Algorithmic randomness over general spaces

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Summary

We study algorithmic randomness over a computable topological space. We propose two notions of randomness:

- **Measure randomness** is defined by a test concept and is characterized by martingales.
- **Complexity randomness** is defined by complexity.

In general they are different but under a condition they coincide. We prove van Lambalgen’s theorem under the condition.
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Martin-Löf randomness

Cantor space $2^\omega$ is the set of all infinite binary sequences. The topology is the one generated by the cylinder sets $[w] = \{ A \in 2^\omega : w \preceq A \}$. The measure $\mu$ is induced by $\mu([w]) = 2^{-|w|}$.

A open set $W$ is c.e. if $W = \bigcup_{w \in V} [w]$ for some c.e.

Definition 1 (Martin-Löf 1966). A Martin-Löf test is a uniformly c.e. open set $U_n$ with $\mu(U_n) \leq 2^{-n}$.

A sequence $A$ is Martin-Löf random if it passes all Martin-Löf test, that is, $A \notin \bigcap_n U_n$.

Theorem 2. There is a universal Martin-Löf test. The class of Martin-Löf random sequences has measure 1.
The prefix-free Kolmogorov complexity $K$ of $w$ is defined as

$$K(w) = \{|u| : U(u) = w\}$$

where $U$ is the universal prefix-free Turing machine.

**Theorem 3.** A sequence $A$ is Martin-Löf random iff

$$K(A \upharpoonright n) \geq n - O(1).$$

A martingale is a function $d : 2^* \rightarrow \mathbb{R}^+$ satisfying

$$2d(w) = d(w0) + d(w1) \text{ for all } w \in 2^*.$$

**Theorem 4 (Schnorr 1971).** A sequence $A$ is Martin-Löf random random iff no c.e. martingale succeeds on $A$, that is,

$$\sup_n d(A \upharpoonright n) < \infty \text{ for all } d.$$

This coincidence is one of the reasons of the fact that Martin-Löf randomness is considered a natural randomness.
A function $f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ is *computable* if it is computed by Type-2 machine. Informally, a *Type-2 machine* is a Turing machine, which reads from input tapes with finite or infinite inscription, operates on work tapes and write one-way to an output tape.

A *representation* of a set $M$ is a surjective function $\gamma : \subseteq Y \rightarrow M$ where $Y \in \{\Sigma^*, \Sigma^\omega\}$. A point is $\gamma$-*computable* if it has a computable representation by $\gamma$.

A function $f : \subseteq M_1 \rightarrow M_2$ is $(\gamma_1, \gamma_2)$-*computable* if it has a computable realization.
Computable topological spaces

**Definition 5** (Hertling and Weihrauch 2009). A computable topological space or CTS is a 4-tuple $X = (X, \tau, \beta, \nu)$ such that

- $(X, \tau)$ is a topological $T_0$-space,
- $\nu : \subseteq \Sigma^* \rightarrow \beta$ is a notation of a base $\beta$ of $\tau$,
- $\text{dom}(\nu)$ is recursive and
- $\nu(u) \cap \nu(v) = \bigcup\{\nu(w) : (u, v, w) \in S\}$ for all $u, v \in \text{dom}(\nu)$ for some r.e. set $S \subseteq (\text{dom}(\nu))^3$. 

To encode sequences of $\Sigma^*$ in $\Sigma^\omega$, we use the notation $u \ll p$ for $u \in \Sigma^*$ and $p \in \Sigma^\omega$.

**Definition 6.** Let $X = (X, \tau, \beta, \nu)$ be a computable topological space. Define a representation $\delta : \subseteq \Sigma^\omega \to X$ of the points as

$$x = \delta(p) \iff (\forall w \in \Sigma^*)(w \ll p \iff x \in \nu(w))$$

and a representation $\theta : \subseteq \Sigma^\omega \to \tau$ of the set of open sets as

$$W = \theta(p) \iff \begin{cases} w \ll p \Rightarrow w \in \text{dom}(\nu) \\ W = \bigcup \{\nu(w) : w \ll p\} \end{cases}.$$
**Example 7.**  
(i) **(real line)** Define $R = (\mathbb{R}, \tau_\mathbb{R}, \beta, \nu)$ such that $\tau_\mathbb{R}$ is the real line topology and $\nu$ is a canonical notation of the set of all open intervals with rational endpoints. The representation $\delta$ for $R$ is denoted by $\rho$.

(ii) **(lower unit interval)** Define $I_\mathbb{R} = (\mathbb{I}, \tau_\mathbb{I}, \beta_\mathbb{I}, \nu_\mathbb{I})$ such that the representation for $I_\mathbb{I}$ is denoted by $\rho_\mathbb{I}$.

(iii) **(extended real line)** Define $\overline{R}_\mathbb{I} = (\mathbb{R} \cup \{+\infty\}, \tau_\mathbb{R}, \beta_\mathbb{R}, \nu_\mathbb{R})$ such that $\nu_\mathbb{R}(w) = \{x : x > q \text{ and } q \in \nu_\mathbb{Q}\}$. The representation $\delta$ for $\overline{R}_\mathbb{I}$ is denoted by $\overline{\rho}_\mathbb{I}$.
Randomness by a test concept

Definition 8 (essentially Hertling and Weihrauch 1998). A test over $X$ is a uniformly $\theta$-computable sequence $\{U_n\}$ of open sets with $\mu(U_n) \leq 2^{-n}$ for all $n$. A point $x$ is measure $\mu$-random over $X$ if $x \notin \bigcap_n U_n$ for each measure test $\{U_n\}$.

They proved existence of a universal test under the condition that the measure is weakly bounded.

Question 9. Why do we need such a condition?
Randomness over a CPS

A “CMS” is a computable metric space and “CPS” is a CMS with a computable probability measure. Hoyrup and Rojas 2009 gave a characterization by complexity over a CPS. However the characterization cannot be generalized to over a CTS.

**Question 10.** Does there exist a natural randomness by complexity over a CTS?
Questions

- What conditions are needed for the existence of a universal test?
- Does there exist a natural randomness by complexity?
- Does there exist a characterization by martingales?
- Should the measure be probabilistic?
Randomness over a CTS

❖ The space of measures
❖ The case of a CMS
❖ A-topology
❖ The case of CTS
❖ Natural properties
❖ Definition by tests
❖ Definition by function tests
❖ Definition by martingales
❖ Universal complexity of sequences
❖ Definition by complexity
❖ Complete base

When they coincide

Relative randomness

Summary
For a topological space $X$, let $M(X)$ be the space of bounded non-negative Borel measures on $X$ and $P(X)$ be the subclass of probability measures. In this slide we assume that $M(X)$ is equipped with the usual topology of weak convergence. It is known that

- $X$ is countable iff $M(X)$ is countable,
- $X$ is separable iff $M(X)$ is separable,
- $X$ is compact iff $M(X)$ is compact,
- and other similar relations of the descriptive complexity.
The case of a CMS

**Theorem 11** (Hoyrup and Rojas 2009). Let \((X, d, S)\) be a computable metric space. Then \((P(X), p, D)\) is a computable metric space.

A computable measure is defined as a computable point in the computable metric space.
We use A-topology which coincides with the weak topology on a metric space.
We assume that the space $X$ is second-countable.
Then we can characterize A-topology as follows.

**Proposition 12.** The following sets form a countable subbase of the A-topology $\tau_A$:

\[
\{ \mu : \mu(G) > q \}, \quad \{ \mu : \mu(X) < q \},
\]

where $G$ is the finite union of base sets and $q \in \mathbb{Q}$.
The case of CTS

Theorem 13. Let $X = (X, \tau, \beta, \nu)$ be a computable topological space. Let $\beta_A$ be the base generated by the above subbase and $\nu_A$ be a natural computable notation of $\beta_A$. Then $(M(X), \tau_A, \beta_A, \nu_A)$ is a computable topological space.

Definition 14. A measure is computable if it is a computable point in the computable topological space.
Natural properties

Proposition 15. We denote the set of non-negative reals by $\mathbb{R}^+$. Let $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$.

(i) The operation $\text{eval} : \mathcal{M}(X) \times \tau \to \mathbb{R}^+$ such that $\text{eval}(\mu, G) = \mu(G)$ is $(\delta_A, \theta, \rho_\prec)$-computable.

(ii) The integral operation $\int : C(X, \mathbb{R}^+) \times \mathcal{M}(X) \to \overline{\mathbb{R}}^+$ is $([\delta \to \rho_\prec], \delta_A, \bar{\rho}_\prec)$-computable.

(iii) The integral operation $\int : C_b(X, \mathbb{R}^+) \times \mathcal{M}(X) \to \mathbb{R}^+$ is $([\delta \to \rho], \delta_A, \rho)$-computable.
Definition 16 (essentially Hertling and Weihrauch 1998). A measure test over $X$ is a uniformly $\theta$-computable sequence $\{U_n\}$ of open sets with $\mu(U_n) \leq 2^{-n}$ for all $n$. A point $x$ is measure $\mu$-random over $X$ if $x \notin \bigcap_n U_n$ for each measure test $\{U_n\}$.

Note that $\mu$ need not to be probabilistic.
Definition by function tests

Inspired by a uniform test by Gács and Levin, we give characterization by a function test.
Let $\overline{\rho}_<$ be the representation of lower real line with infinity.

**Definition 17.** A function test over $X$ is a $(\delta, \overline{\rho}_<)$-computable function $f : X \to \overline{\mathbb{R}}$ such that $\mu f = \int_X f \, d\mu \leq 1$.

**Theorem 18.** A point $x$ is measure $\mu$-random iff $f(x) < \infty$ for each function test $f$. 
Definition by martingales

Let \((X, \mathcal{A}, \mu)\) be a measure space. A **filtration** is a sequence of sub-\(\sigma\)-algebra \((\mathcal{A}_n)\) such that \(\mathcal{A}_n \subseteq \mathcal{A}_{n+1}\) for each \(n\).

A sequence of \(\mathcal{A}\)-measurable functions \((f_n, \mathcal{A}_n)\) is called a **supermartingale** if \(\int f_n d\mu < \infty\) and \(\int_A f_n d\mu \geq \int_A f_{n+1} d\mu\) for all \(A \in \mathcal{A}_n\).

**Theorem 19.** A point \(x\) is measure \(\mu\)-random iff
\[
\sup_n f_n(x) < \infty \text{ for each } ([\nu, \delta], \bar{\rho}_\prec)-\text{computable supermartingale } (f_n, \mathcal{A}_n)
\]

**Proof idea.** Let \(U_{k,m} = \{y : \sup_{n \leq m} f_n(y) > 2^k\}\) and use Doob’s maximal inequality. \(\square\)
Let $f : \subseteq 2^* \rightarrow \Sigma^\omega$ be a prefix-free computable function.

$$K_f(p) = \min \{ \sigma : f(\sigma) = p \}.$$ 

If $p$ is not computable, $K_f(p) = \infty$ for all $f$.

**Theorem 20.** There exists a prefix-free computable function $U : \subseteq 2^* \rightarrow \Sigma^\omega$ such that

$$(\forall f)(\exists c)(\forall p)(\exists q) \theta(p) = \theta(q) \text{ and } K_U(q) \leq K_f(p) + c.$$ 

In the following we write $K$ to mean $K_U$. 


Definition by complexity

Let $\psi^-(p) = X \setminus \theta(p)$.

**Definition 21.** A point $x$ is complexity $\mu$-random if

$$x \in \psi^-(p) \Rightarrow K(p) \geq -\log \mu \psi^-(p) - O(1).$$

This definition is not a straightforward generalization.
**Complete base**

**Definition 22.** The base $\beta = \{\nu(i)\}$ is complete if all equivalent bases are computably reducible to $\nu$.

There exists such a complete base.

**Theorem 23.** A point $x$ is complexity $\mu$-random iff

$$x \in \xi(u) \Rightarrow K(u) \geq -\log \mu \xi(u) - O(1)$$

where $\xi(u) = \nu(u)^c$. 
When they coincide

❖ When they coincide
❖ Almost decidability
❖ The space has almost decidability
❖ Effectively regular
❖ One direction
❖ The other direction
❖ Some natural properties

Relative randomness

Summary
Do measure randomness and complexity randomness coincide?
In general they are different.

**Example 24.** For lower unit interval $I_<$ and Lebesgue measure $\mu$,

- the set of measure $\mu$-random points is $I \setminus \{1\}$ and
- the set of complexity $\mu$-random points is $I \setminus \{0\}$.

However they coincide on a computable metric space with a computable measure, so on a Cantor space too. We shall see the conditions on which they coincide.
**Almost decidability**

Preliminary

Randomness over a CTS

When they coincide

❖ When they coincide

❖ Almost decidability

❖ The space has almost decidability

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Relative randomness

Summary

**Definition 25** (representation of $G_\delta$-set). *Define a representation* $\psi_2^- : \sum^\omega \to A$ of the set of $G_\delta$-sets as

$$\psi_2^-(p) = \bigcap_i \theta(p_i)$$

*where* $p = \langle p_1, p_2, \ldots \rangle$.

Let $\delta_2$ be a representation of the points in $\{0, 1\}$.

**Definition 26** (almost decidability; adapted from Gács, Hoyrup and Rojas 2009). *A set* $A \subseteq X$ *is almost decidable if the function* $1_A : X \to \{0, 1\}$ *is* $(\delta, \delta_2)$-*computable in a* $\psi_2^-(p)$-*computable set with measure one.*
A set $A$ is *almost decidable* iff there are two $\theta$-computable open sets $U$ and $V$ such that:

$$U \subseteq A, \ V \subseteq A^c, \ U \cup V \text{ has measure one.}$$

A CTS with a measure has the property of almost decidability if there is an equivalent basis that is uniformly almost decidable. In a CPS, such a basis exists.
Effectively regular

**Definition 27** (regular). A non-negative measure $\mu$ is regular if for every $A \in \mathcal{A}$ and every $\epsilon > 0$, there exists a closed set $F_\epsilon$ such that $F_\epsilon \subset A$, $A \setminus F_\epsilon \in \mathcal{A}$ and $\mu(A \setminus F_\epsilon) < \epsilon$.

**Definition 28** (effectively regular). A non-negative measure $\mu$ is effectively regular if there exist closed sets $F_{u,i}$ uniformly in $u$ and $i$ such that $F_{u,i} \subset \nu(u)$ and $\mu(\nu(u) \setminus F_{u,i}) < 2^{-i}$.

This property also holds in a CPS for a uniformly almost decidable basis.
Suppose $\mu$ is computable. This is an essential hypothesis.

**Theorem 29.** With almost decidability, complexity \( \mu \)-randomness implies measure \( \mu \)-randomness.

Each measure test is as a computably countable union of almost decidable base elements. Then \( i \)-th element is covered by a complement of a base element.

By getting rid of the union of base elements until \( i - 1 \), we get a sequence of closed sets containing all non-random points. The sum of measures of such sets is at most 1.
The other direction

Theorem 30. With effective regularity, measure $\mu$-randomness implies complexity $\mu$-randomness.

Each closed set can be covered by a base element with as little loss as you want.

Theorem 31. With effective regularity, there exists a universal test.

Remark 32. Grubba and Weihrauch (2007) proved that a computably regular space is computably metrizable.
Some natural properties

Theorem 33. The set of complexity random points has measure 1.

It is obvious for measure randomness, but it needs another proof for complexity randomness. Recall that computable permutations preserves randomness.

Theorem 34. Let \( f : X_1 \rightarrow X_2 \) s.t. \( \mu_1(f^{-1}(V)) \leq C\mu_2(V) \) for all open \( V \subseteq X_2 \).

If \( x \in \text{dom}(f) \) is complexity \( \mu_1 \)-random then \( f(x) \) is complexity \( \mu_2 \)-random.
Relative randomness

Definition

Coincide

Van Lambalgen’s Theorem

Situation

Summary

Relative randomness

Algorithmic randomness over general spaces
Definition

For $i = 1, 2$ let $X_i = (X_i, \tau_i, \beta_i, \nu_i)$ be computable topological spaces with complete bases and $\mu_i$ be a measure on $X_i$. Let $x_1 \in X_1$.

Definition 35. A $x_1$-measure $\mu$-test over $X_2$ is a sequence $\{t_n\}$ of uniformly $(\delta, \theta)$-computable functions with $\mu(t_n(x_1)) \leq 2^{-n}$.

$y \in X_2$ is $x_1$-measure $\mu$-random if $x \not\in \bigcap_n t_n(x_1)$ for each measure test.

Definition 36. A $x$ is $x_1$-complexity $\mu$-random if

$$x \in \xi(u) \Rightarrow K_f(x_1)(u) \geq -\log(\mu(\xi(u))) - O(1)$$

for all $(\delta, \eta^*)$-computable functions $f \subseteq X \to F^*$ such that $\text{dom}(f(x_1)) \subseteq 2^*$ and $f(x_1)$ is prefix-free.
Coincide

Note that if $x_1$ is computable, then $x_1$-randomness coincides with non-relativized randomness. Almost all properties can be relativized.

**Theorem 37.** *With almost decidability and effective regularity, measure randomness and complexity randomness coincide.*
Van Lambalgen’s Theorem

Theorem 38. If \( \langle x_1, x_2 \rangle \in \overline{X} \) is measure \( \mu \)-random, then \( x_1 \) is measure \( \mu_1 \)-random.

Theorem 39. If \( \mu_2 \) is effective regular and \( \langle x_1, x_2 \rangle \in \overline{X} \) is measure \( \overline{\mu} \)-random, then \( x_2 \) is \( x_1 \)-complexity \( \mu_2 \)-random.

Theorem 40. If \( x_1 \) is measure \( \mu_1 \)-random and \( x_2 \) is \( x_1 \)-measure \( \mu_2 \)-random, then \( \langle x_1, x_2 \rangle \in \overline{X} \) is measure \( \overline{\mu} \)-random.

Theorem 41. With almost decidability and effective regularity, van Lambalgen’s Theorem holds.
I do not know whether van Lambalgen’s Theorem holds with no conditions. Recall that van Lambalgen’s theorem is a criterion of natural randomness. So the condition is a sufficient condition for natural randomness.
Preliminary
Randomness over a CTS
When they coincide
Relative randomness
Summary
❖ Discussion
❖ End

Summary
We proposed two randomnesses: measure randomness and complexity randomness.

The measure need not to be probabilistic for the definition of randomness.

With effective regularity, there exists a universal test.

With a condition, two randomnesses coincide.

With a condition, van Lambalgen’s theorem holds for it.

The condition is a sufficient condition of the space where natural randomness can be defined.
Thank you!