Randomness in a dynamical system

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Randomness on a Cantor space

AIT

Algorithmic randomness is a subfield of Algorithmic Information Theory.

> AIT is the result of putting Shannon's information theory and Turing's computability theory into a cocktail shaker and shaking vigorously. (G. J. Chaitin)

- From "for almost every" to "for all random points"
- Random deficiency
- Computable points with statistical properties
- Three approaches to a concept.

Random sequence

2^ω: Cantor space of infinite binary sequences
μ: uniform Bernoulli measure on 2^ω
T: left shift, $T(x_0x_1x_2\ldots) = x_1x_2x_3\ldots$
T preserves μ.
For each $A \subseteq 2^\omega$, frequency of $T^n(x) \in A$ is $\mu(A)$ a.e.

Theorem 1 (Kučera 1985, Bienvenu et al.). *A point $x \in 2^\omega$ is Martin-Löf random iff $(\exists n)T^n(x) \notin A$ for all c.e. open set A with $\mu(A) < 1$.*

Moreover there exists universal c.e. open set A.

Questions?

- New interpretation via three approaches
- Another relation between a property in ergodic theory and a random concept
- Random deficiency
- Other operator T
- General space

Computability

A set $A \subseteq \mathbb{N}$ is *computable* iff $1_A : \mathbb{N} \to \{0,1\}$ is computable.
A set A is *c.e.* iff $A = \mathrm{dom}(f)$ for a partial computable function.
A real $r \in [0,1]$ is *computable* iff its binary expantion is computable.
A real r is *c.e.* iff $\{q \in \mathbb{Q} \ : \ q < r\}$ is c.e.
There exist a c.e. set and a c.e. real that are not computable.
base: a set of $[\sigma] = \{A \ : \ \sigma \preceq A\}$
A open set U is *c.e.* iff $U = \bigcup_{\sigma \in S}[\sigma]$ for a c.e. set S.
μ: generated from $\mu([\sigma]) = 2^{-|\sigma|}$
We often identify 2^ω with $[0,1]$.

Martin-Löf randomness

Typicalness: No effective null set contains the point.

Definition 2 (Martin-Löf 1966). $\{U_n\}$*: a seq. of open sets* $\{U_n\}$ *is a* test *iff it is unifomly c.e. and* $\mu(U_n) \leq 2^{-n}$.
A real A passes the test *iff* $A \notin \bigcap_n U_n$.
A *is* ML-radom *iff* A *passes all tests.*

There exists a universal test.
The set of ML-random reals has measure 1.

Complexity

Incompressibility: The prefix is hard to compress or describe.

Definition 3. *A set* $A \subseteq 2^*$ *is* prefix-free *iff no string is a prefix of another.*
A function $f : 2^* \to 2^*$ *is* prefix-free *iff its domain is prefix-free.*
A machine *is a partial computable function.*
The Kolmogorov complexity *of* σ *is*

$$K(\sigma) = \min\{|\tau| \ : \ V(\tau) = \sigma\}.$$

where V *is a universal prefix-free machine.*

Proposition 4 (Levin, Schnorr). A *is ML-random iff* $K(A \upharpoonright n) > n - O(1)$.

Martingale

Unpredictability: It is hard to predict the next bit.

Definition 5. *A function* $d : 2^* \to \mathbb{R}^+ \cup \{0\}$ *is a* martingale *iff*

$$d(\sigma) = \frac{d(\sigma 0) + d(\sigma 1)}{2}.$$

Proposition 6 (Schnorr 1971)**.** A *is ML-random iff* $\sup_n d(A \upharpoonright n) < \infty$ *for all c.e. martingales.*

There exists a universal c.e. martingale.

van Lambalgen's Theorem

Two parts of random sequence is relatively random each other.
Let $A \oplus B = A(0)B(0)A(1)B(1)\cdots$.

Theorem 7 (van Lambalgen's Theorem 1987). *$A \oplus B$ is ML-random iff A is ML-random and B is A-ML-random.*

This theorem holds for many randomnesses and is a criterion for a proper randomness.

An extension

Theorem 8 (M. 2010). *A_i: ML-random relative to $\oplus_{j<i} A_j$ $\exists B_i$ s.t. $B_i =^* A_i$ and $\oplus B_i$ is ML-random.*

Theorem 9 (Bienvenu et al.). *In the above theorem we can replace $B_i =^* A_i$ with deleting finite prefixes or adding finite strings.*

Random closed set

s-gale

d: a martingale
h: an order (unbounded non-decreasing)

Definition 10 (h-success set of d; Schnorr 1971)**.**

$$S_h[d] = \{A \ : \ \limsup_n \frac{d(A \restriction n)}{h(n)} = \infty\}.$$

Definition 11 (s-gale: Lutz 2000)**.**

$$f(\sigma) = 2^{-s}(f(\sigma 0) + f(\sigma 1)).$$

Theorem 12 (Lutz 2000)**.** $X \subseteq 2^\omega$

$$\begin{aligned} \dim_H(X) &= \inf\{s \ : \ (\exists f) X \subseteq S[f]\} \\ &= \inf\{s \ : \ (\exists d) X \subseteq S_{2^{(1-s)n}}[d]\} \end{aligned}$$

Effective Hausdorff dimension

Definition 13 (Effective Hausdorff dimension; Lutz 2000)**.**
d: a c.e. martingale

$$\dim(X) = \inf\{s \; : \; (\exists d) X \subseteq S_{2^{(1-s)n}}[d]\}$$

We also have a direct characterization with a cover.

Theorem 14 (Mayordomo 2002)**.** *For $A \in 2^\omega$,*

$$\dim(X) = \liminf_n \frac{K(A \upharpoonright n)}{n}.$$

Weakly s-ML-randomness

Definition 15 (Tadaki 2002). $\{U_n\}$ *is a* test for weak s-ML-randomness *iff they are uniformly c.e. sets of strings with* $\sum_{\sigma \in V_n} 2^{-s|\sigma|} \leq 2^{-n}$. A *is* weakly s-ML-random *iff* $A \notin \bigcap_n [\![V_n]\!]$.

A is weakly s-random if $K(A \upharpoonright n) \geq sn - O(1)$.

Theorem 16 (Tadaki 2002). *A is weakly s-ML-random iff A is weakly s-random.*

Proposition 17.

$$\dim(A) = \sup\{s \ : \ A \textit{ is weakly } s\textit{-random}\}.$$

A characterization by martingales is unknown.

Other dimensions

Some researchers also study the relation with the following dimension.

- box counting dimension
- packing dimension

Random closed set

- Barmpalias et al. (2007) introduced it.
- Kjos-Hanssen (2009) uses Galton-Watson trees to obtain a similar notion.
- Axon uses a notion of “random closed set” in probability theory.

Diamondstone et al. (2009) proved the equivalence of the first two.
Axon proved the equivalence on 2^ω.

Generalization

Type-2 Theory of Effectivity

Let $Y_0, \ldots, Y_n \in \{\Sigma^*, \Sigma^\omega\}$ and $Y = Y_1 \times \ldots \times Y_n$.
A function $f :\subseteq Y \to Y_0$ is *computable* if it is computed by Type-2 machine.
Informally, a *Type-2 machine* is a Turing machine, which reads from input tapes with finite or infinite inscription, operates on work tapes and write one-way to an output tape.
On Σ^* we consider the discrete topology.
On Σ^ω we consider the topology generated by the base $\{w\Sigma^\omega \ : \ w \in \Sigma^*\}$ of open sets.
Every computable function is continuous.

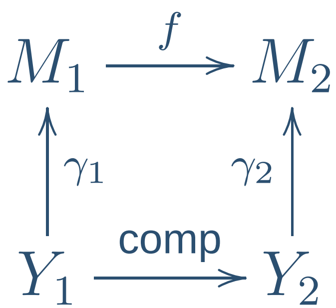
Representation

A *representation* of a set M is a surjective function $\gamma :\subseteq Y \to M$ where $Y \in \{\Sigma^*, \Sigma^\omega\}$.

Example 18. (i) $\nu_{\mathbb{Q}} :\subseteq \Sigma^* \to \mathbb{Q}$

(ii) $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$

A point is γ-*computable* if it has a computable representation by γ.
A function $f :\subseteq M_1 \to M_2$ is (γ_1, γ_2)-*computable* if it has a computable realization.



CTS

Definition 19 (Hertling and Weihrauch 2009). *A* computable topological space *is a 4-tuple* $\mathbf{X} = (X, \tau, \beta, \nu)$ *such that*

- (X, τ) *is a topological* T_0*-space,*
- $\nu :\subseteq \Sigma^* \to \beta$ *is a notation of a base* β *of* τ*,*
- *dom*(ν) *is recursive and*
- $\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \ : \ (u, v, w) \in S\}$ *for all* $u, v \in$ *dom*(ν) *for some r.e. set* $S \subseteq ($*dom*$(\nu))^3$*.*

Representations

Definition 20. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space.*
Define a representation $\delta :\subseteq \Sigma^\omega \to X$ of the points as

$$x = \delta(p) \iff (\forall w \in \Sigma^*)(w \ll p \iff x \in \nu(w))$$

and a representation $\theta :\subseteq \Sigma^\omega \to \tau$ of the set of open sets as

$$W = \theta(p) \iff \begin{cases} w \ll p \Rightarrow w \in \mathit{dom}(\nu) \\ W = \bigcup\{\nu(w) \ : \ w \ll p\} \end{cases} .$$

ML-randomness over a CTS

Definition 21 (Hertling and Weihrauch 2003)**.** $\{U_n\}$: *open sets*
$\{U_n\}$ *is a* test *iff uniformly* θ*-comp. with* $\mu(U_n) \leq 2^{-n}$.
x *is* measure μ-random *iff* $x \notin \bigcap_n U_n$ *for each test.*

Universality needs an unnatural condition.

Randomness over a CMS

Randomness over a computable metric space gets much attention.

- Gács (2005) first gave a restricted characterization by complexity.
- Hoyrup and Rojas (2009) gave a complete characterization by complexity.

The most important point was computability of measures.

Computability of measures

$M(X)$: the space of bounded non-negative Borel measures

Proposition 22. *G: the finite union of base sets*
A countable subbase of the A-topology τ_A:

$$\{\mu \ : \ \mu(G) > q\}, \ \{\mu \ : \ \mu(X) < q\}$$

Theorem 23. $(M(X), \tau_A, \beta_A, \nu_A)$ *is a CTS.*

Definition 24. *A measure is* computable *if it is a computable point in the CTS.*

By martingales

Let $(X, \mathcal{A}, \mu)$ be a measure space.
A *filtration* is a sequence of sub-σ-algebra $(\mathcal{A}_n)$ such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ for each n.
A sequence of $\mathcal{A}$-measurable functions $(f_n, \mathcal{A}_n)$ is called a *supermartingale* if $\int f_n d\mu < \infty$ and $\int_A f_n d\mu \geq \int_A f_{n+1} d\mu$ for all $A \in \mathcal{A}_n$.

Theorem 25. *A point x is measure μ-random iff $\sup_n f_n(x) < \infty$ for each $([\nu_{\mathbb{N}}, \delta], \overline{\rho}_<)$-computable supermartingale $(f_n, \mathcal{A}_n)$*

Proof idea. Let $U_{k,m} = \{y \; : \; \sup_{n \leq m} f_n(y) > 2^k\}$ and use Doob's maximal inequality. □

Complexity randomness

Suppose that the base is complete.

Definition 26. *A point* x *is* complexity μ-random *iff*

$$x \in \xi(u) \Rightarrow K(u) \geq -\log \mu\xi(u) - O(1)$$

where $\xi(u) = \nu(u)^c$.

Theorem 27. *Suppose that the space has almost decidability and the measure is effectively regular. Then two randomnesses coincide and van Lambalgen's Theorem holds for it.*

Randomness and Ergodic Theory

Effective Birkhoff's ergodic theorem

Definition 28. *T is $\ln^2$-ergodic if* $\forall f, g$
$\gamma_n(f,g) = |\frac{1}{n}\sum_{i<n}\int f \circ T^i \cdot g d\mu - \int f d\mu \int g d\mu| \leq \frac{c_{f,g}}{(\ln(n))^2}$

Theorem 29 (Birkhoff's ergodic theorem for random points; V'yugin '97, Nandakumar '08, Hoyrup and Rojas '09)**.**
(X,μ): a CPS
$T : X \to X$ an effectively measurable measure-preserving map
$f \in L^1(X.\mu)$ an effectively measurable function
T: $\ln^2$-ergodic
Then $\exists n(c,\epsilon)$ comp. function s.t.
if $x \in K_c$ then for all $n > n(c,\epsilon)$,
$|\frac{1}{n}\sum_{i<n} f \circ T^i \cdot g d\mu - \int f d\mu \int g d\mu| < \epsilon$

The effectiveness of the convergence speed can be also seen in Davie('01).

Other results

- Under a appropriate condition computable typical points are dense.
- Not all computable dynamical systems have a computable invariant measure.
- The complexity of the orbits of random points equals the Kolmogorov-Sinaï entropy of the system,

Probability

What is probability?

- Frequency probability
- Subjective probability
- Propensity probability

Modern mathematical formalization is by Kolmogorov (1933).

Von Mises's philosophy

Von Mises took a very different approach.

- His first paper on this subject is in 1919.
- His standpoint is empiricism.
- He claims that probability can be defined only for repetitive events.
- First the collective, then the probability (as its frequency).

Collective

A collective is a set of sequences s.t.

- their limits of relative frequency exist and are the same
- relative frequency are not affected by appropriate place selections

Ville (1940) constructed a sequence for which LIL does not hold.

Sigma-algebra

Σ_0 is an *algebra* on S iff

(i) $S \in \Sigma_0$,

(ii) $F \in \Sigma_0 \Rightarrow F^c = S \backslash F \in \Sigma_0$,

(iii) $F, G \in \Sigma_0 \Rightarrow F \cup G \in \Sigma_0$.

Σ is a σ-*algebra* iff it is an algebra and

$$F_n \in \Sigma \Rightarrow \bigcup_n F_n \in \Sigma$$

Probability axioms

P is a *finite probability function* on Σ_0 iff

(i) $P(A) \geq 0$ for $A \in \Sigma_0$,

(ii) $P(S) = 1$,

(iii) (finite additivity)
$A \cap B = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$.

P is a *probability function* on Σ iff it satisfies above and (countable additivity) $P(\bigcup F_n) = \sum_n P(F_n)$ for pairwise disjoint $F_n \in \Sigma$.
Note that P need not to be a measure.

Random sequence

Now we can consider a CTS $\mathbf{X}^{\mathbb{N}}$.
We define frequency as

$$F(A) = \lim_n \frac{\#\{i \leq n \ : \ x_i \in A\}}{n}.$$

Theorem 30. *Suppose that the base is uniformly almost decidable.*
$\mathcal{A}$*: the minimal algebra containing the base*

$$F|\mathcal{A} = \mu|\mathcal{A}$$

We can not extend to Borel sets.

Practically impossibile

$E \in \mathcal{A}$: an observable set

$$\mu(E) = 0 \Rightarrow E \cap \mathrm{cRND}(\mu) = \phi$$
$$\mu(E) = 1 \Rightarrow \mathrm{cRND}(\mu) \subseteq E$$

We need restricted countable additivity.

End

Thank you!