Degree of non-randomness and uniform Solovay reducibility

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Motivation

The degree of non-randomness

- 1001100111111001110110010111001010111...
- 0100000101000000101010101010000010101...
- 0000000000000000000000000000000000000...

Which sequence is random?
How random?

The speed of convergence

Let $K^c = \{A \ : \ K(A \upharpoonright n) > n - c\}$.

Theorem 1 (Davie '01)**.** *There exists a computable function $n(c, \epsilon)$ s.t. if $A \in K^c$ then for all $n > n(c, \epsilon)$,*

$$\left| \frac{S_n(A)}{n} - \frac{1}{2} \right| < \epsilon.$$

For c and $\lambda > 1$, there exists a computable function $n(c, \lambda)$ s.t. if $A \in K^c$ then for all $n > n(c, \lambda)$,

$$S_n(A) \leq \frac{n}{2} + \lambda \sqrt{\frac{n}{2} \log \log n}.$$

Layerwise computability

Let $K_c = X \backslash U_n$.

Definition 2. T *is* $\ln^2$*-ergodic if* $\forall f, g$
$\gamma_n(f,g) = |\frac{1}{n}\sum_{i<n}\int f \circ T^i \cdot g d\mu - \int f d\mu \int g d\mu| \leq \frac{c_{f,g}}{(\ln(n))^2}$

Theorem 3 (Birkhoff's ergodic theorem for random points; V'yugin '97, Nandakumar '08, Hoyrup and Rojas '09)**.**
(X,μ)*: a CPS*
$T : X \to X$ *an effectively measurable measure-preserving map*
$f \in L^1(X.\mu)$ *an effectively measurable function*
T*:* $\ln^2$*-ergodic*
Then $\exists n(c,\epsilon)$ *comp. function s.t.*
if $x \in K_c$ *then for all* $n > n(c,\epsilon)$*,*
$|\frac{1}{n}\sum_{i<n} f \circ T^i \cdot g d\mu - \int f d\mu \int g d\mu| < \epsilon$

The degree of non-randomness

Martin-Löf randomness

A sequence $\{U_n\}$ of open sets is a *ML-test* if it is uniformly c.e. and $\mu(U_n) \leq 2^{-n}$.
A binary sequence A is *ML-random* if $A \notin \bigcap_n U_n$ for all ML-tests.
A ML-test $\{U_n\}$ is *universal* if $\bigcap_n U_n \supseteq \bigcap_n V_n$ for each ML-test $\{V_n\}$.
There exists a universal ML-test.
A is ML-random iff $A \notin \bigcap_n U_n$ for a universal ML-test.

Definition

Definition 4. $U = \{U_n\}$*: a ML-test*

$$t_U(\sigma) = \min\{n \ : \ [\sigma] \subseteq U_n \text{ does not hold.}\}$$

$$t_U(A) = \min\{n \ : \ A \subseteq U_n \text{ does not hold.}\}$$

Proposition 5.

$$t_U(A) = \sup_n t_U(A \upharpoonright n)$$

Definition 6. *A ML-test* $U = \{U_n\}$ *is* optimal *if, for any ML-test* $V = \{V_n\}$, $t_U(\sigma) \geq t_V(\sigma) - O(1)$.

Proposition 7. *There exists an optimal ML-test.*

Optimality and Universality

A ML-test is *universal* iff, for each ML-test $V = \{V_n\}$, $t_V(A) = \infty \Rightarrow t_U(A) = \infty$ for all $A \in 2^\omega$.
Hence optimality implies universality.

Proposition 8. *Let $\{U_n\}$ be an optimal ML-test. Then $\mu(U_n) \geq 2^{-n-O(1)}$.*

Proposition 9. *There exists a ML-test such that it is universal but not optimal.*

Proof. Let $\{U_n\}$ be a universal Martin-Löf test. Then $V = \{V_n\} = \{U_{2n}\}$ is also a universal Martin-Löf test and we have $\mu(V_n) = \mu(U_{2n}) \leq 2^{-2n}$. By the proposition above, V is not optimal. □

Other complexity

Theorem 10. *Let d be an optimal c.e. supermartingale.*

$$\max_{\tau \preceq \sigma}(|\tau| - K(\tau)) \leq \max_{\tau \preceq \sigma} \log d(\tau) + O(1) \leq t(\sigma) + O(1).$$

Theorem 11.

$$t(A) \leq 2 \sup_{\tau \preceq A}(|\tau| - K(\tau)) + O(1).$$

It seems diffi cult to characterize t by a martingale or complexity.
However it may be possible over a computable topological space.

Uniform Solovay reducibility

Solovay reducibility

$\alpha \leq_s \beta$ if $\exists c$ and a partial comp. func. $f : \mathbb{Q} \to \mathbb{Q}$ s.t. $q \in \mathbb{Q}$ and $q < \beta \Rightarrow f(q) \downarrow < \alpha$ and $\alpha - f(q) < c(\beta - q)$.
A left-c.e. α is *Solovay complete or Ω-like* if $\beta \leq_s \alpha$ for all left-c.e. reals β.

Theorem 12 (Solovay '75, Kučera and Slaman '02)**.** *For left-c.e. reals α, the following are equivalent.*

(i) α *is 1-random.*

(ii) α *is Solovay complete.*

(iii) $\Omega \leq_s \alpha$.

(iv) α *is the measure of a component of a universal ML-test.*

Definition

$\{\alpha_k\}$, $\{\beta_k\}$: sequences of reals

Definition 13 (uniformly Solovay reducible or us-reducible)**.** $\alpha_k \leq_{us} \beta_k$ *if* $\exists c$ *and unif. partial comp. func.* $f_k : \mathbb{Q} \to \mathbb{Q}$ *s.t.* $f_k(0) = 0$ *and* $q \in \mathbb{Q}$ *and* $q < \beta_k$ *implies* $f_k(q) \downarrow < \alpha_k$ *and* $\alpha_k - f_k(q) < c(\beta_k - q)$.

The condition $f_k(0) = 0$ is equivalent to $\alpha_k < c\beta_k$ for some c.
$\alpha_k \leq_{us} \beta_k$ implies $\frac{\alpha_k}{\beta_k}$ is bounded.
Solovay reducibility does not need such a condition.

Some properties

$\{\alpha_k\}$, $\{\beta_k\}$: sequences of uniformly left-c.e. reals

Theorem 14. *TFAE:*

(i) $\alpha_k \leq_{us} \beta_k$.

(ii) *For any uniformly computable sequences* $b_1^k, b_2^k, \cdots$ *of non-negative rationals such that* $\beta_k = \sum_n b_n^k$*, there are a constant* c *and uniformly computable sequences of rationals* $\epsilon_{k,n} \in [0, c]$ *for all* n *such that* $\alpha_k = \sum_n \epsilon_{k,n} b_{k,n}$ *for all* k.

(iii) *There are a constand* c *and uniformly left-c.e. reals* γ_k *such that* $c\beta_k = \alpha_k + \gamma_k$ *for all* k.

us-complete sequence

Definition 15. *Let*

$$\Omega_k = \sum_{i=0}^{\infty} \sum_{M_i(0^k 1\sigma)\downarrow} 2^{-i-1-k-|\sigma|}.$$

Theorem 16. *The sequence $\{2^k \Omega_k\}$ is us-complete.*

Theorem 17. *If α_k is us-complete then $t(\alpha_k) < \infty$.*

Theorem 18. *The sequence $\{\alpha_k\}$ such that $\alpha_k = \alpha$ for all k is not us-complete.*

Measures of an optimal ML-test

Measures of a universal ML-test

Kučera and Slaman (2002) showed measure of any component of a universal ML-test is ML-random. Conversely Merkle, Mihailović and Slaman (2006) showed that for any uniformly c.e. ML-random reals $r_n \leq 2^{-n}$ there is a universal ML-test $\{U_n\}$ such that $\mu(U_n) = r_n$.

Theorem 19. *Let r_n be uniformly left-c.e. reals such that $r_n \leq 2^{-n}$. TFAE:*

(i) *There exists an optimal ML-test U_n such that $\mu(U_n) = r_n$.*

(ii) $\{2^n r_n\}$ *is us-complete.*

Proof

(i)⇒(ii). $\{U_n\}$: an optimal ML-test
$r_n = \mu(U_n)$
α_n: a uniformly left-c.e. sequence of reals with $\alpha_n \leq 2^{-n}$.
We will show $\alpha_n \leq_{us} r_n$.
For each $0 < m < n$, construct A_n^m.
If $A_n^m[s] \not\subseteq U_m[s]$ then do nothing.
Otherwise enumerate into A_n^m a set of strings $\{\sigma_i\}$ s.t. pairwise disjoint, $\sum_i 2^{-|\sigma_i|} = 2^{2(m-n)}(\alpha_{m,s} - \alpha_{m,t})$ and $\sigma_i \notin \mathrm{dom} U_m[s] \cup A_n^m[s]$.
Let $A_n = \bigcup_{m<n} A_n^m$.
Then
$\mu(A_n) \leq \sum_{m<n} 2^{2(m-n)}\alpha_m \leq \sum_{m<n} 2^{m-2n} \leq 2^{n-2n} = 2^{-n}$.
Hence $\{A_n\}$ is a ML-test.
By optimality, $U_n \supseteq A_{n+d} = \bigcup_{m<n} A_{n+d}^m \supseteq A_{n+d}^n$.
Then $r_{n,s_{i+1}} - r_{n,s_i} > 2^{-2d}(\alpha_{n,s_i} - \alpha_{n,s_{i-1}})$, so $\alpha_n \leq_{us} r_n$.

Proof 2

(ii)⇒(i).
Suppose that $\Omega_n \leq_{us} r_n$.
Let $u_n = \mu(U_n)$ where $\{U_n\}$ is an optimal ML-test.
Then there exists another optimal ML-test $\{V_n\}$ such that $\mu(V_n) = v_n = \sum_{m=1}^{\infty} m \cdot u_{n+2m}$ by adding extra strings.
Note that $v_n \leq 2^{-n} \sum_{m=1}^{\infty} m \cdot 2^{-2m} \leq 2^{-n}$.
Furthermore $v_n \leq_{us} \Omega_n \leq_{us} r_n$.
Then $dr_n = v_n + \gamma_n$ for some d and uniformly left-c.e. reals γ_n.
It follows that $dr_n = \sum_{m=1}^{\infty} m \cdot u_{n+2m} + \gamma_n$ and $r_n = u_{n+2d} + \sum_{m \neq d} \frac{m}{d} \cdot u_{n+2m} + \gamma_n$.
Hence we can construct a ML-test $\{W_n\}$ such that $W_n \supseteq U_{n+2d}$ and $\mu(W_n) = r_n$ by adding extra strings.
Furthermore $\{W_n\}$ is optimal by $W_n \supseteq U_{n+2d}$.

1-randomness

Theorem 20 (Solovay '75, Kučera and Slaman '02)**.** *For left-c.e. reals α, the following are equivalent.*

(i) α *is 1-random.*

(ii) α *is Solovay complete.*

(iii) $\Omega \leq_s \alpha$.

(iv) α *is the measure of a component of a universal ML-test.*

Summary

Theorem 21. *For a sequence $\{\alpha_k\}$ of uniformly left-c.e. reals, the following are equivalent.*

(i) $\{\alpha_k\}$ *is uniform Solovay complete.*

(ii) $\Omega_k \leq_{us} \alpha_k$.

(iii) $\{\alpha_k\}$ *is the measures of an optimal ML-test.*

Unfortunately, "$t(\alpha_k) < \infty$" is not equivalent to these. Neither is "$\{\alpha_k\}$ is 1-random".

End

Thank you!