose $B_n \to \infty$. Suppose also that there exists a sequence of positive constants $\{M_n\}$ such that*

$$M_n = o\left(\left(\frac{B_n}{\log\log B_n}\right)^{1/2}\right)$$

and

$$|X_n| \leq M_n \ a.s.$$

Then

$$\limsup \frac{S_n}{(2B_n \log\log B_n)^{1/2}} = 1 \ a.s.$$

To prove this, they use some lemmas.

Lemma 5.16. *We put $q_n(x) = P(S_n \geq x)$. If $0 \leq xM_n \leq B_n$, then*

$$q_n(x) \leq \exp\left\{-\frac{x^2}{2B_n}\left(1 - \frac{xM_n}{2B_n}\right)\right\}.$$

If $xM_n \geq B_n$, we have

$$q_n(x) \leq \exp\left\{-\frac{x}{4M_n}\right\}.$$

Lemma 5.17. *If $x = x(n) > 0$, $xM_n/B_n \to 0$, and $x^2/B_n \to \infty$, then for every fixed $c > 0$ and all sufficiently large n we have*

$$q_n(x) \geq \exp\left\{-\frac{x^2}{2B_n}(1 + c)\right\}.$$

By the proof, more precisely, the following conditions are enough to hold the inequality: $t = x/(1-\delta)B_n$, $tM_n < \alpha$, $tM_n \le 1/8$, $\exp\left\{\frac{B_n}{2}t^2(1-\alpha)\right\} \ge 8$, $\frac{yM_n}{2B_n} < \beta$ for all y such that $0 \le yM_n \le B_n$, $t(1-\delta)B_n < \frac{tB_n}{1-\beta} < t(1+\delta)B_n$, $\beta < \delta^2/2(1+\delta)^2$, $32t^2B_n \le \exp\left\{\frac{t^2B_n}{8}\delta^2\right\}$, $tM_n < \frac{\delta^2}{4}$, $\delta < 1/2$, $\left(1+\alpha+2\delta+\frac{\delta^2}{4}\right)(1-\delta)^{-2} < 1+c$. Then if $x(n), M_n, B_n$ are computable functions and c is rational, we can compute N such that the inequality holds for $n \ge N$.

Put $\chi(n) = (2B_n \log\log B_n)^{1/2}$.

Lemma 5.18. *If the conditions of Theorem 5.15 are satisfied, then for all positive constants b and c and for all sufficiently large n the following inequalities hold:*

$$(\log B_n)^{-(1+c)b^2} \le P(S_n \ge b\chi(n)) \le (\log B_n)^{-(1-c)b^2}.$$

Just let $x = b\chi(n)$ in Lemma 5.17 to get the proof. Hence if b is computable too, we can compute N such that the inequality holds for $n \ge N$.

Now we prove randomness version of LIL.

Theorem 5.19 (LIL). *Let $\mu = \prod \mu_i$ be a measure on $\mathbb{R}^{\mathbb{N}}$ such that $m_i = \int_{X_i} x_i\mu_i(dx_i)$ and $v_i = \int_{X_i}(x_i - m_i)^2\mu_i(dx_i)$ are uniformly computable and $B_n = \sum_{i=1}^n v_i \to \infty$. If there exists a positive sequence $\{M_n\}$ such that*

$$M_n = o\left(\sqrt{\frac{B_n}{\ln\ln B_n}}\right) \tag{6}$$

and

$$\mu_i(\{z : |z| \le M_n\}) = 1, \tag{7}$$

then

$$\{x_i\} \text{ is } \mu\text{-random} \Rightarrow \limsup_n \frac{\sum(x_i - m_i)}{\sqrt{2B_n \ln\ln B_n}} = 1.$$

Proof. The proof is based on [27].

We can assume $m_n = 0$ for all n without loss of generality. Suppose that $y = \{y_i\}$ is a μ-random sequence.

First we shall prove that

$$\limsup_n \frac{S_n(y)}{\chi(n)} \le 1 \text{ where } S_n(x) = \sum_{x=1}^n x_i. \tag{8}$$

It suffices to show that $S_n(y) > (1+\epsilon)\chi(n)$ holds for only finite n.

Let d_n such that $M_n^2 = d_nB_n/\ln\ln B_n$. By (6), $d_n \to 0$ as $n \to \infty$. Hence there exists D such that $d_n \le D$. By (7),

$$v_n = \int z^2 d\mu_n \le M_n^2.$$

Then

$$\frac{B_n}{B_{n+1}} = 1 - \frac{v_{n+1}}{B_{n+1}} \ge 1 - \frac{M_{n+1}^2}{B_{n+1}} = 1 - \frac{d_{n+1}}{\ln\ln B_{n+1}} \ge 1 - \frac{D}{\ln\ln B_{n+1}}.$$

It follows that

$$\frac{v_{n+1}}{B_n} \leq \frac{1}{1 - D/\ln\ln B_{n+1}} - 1 \to 0. \tag{9}$$

Hence for each $\tau > 0$ there exists N such that $B_{n+1} \leq (1+\tau)B_n$ for all $n \geq N$. Then there exists a non-decreasing computable sequence of integers $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$ and

$$B_{n_k-1} < (1+\tau)^k < B_{n_k+1} \tag{10}$$

for all $k \geq K$ for some K. (Here for large k, there exists n_k such that $B_{n_k} \leq (1+\tau)^k < B_{n_k+1}$. However the sequence n_k may not be computable because there may be n_k such that $B_{n_k} = (1+\tau)^k$. In order to make n_k to be computable, we consider the inequality above.) Then $B_{n_k} < (1+\tau)^{k+1}$ for all $k \geq K$ for some K.

We write $\overline{S}_{n_k} = \max_{n \leq n_k} S_n$. We shall show that

$$\overline{S}_{n_k}(y) \leq (1+\gamma)\chi(n_k) \tag{11}$$

for all but finite k for every computable γ. Using Kolmogorov inequality,we obtain

$$\mu(\{x : \overline{S}_{n_k}(x) > (1+\gamma)\chi(n_k)\}) \leq 2\mu(\{x : S_{n_k}(x) > (1+\gamma)\chi(n_k) - \sqrt{2B_{n_k}}\}).$$

Let γ_1 be computable such that $0 < \gamma_1 < \gamma$. Then we can compute K_1 such that, for all $k \geq K_1$,

$$(1+\gamma)\chi(n_k) - \sqrt{2B_{n_k}} \geq (1+\gamma_1)\chi(n_k).$$

Let c be computable such that $\gamma_1(1-c)(1+\gamma_1)^2 > 1$. Then by Lemma 5.18 we can compute K_2 such that, for all $k \geq K_2$,

$$\mu(\{x : S_{n_k} > (1+\gamma_1)\chi(n_k)\}) \leq (\log B_{n_k})^{-(1-c)(1+\gamma_1)^2} \leq ((k+1)\log(1+\tau))^{-(1-c)(1+\gamma_1)^2}.$$

Hence $\sum_k \mu(\{x : S_{n_k} > (1+\gamma_1)\chi(n_k)\}) < \infty$. By BC1, (11) holds.

Suppose for a contradiction that

$$S_n(y) > (1+\epsilon)\chi(n) \tag{12}$$

for infinitely many n for a computable $\epsilon > 0$. Then

$$\max_{n_{k-1} \leq n \leq n_k} S_n > (1+\epsilon)\chi(n_{k-1}) \text{ for inf } n \Rightarrow \overline{S}_{n_k} > (1+\epsilon)\chi(n_{k-1}) \text{ for inf } n.$$

Since $\chi(n_k)/\chi(n_{k-1}) < (1+2\tau)^{1/2}$ for sufficiently large k by (10),

$$\overline{S}_{n_k} > (1+\epsilon)(1+2\tau)^{-1/2}\chi(n_k) \text{ for inf } n$$

Recall that τ is an arbitrary computable reals. So we can assume further that $(1+\epsilon)(1+2\tau)^{-1/2} > 1+\epsilon/2$. Then

$$\overline{S}_{n_k}(y) > (1+\epsilon/2)\chi(n_k) \text{ for inf } n.$$

However this contradicts to (11). Hence (12) holds for only finite n.

Next we shall prove that

$$\limsup_n \frac{S_n(y)}{\chi(n)} \geq 1.$$

The goal is to show that

$$S_n(x) > (1-\epsilon)\chi(n) \text{ for inf } n \tag{13}$$

for every computable $\epsilon > 0$. We write

$$\psi(n_k) = (2(B_{n_k} - B_{n_k-1}) \log\log(B_{n_k} - B_{n_k-1}))^{1/2}.$$

Note that

$$\log(B_{n_k} - B_{n_k-1}) < \log B_{n_k} < (k+1)\log(1+\tau)$$

for all $k \geq K_1$. Furthermore, there exists K_3 such that $\psi(n_k)/\chi(n_{k-1}) > \frac{2\tau^{1/2}}{3}$ for all $k \geq K_3$. For every computable positive $\gamma < 1$ we have

$$\begin{aligned}
&\mu(\{x : S_{n_k} - S_{n_{k-1}} \geq (1-\gamma)\psi(n_k))\\
&\geq \mu(\{x : S_{n_k} > (1-\gamma/2)\psi(n_k))\&S_{n_{k-1}} \leq \gamma\psi(n_k)/2\})\\
&\geq \mu(\{x : S_{n_k} > (1-\gamma/2)\psi(n_k)\}) - \mu(\{x : S_{n_{k-1}} > \gamma\psi(n_k)/2\\
&\geq \mu(\{x : S_{n_k} > (1-\gamma/2)\psi(n_k)\}) - \mu(\{x : S_{n_{k-1}} > \gamma\tau^{1/2}\chi(n_k)/3\\
&\geq (\log B_{n_k})^{-(1+c)(1-\gamma/2)^2} - (\log B_{n_{k-1}})^{-\gamma^2\tau/10}
\end{aligned}$$

by Lemma 5.18. Let c and τ be computable such that

$$(1+c)(1-\gamma/2)^2 < 1 \text{ and } (1+c)(1-\gamma/2)^2 < \gamma^2\tau/10.$$

Then $\mu(\{x : S_{n_k} - S_{n_{k-1}} \geq (1-\gamma)\psi(n_k))$ has a computable lower-bound whose sums tend to infinity. By BC2, $S_{n_k}(y) - S_{n_{k-1}}(y) \geq (1-\gamma)\psi(n_k)$ for infinitely many n. Since $S_{n_{k-1}}(y) \geq -2\chi(n_{k-1})$ for all but finite n,

$$S_{n_k}(y) \geq (1-\gamma)\psi(n_k) - 2\chi(n_{k-1})$$

for infinitely many n. Note that

$$\frac{(1-\gamma)\psi(n_k) - 2\chi(n_{k-1})}{\chi(n_k)} \to (1-\gamma)\tau^{1/2}(1+\tau)^{-1/2} - 2(1+\tau)^{-1/2}.$$

For each computable $\epsilon > 0$, choose computable positive γ and τ such that

$$(1-\gamma)\tau^{1/2}(1+\tau)^{-1/2} - 2(1+\tau)^{-1/2} > 1-\epsilon.$$

Then (13) holds. □

Finally we prove randomness version of Hartman-Wintner LIL. The proof is also based on [27]. We set $\mathrm{LL}n = \log\log n$ for $n \geq 3$, $\mathrm{LL}n = 1$ for $n = 1$ and $n = 2$. Let $a_n = (2n\mathrm{LL}n)^{1/2}$.

Lemma 5.20. *Let μ_1 be a continuous computable measure on* **R** *such that $\int z\mu_1(dz) = 0$ and $\int z^2\mu_1(dz) = v$ are computable. Then there exists a computable sequence $\{\tau_n\}$ of reals such that $\tau \downarrow 0$, $\tau_n(n/LLn)^{1/2} \uparrow \infty$, and*

$$\sum_n \frac{\int_{I_n} |z| d\mu}{a_n} < \infty$$

where $I_n = \{z \ : \ |z| \geq \tau_n(n/LLn)^{1/2}\}$.

Proof. Let

$$f(y) = \int_{|z|<y} z^2 d\mu_1 \text{ and } g(y) = -\log(1 - f(y)/v).$$

Since μ_1 is continuous and computable, f and g are (ρ, ρ)-computable. Furthermore $f(y) \uparrow v$ and $g(y) \uparrow \infty$. Let $\{b_k\}$ be the computable sequence such that $g(b_k) = k$. Note that $f(b_k) = (1 - 2^{-k})v$. Let $\{c_k\}$ is a non-decreasing computable sequence such that $c_0 = 0$, $\{b_k\} \subseteq \{c_k\}$ and $c_{k+1}/c_k < r$ for all k. Then

$$\begin{aligned}
\int z^2 g(z) d\mu_1 &\leq \sum_k c_{k+1}^2 g(c_{k+1})\mu_1(\{z \ : \ c_k \leq |z| < c_{k+1}\}) \\
&\leq \sum_k \frac{c_{k+1}^2}{c_k^2} g(c_{k+1}) \int_{c_k \leq |z| < c_{k+1}} z^2 d\mu_1 \\
&\leq \sum_k r^2 g(b_{k+1}) \int_{b_k \leq |z| < b_{k+1}} z^2 d\mu_1 \\
&\leq \sum_k r^2 (k+1)(f(b_{k+1}) - f(b_k)) \\
&\leq \sum_k r^2 (k+1) 2^{-(k+1)} < \infty.
\end{aligned}$$

Let h be a (ρ, ρ)-computable function such that

$$h(n^{1/3}) = \min\left\{ g(n^{1/3}), \left(\frac{n}{\text{LL}n}\right)^{1/2} n^{-1/3} \right\}.$$

Then $h(y) \uparrow \infty$ and $\int z^2 h(z) d\mu_1 < \infty$.

We put $\tau_n = 1/h(n^{1/3})$ and $M_n = \tau_n(n/\text{LL}n)^{1/2}$. We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\int_{I_n} |z| d\mu_1}{a_n} &= \sum_{n=1}^{\infty} a_n^{-1} \sum_{m=n}^{\infty} \int_{M_m \leq |z| < M_{m+1}} |z| d\mu_1 \\
&\leq \sum_{n=1}^{\infty} a_n^{-1} \sum_{m=n}^{\infty} M_{m+1} \mu_1(\{z \ : \ M_m \leq |z| < M_{m+1}\}) \\
&= \sum_{m=1}^{\infty} M_{m+1} \mu_1(\{z \ : \ M_m \leq |z| < M_{m+1}\}) \sum_{n=1}^{m} a_n^{-1}.
\end{aligned}$$

Here there exists c such that $\sum_{n=1}^{m} a_n^{-1} \leq cM_m/\tau_m$ and $M_{m+1} \leq cM_m$ for all m. Therefore,

$$\sum_{n=1}^{\infty} \frac{\int_{I_n} |z| d\mu_1}{a_n} \leq c' \sum_{m=1}^{\infty} \frac{M_m^2}{\tau_m} \mu_1(\{z \ : \ M_m \leq |z| < M_{m+1}\}) < \infty,$$

since $M_m^2/\tau_m = M_m^2 h(m^{1/3}) \leq M_m^2 h(M_m)$ and $\int z^2 h(z) d\mu_1 < \infty$. □

Theorem 5.21 (Hartman-Wintner LIL)**.** *Let* $\mu = \prod \mu_1$ *be a computable measure on* $\mathbb{R}^{\mathbb{N}}$ *such that* μ_1 *is continuous,* $\int z\mu_1(dz) = 0$ *and* $\int z^2 \mu_1(dz) = \sigma^2$ *is computable. Then*

$$\{y_i\} \text{ is } \mu\text{-random} \Rightarrow \limsup_n \frac{\sum y_i}{\sqrt{2n \ln \ln n}} = \sigma.$$

Proof. Let $\{\tau_n\}$ be a sequence from Lemma 5.20 and

$$M_n = \tau_n (n/\mathrm{LL}n)^{1/2}.$$

For $z \in \mathbb{R}$, let

$$f_n(z) = \begin{cases} z & \text{if } |z| \geq M_n \\ 0 & \text{otherwise} \end{cases} \text{ and } \overline{f}_n(z) = z/a_n.$$

Note that $\overline{f}_n$ are uniformly (ρ, ρ)-computable. For each n, let μ_n^f be the computable measures such that

$$\mu_n^f(A) = \mu_1(\overline{f}_n^{-1}(A))$$

for all open sets A. Since $\{y_n\}$ is μ-random, $\{\overline{f}_n(y_n)\}$ is $\prod \mu_n^f$-random by Theorem 5.13. Furthermore $m_n = \int |z| d\mu_n^f$ are uniformly computable because μ_1 is continuous. By Lemma 5.20, $\sum_n m_n < \infty$. Hence $\sum \overline{f}_n(y_n) = \sum_{n=1}^{\infty} f_n(y_n)/a_n$ converges by Corollary 5.2. By Kronecker's lemma,

$$\frac{\sum_{i=1}^{n} f_i(y_i)}{a_n} \to 0. \tag{14}$$

Next let

$$g_n(z) = \begin{cases} z & \text{if } |z| < M_n \\ 0 & \text{otherwise} \end{cases}.$$

Note that g is (ρ, ρ)-computable. Let μ^g be a computable measures such that

$$\mu_n^g(A) = \mu_1(g_n^{-1}(A))$$

for all open sets A. Again $m_n^g = \int z d\mu_n^g$ and $v_n^g = \int (z - m_n^g)^2 d\mu_n^g$ are uniformly computable. Since $v_n^g \to \sigma^2$, we obtain $B_n = \sum_{i=1}^{n} v_n^g \to \infty$. Hence by randomness version of LIL (Theorem 5.19) we obtain

$$\limsup_n \frac{\sum_{i=1}^{n} (g_i(y_i) - m_i^g)}{\sqrt{2B_n \ln \ln B_n}} = 1. \tag{15}$$

Here $B_n/n \to \sigma^2$ and

$$\frac{\sqrt{2B_n \ln\ln B_n}}{a_n} \to \sigma. \tag{16}$$

Furthermore

$$\int z d\mu_n^g = \int_{|z|<M_n} z d\mu_1 = -\int_{|z|\geq M_n} z d\mu_1.$$

Hence

$$\sum_{n=1}^{\infty} \frac{m_n^g}{a_n} \leq \sum_{n=1}^{\infty} \frac{|\int z d\mu_n^g|}{a_n} \leq \sum_{n=1}^{\infty} \frac{\int_{|z|\geq M_n} |z| d\mu_1}{a_n} < \infty$$

by the results stated in the proof of Lemma 5.20. By Kronecker's lemma, we obtain

$$\frac{\sum_{n=1}^{\infty} m_n^g}{a_n} \to 0. \tag{17}$$

By (15), (16) and (17),

$$\limsup_n \frac{\sum_{i=1}^n g_i(y_i)}{a_n} = \sigma.$$

Furthermore by (14),

$$\limsup_n \frac{\sum_{i=1}^n y_i}{a_n} = \limsup_n \frac{\sum_{i=1}^n (f_i(y_i) + g_i(y_i))}{a_n} = \sigma.$$

□

Acknowledgement

The author thanks Masahito Hasegawa for encouragements and comments. This work was partly supported by a Global COE Program funded by the Ministry of Education, Culture, Sports, Science and Technology.

References

[1] P. Billingsley. *Convergence of Probability Measures*. John Wiley, New York, 1968.

[2] V. I. Bogachev. *Measure Theory*, volume 1,2. Springer, 2007.

[3] V. Brattka, P. Hertling, and K. Weihrauch. A tutorial on computable analysis. *New Computational Paradigms*, pages 425–491, 2008.

[4] G. Chaitin. Information-theoretical characterizations of recursive infinite strings. *Theoretical Computer Science*, 2:45–48, 1976.

[5] S. B. Cooper. *Computability theory*. CRC Press, 2004.

[6] R. Downey and D. R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer, Berlin, 2008.

[7] P. Gács. Uniform test of algorithmic randomness over a general space. *Theoretical Computer Science*, 341:91–137, 2005.

[8] P. Gács, M. Hoyrup, and C. Rojas. Randomness on computable probability spaces - a dynamical point of view. *Symposium on Theoretical Aspects of Computer Science*, pages 469–480, 2009.

[9] P. Hartman and A. Wintner. On the law of the iterated logarithm. *American J. Math.*, 63:169–176, 1941.

[10] P. Hertling and K. Weihrauch. Randomness spaces. In *Automata, Languages and Programming, Proceedings of the 25th International Colloquium, ICALP'98*, pages 796–807. Springer-Verlag, 1998.

[11] M. Hoyrup and C. Rojas. Applications of effective probability theory to martin-löf randomness. In *ICALP (1)*, pages 549–561, 2009.

[12] M. Hoyrup and C. Rojas. Computability of probability measures and martin-löf randomness over metric spaces. *Information and Computation*, 207(7):830–847, 2009.

[13] S. Kautz. *Degrees of Random Sets*. PhD thesis, Cornell University, 1991.

[14] A. Y. Khinchin. über einen satz der wahrscheinlichkeitrechnung. *Fund. Mat.*, 6:9–20, 1924.

[15] A. Kolmogorov. *Grundbegriffe der Wahrscheinlichkeitsrechnung*. Springer, 1933.

[16] A. N. Kolmogorov. über das gesetz des iterierten logarithmus. *Math. Ann.*, 101:126–135, 1929.

[17] A. N. Kolmogorov. Sur la loi forte des grands nombres. *CRAS Paris*, 191:910–912, 1930.

[18] C. Kreitz and K. Weihrauch. Theory of representations. *Theoretical Computer Science*, 38:35–53, 1985.

[19] L. A. Levin. *Some Theorems on the Algorithmic Approach to Probability Theory and Information Theory*. PhD thesis, Moscow, 1971.

[20] M. Li and P. Vitányi. *An introduction to Kolmogorov complexity and its applications*. Graduate Texts in Computer Science. Springer-Verlag, New York, 2007.

[21] P. Martin-Löf. The definition of random sequences. *Information and Control*, 9(6):602–619, 1966.

[22] J. S. Miller. Degrees of unsolvability of continuous functions. *The Journal of Symbolic Logic*, 69(2):555–584, 2004.

[23] K. Miyabe. Algorithmic randomness over general spaces. submitted (you can download a preprint from ”http://www.kurims.kyoto-u.ac.jp/ kmiyabe/papers/randspace.pdf”).

[24] K. Miyabe. An extension of van lambalgen’s theorem to infinitely many relative 1-random reals. to appear in Notre Dame Journal of Formal Logic.

[25] P. Odifreddi. *Classicla Recursion Theory*, volume 1. North-Holland, 1990.

[26] P. Odifreddi. *Classical Recursion Theory*, volume 2. North-Holland, 1999.

[27] V. V. Petrov. *Limit theorems of probability theory: sequences of independent random variables*. Oxford University Press, USA, 1995.

[28] R. L. Schilling. *Measures, integrals and martingales*. Cambridge University Press, 2005.

[29] C. Schnorr. A unified approach to the definition of a random sequence. *Mathematical Systems Theory*, 5:246–258, 1971.

[30] C. P. Schnorr. Process complexity and effective random tests. *Journal of Computer and System Sciences*, 7:376–388, 1973.

[31] G. Shafer and V. Vovk. *Probability and Finance: It’s Only a Game!* Wiley, 2001.

[32] R. I. Soare. *Recursively enumerable sets and degrees*. Perspectives in Mathematical Logic. Springer, Berlin, 1987.

[33] W. Stephen. *General Topology*. Dover Publications, 2004.

[34] R. von Mises. Grundlagen der wahrscheinlichkeistrechnung. *Mathematische Zeitschrift*, 5:52–99, 1919.

[35] V. G. Vovk. The law of the iterated logarithm for random kolmogorov, or chaotic, sequences. *Theory Probab. Appl.*, 32:413–425, 1987.

[36] V. G. Vovk. Kolmogorov-stout law of the iterated logarithm. *Mathematical Notes*, 44(1), 1988.

[37] K. Weihrauch. *Computable Analysis: an introduction*. Springer, Berlin, 2000.

[38] K. Weihrauch. The computable multi-functions on multi-represented sets are closed under programming. *Journal of Universal Computer Science*, 14(6):801–844, 2008.

[39] K. Weihrauch and T. Grubba. Elementary computable topology. *Journal of Universal Computer Science*, 15(6):1381–1422, 2009.

[40] D. Williams. *Probability with Martingales*. Cambridge University Press, 1991.