Algorithmic randomness over general spaces

Kenshi Miyabe

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Preliminary

Summary

We study algorithmic randomness over a computable topological space.
We propose two notions of randomness:

- Measure randomness is defined by a test concept and is characterized by martingales.
- Complexity randomness is defined by complexity.

In general they are different but under a condition they coincide.
We prove van Lambalgen's theorem under the condition.

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Martin-Löf randomness

Cantor space 2^ω is the set of all infinite binary sequences.
The topology is the one generated by the cylinder sets $[w] = \{A \in 2^\omega \ : \ w \preceq A\}$.
The measure μ is induced by $\mu([w]) = 2^{-|w|}$.
A open set W is *c.e.* if $W = \bigcup_{w \in V} [w]$ for some c.e.

Definition 1 (Martin-Löf 1966). *A* Martin-Löf test *is a uniformly c.e. open set U_n with $\mu(U_n) \leq 2^{-n}$.*
A sequence A is Martin-Löf random *if it passes all Martin-Löf test, that is, $A \notin \bigcap_n U_n$.*

Theorem 2. *There is a universal Martin-Löf test.*
The class of Martin-Löf random sequences has measure 1.

Complexity and martingales

The *prefix-free Kolmogorov complexity* K of w is defined as $K(w) = \{|u| \ : \ U(u) = w\}$ where U is the universal prefix-free Turing machine.

Theorem 3. *A sequence A is Martin-Löf random iff $K(A \restriction n) \geq n - O(1)$.*

A martingale is a function $d : 2^* \to \mathbb{R}^+$ satisfying $2d(w) = d(w0) + d(w1)$ for all $w \in 2^*$.

Theorem 4 (Schnorr 1971)**.** *A sequence A is Martin-Löf random iff no c.e. martingale succeeds on A, that is, $\sup_n d(A \restriction n) < \infty$ for all d.*

This coincidence is one of the reasons of the fact that Martin-Löf randomness is considered a natural randomness.

Representations

A function $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ is *computable* if it is computed by Type-2 machine.

Informally, a *Type-2 machine* is a Turing machine, which reads from input tapes with finite or infinite inscription, operates on work tapes and write one-way to an output tape.

A *representation* of a set M is a surjective function $\gamma :\subseteq Y \to M$ where $Y \in \{\Sigma^*, \Sigma^\omega\}$.

A point is γ-*computable* if it has a computable representation by γ.

A function $f :\subseteq M_1 \to M_2$ is (γ_1, γ_2)-*computable* if it has a computable realization.

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \uparrow \gamma_1 & & \uparrow \gamma_2 \\ Y_1 & \xrightarrow{\text{comp}} & Y_2 \end{array}$$

Computable topological spaces

Definition 5 (Hertling and Weihrauch 2009)**.** *A* computable topological space *or* CTS *is a 4-tuple* $\mathbf{X} = (X, \tau, \beta, \nu)$ *such that*

- (X, τ) *is a topological* T_0*-space,*
- $\nu :\subseteq \Sigma^* \to \beta$ *is a notation of a base* β *of* τ*,*
- *dom*(ν) *is recursive and*
- $\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \ : \ (u, v, w) \in S\}$ *for all* $u, v \in$ *dom*(ν) *for some r.e. set* $S \subseteq ($*dom*$(\nu))^3$*.*

Representations for a CTS

To encode sequences of Σ^* in Σ^ω, we use the notation $u \ll p$ for $u \in \Sigma^*$ and $p \in \Sigma^\omega$.

Definition 6. *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be a computable topological space.*
Define a representation $\delta :\subseteq \Sigma^\omega \to X$ *of the points as*

$$x = \delta(p) \iff (\forall w \in \Sigma^*)(w \ll p \iff x \in \nu(w))$$

and a representation $\theta :\subseteq \Sigma^\omega \to \tau$ *of the set of open sets as*

$$W = \theta(p) \iff \begin{cases} w \ll p \Rightarrow w \in \mathit{dom}(\nu) \\ W = \bigcup\{\nu(w) \ : \ w \ll p\} \end{cases} .$$

Examples

Example 7. (i) *(real line) Define* $\mathbf{R} = (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ *such that* $\tau_{\mathbb{R}}$ *is the real line topology and* ν *is a canonical notation of the set of all open intervals with rational endpoints. The representation* δ *for* $\mathbf{R}$ *is denoted by* ρ*.*

(ii) *(lower unit interval) Define* $\mathbf{I}_< = (\mathbb{I}, \tau_<, \beta_<, \nu_<)$ *such that* $\nu_<(w) = \{x \,:\, 0 \leq q < x \leq 1$ *and* $q \in \nu_{\mathbb{Q}}\}$*. The representation* δ *for* $\mathbf{I}_<$ *is denoted by* $\rho_<$*.*

(iii) *(extended real line) Define* $\overline{\mathbf{R}}_< = (\mathbb{R} \cup \{+\infty\}, \tau_<, \beta_<, \nu_<)$ *such that* $\nu_<(w) = \{x \,:\, x > q$ *and* $q \in \nu_{\mathbb{Q}}\}$*. The representation* δ *for* $\overline{\mathbf{R}}_<$ *is denoted by* $\overline{\rho}_<$*.*

Randomness by a test concept

Definition 8 (essentially Hertling and Weihrauch 1998). *A* test *over* $\mathbf{X}$ *is a uniformly* θ*-computable sequence* $\{U_n\}$ *of open sets with* $\mu(U_n) \le 2^{-n}$ *for all* n*.*
A point x *is* measure μ-random over $\mathbf{X}$ *if* $x \notin \bigcap_n U_n$ *for each measure test* $\{U_n\}$*.*

They proved existence of a universal test under the condition that the measure is weakly bounded.

Question 9. *Why do we need such a condition?*

Randomness over a CPS

A “CMS” is a computable metric space and “CPS” is a CMS with a computable probability measure.
Hoyrup and Rojas 2009 gave a characterization by complexity over a CPS.
However the characterization can not be generalized to over a CTS.

Question 10. *Does there exist a natural randomness by complexity over a CTS?*

Questions

- What conditions are needed for the existence of a universal test?
- Does there exist a natural randomness by complexity?
- Does there exist a characterization by martingales?
- Should the measure be probabilistic?

Randomness over a CTS

The space of measures

For a topological space X, let $M(X)$ be the space of bounded non-negative Borel measures on X and $P(X)$ be the subclass of probability measures.
In this slide we assume that $M(X)$ is equipped with the usual topology of weak convergence.
It is known that

- X is countable iff $M(X)$ is countable,
- X is separable iff $M(X)$ is separable,
- X is compact iff $M(X)$ is compact,
- and other similar relations of the descriptive complexity.

The case of a CMS

Theorem 11 (Hoyrup and Rojas 2009). *Let (X, d, S) be a computable metric space. Then $(P(X), p, D)$ is a computable metric space.*

A computable measure is defined as a computable point in the computable metric space.

A-topology

We use A-topology which coincides with the weak topology on a metric space.
We assume that the space X is second-countable.
Then we can characterize A-topology as follows.

Proposition 12. *The following sets form a countable subbase of the A-topology τ_A:*

$$\{\mu \ : \ \mu(G) > q\}, \ \{\mu \ : \ \mu(X) < q\},$$

where G is the finite union of base sets and $q \in \mathbb{Q}$.

The case of CTS

Theorem 13. *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be a computable topological space. Let* β_A *be the base generated by the above subbase and* ν_A *be a natural computable notation of* β_A*. Then* $(M(X), \tau_A, \beta_A, \nu_A)$ *is a computable topological space.*

Definition 14. *A measure is computable if it is a computable point in the computable topological space.*

Natural properties

Proposition 15. *We denote the set of non-negative reals by* $\mathbb{R}^+$. *Let* $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$.

(i) *The operation* $eval : \mathcal{M}(X) \times \tau \to \mathbb{R}^+$ *such that* $eval(\mu, G) = \mu(G)$ *is* $(\delta_A, \theta, \rho_<)$*-computable.*

(ii) *The integral operation* $\int : C(X, \mathbb{R}^+) \times \mathcal{M}(X) \to \overline{\mathbb{R}}^+$ *is* $([\delta \to \rho_<], \delta_A, \overline{\rho}_<)$*-computable.*

(iii) *The integral operation* $\int : C_b(X, \mathbb{R}^+) \times \mathcal{M}(X) \to \mathbb{R}^+$ *is* $([\delta \to \rho], \delta_A, \rho)$*-computable.*

Definition by tests

Definition 16 (essentially Hertling and Weihrauch 1998). *A* measure test *over* $\mathbf{X}$ *is a uniformly* θ*-computable sequence* $\{U_n\}$ *of open sets with* $\mu(U_n) \leq 2^{-n}$ *for all* n*.*
A point x *is measure* μ*-random over* $\mathbf{X}$ *if* $x \notin \bigcap_n U_n$ *for each measure test* $\{U_n\}$*.*

Note that μ need not to be probabilistic.

Definition by function tests

Inspired by a uniform test by Gács and Levin, we give characterization by a function test.
Let $\overline{\rho}_<$ be the representation of lower real line with infinity.

Definition 17. *A* function test *over* $\mathbf{X}$ *is a* $(\delta, \overline{\rho}_<)$*-computable function* $f : X \to \overline{\mathbb{R}}$ *such that* $\mu f = \int_X f d\mu \leq 1$.

Theorem 18. *A point* x *is measure* μ*-random iff* $f(x) < \infty$ *for each function test* f*.*

Definition by martingales

Let $(X, \mathcal{A}, \mu)$ be a measure space.
A *filtration* is a sequence of sub-σ-algebra $(\mathcal{A}_n)$ such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for each n.
A sequence of $\mathcal{A}$-measurable functions $(f_n, \mathcal{A}_n)$ is called a *supermartingale* if $\int f_n d\mu < \infty$ and $\int_A f_n d\mu \geq \int_A f_{n+1} d\mu$ for all $A \in \mathcal{A}_n$.

Theorem 19. *A point x is measure μ-random iff $\sup_n f_n(x) < \infty$ for each $([\nu_{\mathbb{N}}, \delta], \overline{\rho}_<)$-computable supermartingale $(f_n, \mathcal{A}_n)$*

Proof idea. Let $U_{k,m} = \{y \,:\, \sup_{n \leq m} f_n(y) > 2^k\}$ and use Doob's maximal inequality. □

Universal complexity of sequences

Let $f :\subseteq 2^* \to \Sigma^\omega$ be a prefix-free computable function.

$$K_f(p) = \min\{\sigma \;:\; f(\sigma) = p\}.$$

If p is not computable, $K_f(p) = \infty$ for all f.

Theorem 20. *There exists a prefix-free computable function $U :\subseteq 2^* \to \Sigma^\omega$ such that*

$$(\forall f)(\exists c)(\forall p)(\exists q)\theta(p) = \theta(q) \text{ and } K_U(q) \leq K_f(p) + c.$$

In the following we write K to mean K_U.

Definition by complexity

Let $\psi^{-}(p) = X \setminus \theta(p)$.

Definition 21. *A point x is complexity μ-random if*

$$x \in \psi^{-}(p) \Rightarrow K(p) \geq -\log \mu\psi^{-}(p) - O(1).$$

This definition is not a straightforward generalization.

Complete base

Definition 22. *The base $\beta = \{\nu(i)\}$ is complete if all equivalent bases are computably reducible to ν.*

There exists such a complete base.

Theorem 23. *A point x is complexity μ-random iff*

$$x \in \xi(u) \Rightarrow K(u) \geq -\log \mu\xi(u) - O(1)$$

where $\xi(u) = \nu(u)^c$.

When they coincide

When they coincide

Do measure randomness and complexity randomness coincide?
In general they are different.

Example 24. For lower unit interval $\mathbf{I}_<$ *and Lebesgue measure* μ*,*

- *the set of measure* μ*-random points is* $I \backslash \{1\}$ *and*
- *the set of complexity* μ*-random points is* $I \backslash \{0\}$*.*

However they coincide on a computable metric space with a computable measure, so on a Cantor space too.
We shall see the conditions on which they coincide.

Almost decidability

Definition 25 (representation of G_δ-set)**.** *Define a representation* $\psi_2^- : \Sigma^\omega \to \mathcal{A}$ *of the set of* G_δ*-sets as*

$$\psi_2^-(p) = \bigcap_i \theta(p_i)$$

where $p = \langle p_1, p_2, \ldots \rangle$*.*

Let δ_2 be a representation of the points in $\{0, 1\}$.

Definition 26 (almost decidability; adapted from Gács, Hoyrup and Rojas 2009)**.** *A set* $A \subseteq X$ *is* almost decidable *if the function* $1_A : X \to \{0, 1\}$ *is* (δ, δ_2)*-computable in a* $\psi_2^-(p)$*-computable set with measure one.*

The space has almost decidability

A set A is *almost decidable* iff there are two θ-computable open sets U and V such that:

$$U \subset A,\ V \subseteq A^c,\ U \cup V \text{ has measure one.}$$

A CTS with a measure has the property of almost decidability if there is an equivalent basis that is uniformly almost decidable.
In a CPS, such a basis exists.

Effectively regular

Definition 27 (regular)**.** *A non-negative measure* μ *is* regular *if for every* $A \in \mathcal{A}$ *and every* $\epsilon > 0$*, there exists a closed set* F_ϵ *such that* $F_\epsilon \subset A$*,* $A \backslash F_\epsilon \in \mathcal{A}$ *and* $\mu(A \backslash F_\epsilon) < \epsilon$*.*

Definition 28 (effectively regular)**.** *A non-negative measure* μ *is* effectively regular *if there exist closed sets* $F_{u,i}$ *uniformly in* u *and* i *such that* $F_{u,i} \subset \nu(u)$ *and* $\mu(\nu(u) \backslash F_{u,i}) < 2^{-i}$*.*

This property also holds in a CPS for a uniformly almost decidable basis.

One direction

Suppose μ is computable. This is an essential hypothesis.

Theorem 29. *With almost decidability, complexity μ-randomness implies measure μ-randomness.*

Each measure test is as a computably countable union of almost decidable base elements.
Then i-th element is covered by a complement of a base element.
By getting rid of the union of base elements until $i - 1$, we get a sequence of closed sets containing all non-random points.
The sum of measures of such sets is at most 1.

The other direction

Theorem 30. *With effective regularity, measure μ-randomness implies complexity μ-randomness.*

Each closed set can be covered by a base element with as little loss as you want.

Theorem 31. *With effective regularity, there exists a universal test.*

Remark 32. Grubba and Weihrauch (2007) proved that a computably regular space is computably metrizable.

Some natural properties

Theorem 33. *The set of complexity random points has measure 1.*

It is obvious for measure randomness, but it needs another proof for complexity randomness.
Recall that computable permutations preserves randomness.

Theorem 34. *Let $f : X_1 \to X_2$ s.t. $\mu_1(f^{-1}(V)) \leq C\mu_2(V)$ for all open $V \subseteq X_2$.*
If $x \in dom(f)$ is complexity μ_1-random then $f(x)$ is complexity μ_2-random.

Relative randomness

Definition

For $i = 1, 2$ let $\mathbf{X}_i = (X_i, \tau_i, \beta_i, \nu_i)$ be computable topological spaces with complete bases and μ_i be a measure on X_i. Let $x_1 \in X_1$.

Definition 35. *A x_1-measure μ-test over $\mathbf{X_2}$ is a sequence $\{t_n\}$ of uniformly (δ, θ)-computable functions with $\mu(t_n(x_1)) \leq 2^{-n}$.*
$y \in X_2$ is x_1-measure μ-random if $x \notin \bigcap_n t_n(x_1)$ for each measure test.

Definition 36. *A x is x_1-complexity μ-random if*

$$x \in \xi(u) \Rightarrow K_{f(x_1)}(u) \geq -\log(\mu(\xi(u))) - O(1)$$

for all (δ, η^{**})-computable functions $f :\subseteq X \to F^{**}$ such that dom$(f(x_1)) \subseteq 2^*$ and $f(x_1)$ is prefix-free.

Coincide

Note that if x_1 is computable, then x_1-randomness coincides with non-relativized randomness.
Almost all properties can be relativized.

Theorem 37. *With almost decidability and effective regularity, measure randomness and complexity randomness coincide.*

Van Lambalgen's Theorem

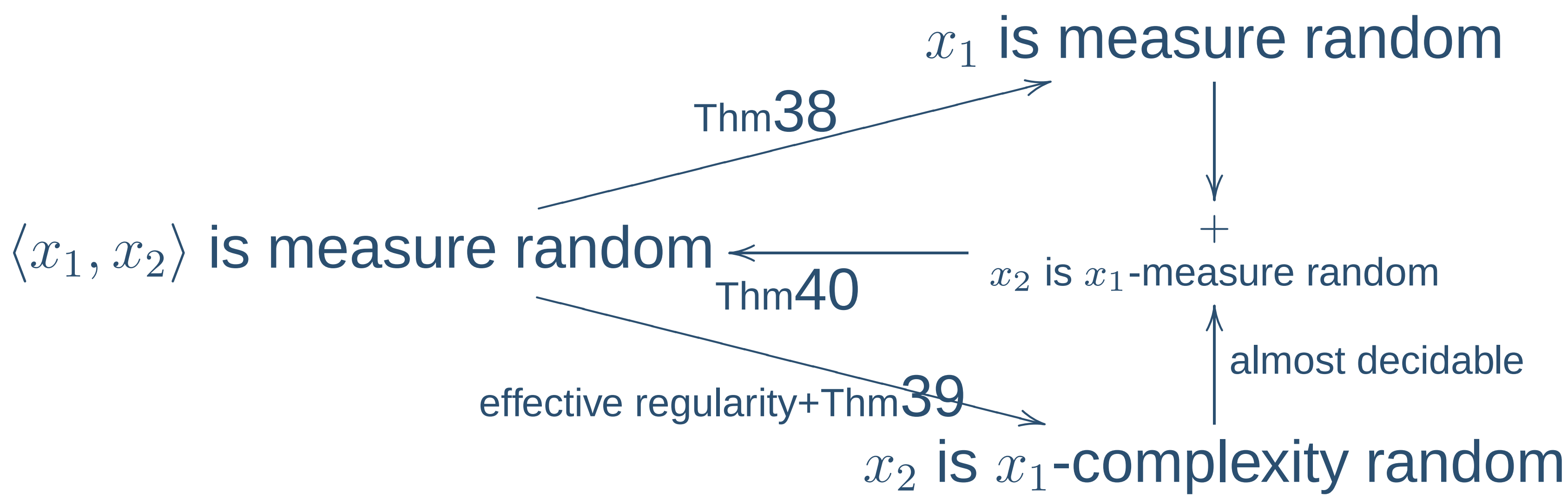
Theorem 38. *If* $\langle x_1, x_2 \rangle \in \overline{X}$ *is measure* $\overline{\mu}$*-random, then* x_1 *is measure* μ_1*-random.*

Theorem 39. *If* μ_2 *is effective regular and* $\langle x_1, x_2 \rangle \in \overline{X}$ *is measure* $\overline{\mu}$*-random, then* x_2 *is* x_1*-complexity* μ_2*-random.*

Theorem 40. *If* x_1 *is measure* μ_1*-random and* x_2 *is* x_1*-measure* μ_2*-random, then* $\langle x_1, x_2 \rangle \in \overline{X}$ *is measure* $\overline{\mu}$*-random.*

Theorem 41. *With almost decidability and effective regularity, van Lambalgen's Theorem holds.*

Situation



I do not know whether van Lambalgen's Theorem holds with no conditions.
Recall that van Lambalgen's theorem is a criterion of natural randomness.
So the condition is a sufficient condition for natural randomness.

Summary

Discussion

- We proposed two randomnesses: measure randomness and complexity randomness.
- The measure need not to be probabilistic for the definition of randomness.
- With effective regularity, there exists a universal test.
- With a condition, two randomnesses coincide.
- With a condition, van Lambalgen's theorem holds for it.
- The condition is a sufficient condition of the space where natural randomness can be defined.

End

Thank you!