preamble

Computability of measures and A-topology

Kenshi Miyabe

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Questions

Computable function

What is ”computation”?
A computable function is defined by

- a recursive function $(\mathbb{N} \to \mathbb{N})$,
- a Turing machine $(2^* \to 2^*)$,
- and so on.

The notion of computability is highly and directly related with a function.

Remark 1. *Computable functions contain partial functions and total functions.*

Computable set

A computable function $\mathbb{N} \to 2$ can be identified with a subset of $\mathbb{N}$.

Definition 2. *A set* $A \subseteq \mathbb{N}$ *is* computable *if the characteristic function* $1_A : \mathbb{N} \to 2$ *is computable.*

It seems to be natural ... but ...

Definition 3 (or Proposition). *A set* $A \subseteq \mathbb{N}$ *is* c.e.
iff $\exists f$ *:p.c.f. s.t.* $\mathrm{dom}(f) = A$
iff $\exists f$ *:p.c.f. s.t.* $\mathrm{ran}(f) = A$*.*

Which is a natural definition??

Computable real

Definition 4 (or Proposition). *A real* $r \in [0,1]$ *is* computable
iff its binary expansion is computable
iff $\exists q_n$*:computable seq. of rationals s.t.* $|r - q_n| \leq 2^{-n}$*.*

Definition 5 (or Proposition). *A real* $r \in [0,1]$ *is* left-c.e. *or* c.e.
iff $\exists q_n$*:inc. comp. seq. of rationals s.t.* $\lim_n q_n = r$
iff $\{q \in \mathbb{Q} \ : \ q < r\}$ *is c.e.*

How to define

Question 6. *How should we define a "computable" object?*

We need a unified and abstract manner to define computability.
Recall that

- computable = Δ^0_1,
- c.e. = Σ^0_1.

It suggests that the notion of c.e. is more basic than the notion of computable.

Naming system

Why do we need a name, a notation or a representation?

$$\sqrt{2} = 1.41421356\cdots$$

The left-hand side is the number whose square is 2.
The right-hand side is the number which is contained in the following intervals

$$[1, 2),\ [1.4, 1.5),\ [1.41, 1.42),\ \cdots .$$

We can not see nor count a real.
All we can do is to represent a real by some formula.
That formula is nothing but a name.

A program and its value

Many naming systems are possible:

- TeX code,
- $\lim a_n$,
- $\{x : x \in x\}$,
- programs written in any languages.

If a mathematician writes a formula, it means the value exists.
If a programer writes a program, it does not mean it has a value.

Representation

We can use arbitrary representations.
However since we are interested in computability of M, we often restrict ourseleves to use ”computable” representations.
Note that the word ”computable” is not defined.
Then we need to consider computability.

Question 7. *How do we find a natural representation?*

Continuity

- Every computable function is continous.
- A function is continuous iff it is computable relative to an oracle.

The relation is similar to that between Σ^0_1 and $\boldsymbol{\Sigma^0_1}$.

Non-continuous function

Then continuity is a weeker notion of computability.
Continuity is based on computability not on a representation approach.
Suppose the following function.

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}.$$

The relation $f(x) = 0$ or $x = 0$ is Π^0_1.

TTE

Turing machine

A Turing computer (or a machine) receives an input as a string $\sigma \in \Sigma^*$ and two results are possible.

- It outputs a string $\tau \in \Sigma$ and terminate.
- It does not output anything and does not terminate.

If we allow it to output sequences, there are other possibilities.

- It outputs a sequence.
- It outputs a finite string (maybe the empty string) and does not output anymore.

The latter case is regarded as no output.

Type-2 machine

We often watch a movie on PC by streaming, which is a kind of computation too.
A Type-2 machine can receive an input as a sequence $\in \Sigma^\omega$.
Hence a Type-2 machine is a computable function from $\{\Sigma^*, \Sigma^\omega\}$ to $\{\Sigma^*, \Sigma^\omega\}$.
Thesis: A function from $\{\Sigma^*, \Sigma^\omega\}$ to $\{\Sigma^*, \Sigma^\omega\}$ informally or by a physical device, if and only if it can be computed by a Type-2 machine.

Topology

On Σ^* we consider the discrete topology.
On Σ^ω we consider the topology generated by the base $\{w\Sigma^\omega \ : \ w \in \Sigma^*\}$ of open sets.

Proposition 8.
- *Every computable function is continous.*
- *A function is continuous iff it is computable relative to an oracle.*

The set $W \subseteq Y$ is r.e. in Z if M halts on input $y \iff y \in W$.
(Compare to the notion of open.)

Representation

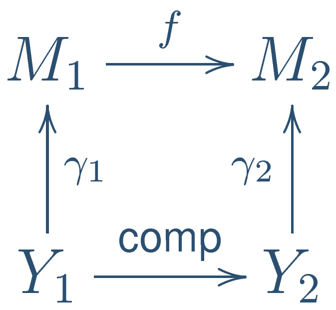
Definition 9. *A notation of M is a surjective function $\gamma :\subseteq \Sigma^* \to M$.*
A representation of M is a surjective function $\gamma :\subseteq \Sigma^\omega \to M$.
A naming system of M is a notation or a representation.

Example 10.

- $\gamma_{\mathbb{N}} :\subseteq \Sigma^* \to \mathbb{N}$
- $\gamma_{\mathbb{Q}} :\subseteq \Sigma^* \to \mathbb{Q}$
- $\gamma_{10} :\subseteq \Sigma^\omega \to \mathbb{R}$

Realization

A partial function $h :\subseteq Y_1 \to Y_2$ realizes a function $f :\subseteq M_1 \to M_2$ if $h(y)$ is the name of $f(x)$ where y is the name of $x \in \mathrm{dom}(f)$.
In other words $\delta_1(y) = x$ implies $\delta_2 \circ h(y) = f(x)$.
A function f is (γ_1, γ_2)-computable (-continuous) if it has a computable (continuous) realization.



r.e. relation

A point $x \in M$ is γ-computable iff $x \in \gamma(p)$ for some computable p.
A set S is γ-r.e. (-open) if there is an r.e. (open) set $W \subseteq Y$ such that $x \in S \iff y \in W$ for all x, y such that $x = \gamma(y)$.

reducibility

$\gamma_1 \leq \gamma_0$ if $M_1 \subseteq M_0$ and the identity $\mathrm{id} : M_1 \to M_0$ is (γ_1, γ_0)-computable.
This means some computable function h translates γ_1-names to γ_0-names.
$\gamma_1 \equiv \gamma_0$ iff $\gamma_1 \leq \gamma_0$ and $\gamma_0 \leq \gamma_1$.
Continuous reducibility $\gamma_1 \leq_t \gamma_0$ defined accordingly by means of continuous functions.

Computable topological space

Open sets, base, subbase

A topology on a set is a class of subsets (open sets) s.t.

- it is closed under any union,
- it is closed under finite intersetion,
- the empty set and the whole set belong to it.

A base for the topology is a class of subsets s.t. the class of all possible unions is the topology.
A subbase for the topology is a class of all finite intersections forms a base for the topology.
The space is a set equipped with a topology etc.
The space is second-countable if it has a countable base.

Separation axiom

Let O be a topology of the space.
The space is T_0 if for any $x, y \in X$ there exists $U \in O$ s.t. $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.
The space is T_1 if for any $x, y \in X$ there exists $U, V \in O$ s.t. $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$.
The space is T_2 or Hausdorff if for any $x, y \in X$ there exists $U, V \in O$ s.t. $x \in U$, $y \in V$ and $U \cap V = \phi$.

Example

The topology on real line $\mathbb{R}$ is generated by the base $\{(a, b) \;:\; a, b \in \mathbb{R}\}$.
The set $\{(p, q) \;:\; p, q \in \mathbb{Q}\}$ forms a countable base.
The topology on lower extended real line $\mathbb{R} \cup \{+\infty\}$ is $\{(a, +\infty] \;:\; a \in \mathbb{R}\}$.
The set $\{(p, +\infty] \;:\; p \in \mathbb{Q}\}$ forms a countable base.

Notation

$|w|$ is the length of the word $w \in \Sigma^*$.
The "wrapping function" $\iota : \Sigma^* \to \Sigma^*$ is the one such that

$$\iota(a_1 a_2 \ldots a_k) = 110 a_1 0 a_2 0 \ldots a_k 011.$$

$u \ll w$ iff $i(u)$ is a subword of w for $u \in \Sigma^*$ and $w \in \Sigma^* \cup \Sigma^\omega$.

Computable topological space

Definition 11 (Hertling and Weihrauch 2009). *An* effective topological space *is a 4-tuple* $\mathbf{X} = (X, \tau, \beta, \nu)$ *such that*

- (X, τ) *is a topological* T_0*-space,*
- $\nu :\subseteq \Sigma^* \to \beta$ *is a notation of a base* β *of* τ*.*

$\mathbf{X}$ *is a* computable topological space *or* CTS *if*

- $\mathrm{dom}(\nu)$ *is recursive and*
- $\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \ : \ (u, v, w) \in S\}$ *for all* $u, v \in dom(\nu)$ *for some r.e. set* $S \subseteq (dom(\nu))^3$*.*

Example

Example 12. (i) *(real line) Define* $\mathbf{R} = (\mathbb{R}, \tau_{\mathbb{R}}, \beta, \nu)$ *such that* $\tau_{\mathbb{R}}$ *is the real line topology and* ν *is a canonical notation of the set of all open intervals with rational endpoints.*

(ii) *(lower unit interval) Define* $\mathbf{I}_< = (\mathbb{I}, \tau_<, \beta_<, \nu_<)$ *such that* $\nu_<(w) = \{x \ : \ 0 \leq q < x \leq 1$ *and* $q \in \nu_{\mathbb{Q}}\}$*. The representation* δ *for* $\mathbf{I}_<$ *is denoted by* $\rho_<$*.*

Open sets

Define a representation $\theta :\subseteq \Sigma^\omega \to \tau$ of the set of open sets as

$$W = \theta(p) \iff \begin{cases} w \ll p \Rightarrow w \in \mathrm{dom}(\nu) \\ W = \bigcup\{\nu(w) \ : \ w \ll p\}. \end{cases}$$

Proposition 13. *A open set on Cantor space is c.e. iff it is θ-computable with the canonical base.*

Closed sets and points

Define a representation $\psi :\subseteq \Sigma^\omega \to \mathcal{A}$ of the set of closed sets as

$$A = \psi(p) \iff (\forall w \in \Sigma^*)(w \ll p \iff A \cap \nu(w) \neq \phi).$$

Define a representation $\delta :\subseteq \Sigma^\omega \to X$ of the points as

$$x = \delta(p) \iff (\forall w \in \Sigma^*)(w \ll p \iff x \in \nu(w)).$$

A point is δ-computable if it has a computable δ-representation.

Proposition 14. *A real is δ-computable iff it is computable in the usual sense.*

Real line

Let $r \in \mathbb{R}$.
TFAE:

- A real r is ρ-computable.
- There exists a computable sequence (p_n, q_n) of pairs of rationals s.t. $p_n < r < q_n$ and $q_n - p_n \leq 2^{-n}$.
- There exists a computable sequence p_n of rationals s.t. $|r - p_n| \leq 2^{-n}$.

Lower unit interval

Let $r \in [0,1]$.
TFAE:

- r is $\rho_<$-computable.
- $\{q \; : \; q < r\}$ is c.e.
- There exists an increasing computable sequence q_n of rationals s.t. $\lim_n q_n = r$.

Extended lower real line

Let $r \in \mathbb{R} \cup \{+\infty\}$.
TFAE:

- r is $\overline{\rho}_<$-computable.
- r is $\rho_<$-computable or $r = +\infty$.

Extended real line

Let $r \in \mathbb{R} \cup \{\pm\infty\}$.
TFAE:

- r is $\overline{\rho}$-computable.
- r is ρ-computable or $r = \pm\infty$.

Boolean operations

- $\nu \leq \theta$
- Finite intersection on open sets is (ν^{fs}, θ)-computable and (θ^{fs}, θ)-computable.
- Union on open sets is (θ^{cs}, θ)-computable.
- $\{(x, U) \in X \times \beta \ : \ x \in U\}$ is (γ, ν)-r.e. iff $\gamma \leq \delta$.
- $\{(x, W) \in X \times \Gamma \ : \ x \in W\}$ is (δ, γ)-r.e. iff $\gamma \leq \theta$.

For a point x and an open set W, $x \in W$ is (δ, θ)-r.e.
W is open iff W is δ-open.

The space of continuous functions

Notations of computable functions

Definition 15 (Notations of computable functions)**.** *For all* $a, b \in \{*, \omega\}$,

(i) $P^{ab} = \{f :\subseteq \Sigma^a \to \Sigma^b \; : \; f$ *is computable*$\}$ *and,*

(ii) *a notation* $\xi^{ab} : \Sigma^* \to P^{ab}$ *is defined as* $\xi^{ab}(w)$ *is the function* $f \in P^{ab}$ *computed by the Type-2 machine with code* w *if the machine is recursive.*

Sets of functions

(i) F^{**} is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^*$.

(ii) $F^{*\omega}$ is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^\omega$.

(iii) $F^{\omega *}$ is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^*$ with open domain.

(iv) $F^{\omega\omega}$ is the set of all partial functions $f :\subseteq \Sigma^* \to \Sigma^*$ with G_δ-domain.

Representations of functions

Definition 16 (Representations of the sets F^{ab}). *Define $\eta^{ab} : \Sigma^\omega \to F^{ab}$ by*

$$\eta^{ab}(\langle x, p\rangle)(y) = \xi_x^{\omega b}\langle p, y\rangle$$

for all $x \in \Sigma^*$, $p \in \Sigma^\omega$ and $y \in \Sigma^a$.

Representations of functions

Definition 17. *Let $\gamma_1 : \Sigma^a \rightrightarrows M_1$ and $\gamma_2 : \Sigma^b \rightrightarrows M_2$ be multi-representations.*
We define a multi-representation $[\gamma_1 \rightrightarrows \gamma_2]$ of the (γ_1, γ_2)-continuous multi-functions $f : M_1 \rightrightarrows M_2$ as $f \in [\gamma_1 \rightrightarrows \gamma_2](p)$ iff $\eta_p^{ab} = \eta^{ab}(p)$ realizes f w.r.t. (γ_1, γ_2).
The restriction ot the single-valued functions is $[\gamma_1 \rightarrow_p \gamma_2]$.
The restriction to the total (γ_1, γ_2)-continuous functions is $[\gamma_1 \rightarrow \gamma_2]$.

Computable Operators

Theorem 18. (i) $eval : (f, x) \mapsto f(x)$ *is* $([\delta_1 \to_p \delta_2], \delta_1, \delta_2)$*-computable.*

(ii) $(f, g) \mapsto g \circ f$ *is* $([\delta_1 \to_p \delta_2], [\delta_2 \to_p \delta_3], [\delta_1 \to_p \delta_3])$*-computable.*

(iii) *The multi-function* $(f, W) \rightrightarrows T$ *mapping every continuous function* $f :\subseteq X_1 \to X_2$ *and every open set* $W \subseteq X_2$ *to some open set* $T \subseteq X_1$ *such that* $f^{-1}[W] = T \cap dom(f)$ *is* $([\delta_1 \to_p \delta_2], \theta_2, \theta_1)$*-computable.*

Continuity

TFAE:

- A partial function $f :\subseteq X_1 \to X_2$ is continuous.
- For every $W \in \tau_2$, $f^{-1}[W]$ is open in $\mathrm{dom}(f)$.
- For every $W \in \tau_2$, $f^{-1}[W] = V \cap \mathrm{dom}(f)$ for some $V \in \tau_1$.
- $(\forall x \in \mathrm{dom}(f), W \in \tau_2)(f(x) \in W \Rightarrow (\exists V \in \tau_1)(x \in V \wedge f[V] \subseteq W))$.
- $f[\mathrm{cls}_{\mathrm{dom}(f)}(C)] \subseteq \overline{f[C]}$ for every $C \subseteq \mathrm{dom}(f)$.
- f has a continous (δ_1, δ_2)-realization.

Representation

Let $\mathrm{CP}(X_1, X_2)$ be the set of partial continuous functions $f :\subseteq X_1 \to X_2$.
Define a representation of the set $\mathrm{CP}(X_1, X_2)$ as
$f \in \vec{\delta_4}(p) \iff$

$$\begin{cases} (w \ll p \Rightarrow (\exists u \in \mathrm{dom}(\nu_1),\ v \in \mathrm{dom}(\nu_2))w = \langle u, v \rangle) \\ \text{and } f^{-1}[\nu_2(v)] = \bigcup_{\langle u,v \rangle \ll p} \nu_1(u) \cap \mathrm{dom}(f). \end{cases}$$

Measures

sigma-algebra

Let S be a set.
A class $\mathcal{A}$ of subsets of S is σ-algebra if

- $S \in \mathcal{A}$,
- $F \in \mathcal{A} \Rightarrow F^c \in \mathcal{A}$,
- $F_n \in \mathcal{A} \Rightarrow \bigcup_n F_n \in \mathcal{A}$.

Measure

Let $\mathcal{A}$ be a σ-algebra on S.
A set function μ to reals is a measure if

- $\mu(F) \geq 1$ for $F \in \mathcal{A}$,
- $\mu(S) = 1$,
- $\mu(\bigcup_n F_n) = \sum_n \mu(F_n)$ for pairwise disjoint $\{F_n\}$.

Borel measure

Borel σ-algebra $\mathcal{B}(X)$ is the class of sets generated by all open sets.
A Borel measure is a measure on Borel σ-algebra.

Proposition 19. *If two Borel measures on a topological space coincide on all open sets, then they coincide on all Borel sets.*

Baire measure

Baire σ-algebra $\mathcal{B}a(X)$ is the class of sets generated by all sets of the form

$$\{x \;:\; f(x) > 0\}$$

where f is a continous function.
A Baire measure is a measure on Baire σ-algebra.

Proposition 20. *Let X be a perfectly normal space. Then we have $\mathcal{B}(X) = \mathcal{B}a(X)$.*

If X is second-countable, regularity is a sufficient condition.

tau-additive measure

The total variation of μ is the measure

$$|\mu| = \mu^+ + \mu^-$$

where $\mu = \mu^+ - \mu^-$ is the Jordan-Hahn decomposition.

Definition 21. *A Borel measure μ is τ-additive if for every increasing net of open sets U_γ,*

$$|\mu|\left(\bigcup_\gamma U_\gamma\right) = \lim_\gamma |\mu|(U_\gamma).$$

If X is second-countable, every Borel measure is τ-additive.

The space of measures

$\mathcal{M}_\sigma(X)$ is the set of all Baire measures,
$\mathcal{M}_\mathcal{B}(X)$ is the set of all Borel measures,
$\mathcal{M}_\tau(X)$ is the set of all τ-additive Borel measures.
$\mathcal{M}^+(X)$ is the subclass of non-negative measures,
$\mathcal{P}(X)$ is the subclass of probability measures.

Weak topology

Definition 22. *The weak topology on the space $\mathcal{M}_\sigma(X)$ of Baire measures is the topology $\sigma(\mathcal{M}_\sigma(X), C_b(X))$, i.e., the subbase of the weak topology consists of the sets*

$$U_{f,\epsilon}(\mu) = \left\{ \nu \ : \ \left| \int_X f d\mu - \int_X f d\nu \right| < \epsilon \right\},$$

where $\mu \in \mathcal{M}_\sigma(X)$, $f_i \in C_b(X)$, $\epsilon > 0$.

Theorem 23. *Let X be completely regular.*

- *X is compact iff $\mathcal{M}^+_\tau(X)$ is compact,*
- *X is metrizable iff $\mathcal{M}^+_\tau(X)$ is metrizable,*
- *X is separable iff $\mathcal{M}^+_\tau(X)$ is separable.*

A-topology

The A-topology on the space $\mathcal{P}(X)$ is defined by means of neighborhoods of the form

$$U(\mu, G, \epsilon) = \{\nu \;:\; \mu(G) < \nu(G) + \epsilon\},$$

where $\mu \in \mathcal{P}(X)$, G is an open set, $\epsilon > 0$.
The A-topology coincides with the weak topology on a completely regular space.
The A-topology on the space $\mathcal{M}^+(X)$ is obtained by adding the neighborhoods

$$U'(\mu, \epsilon) = \{\nu \;:\; |\mu(X) - \nu(X)| < \epsilon\}.$$

Corresponding property

Theorem 24.

- X *is second-countable iff* $\mathcal{M}_\tau^+(X)$ *is second-countable,*
- X *is regular iff* $\mathcal{M}_\tau^+(X)$ *is regular,*
- X *is completely regular iff* $\mathcal{M}_\tau^+(X)$ *is completely regular.*

Computable measures

CTS with the A-topology

Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space.
We consider the space $\mathcal{M}^+(X)$ equipped with the A-topology τ_A.
The following sets form a countable subbase:

$$\{\mu \ : \ \mu(G) > q\}, \ \{\mu \ : \ \mu(X) < q\},$$

where G is the finite union of base sets and $q \in \mathbb{Q}$.
Hence $\mathcal{M}^+(X)$ is second-countable.
Furthermore $\mathcal{M}^+(X)$ is T_0.

The space is a CTS.

Let β_A be the base generated from the subbase.
Let ν_A be a canonical notation of the base β_A.

Proposition 25. $\mathrm{M} = (\mathcal{M}^+(X), \tau_A, \beta_A, \nu_A)$ *is a computable topological space.*

Computable points

Let δ_A be the canonical representation of points on $\mathbf{M}$.
We consider δ_A-representation p of μ.
From p we can enumerate all subbase that contains μ.
Then for each finite union G of base sets we can enumerate all rationals q s.t. $\mu(G) > q$.
Hence $\mu(G)$ is approximated from below.
Similarly $\mu(X)$ is approximated form above.

Evaluation

Proposition 26. *The operation* $eval : \mathcal{M}^+(X) \times \tau \to \mathbb{R}^+$ *such that*

$$eval(\mu, G) = \mu(G)$$

is $(\delta_A, \theta_X, \rho_<)$*-computable.*

A computable measure on Cantor space is usually defined as the following:
μ is computable iff $\mu([\sigma])$ is uniformly computable in $\sigma \in 2^*$.

Integration

Proposition 27. *The integral operation* $\int : C(X, \mathbb{R}^+) \times \mathcal{M}(X) \to \overline{\mathbb{R}}^+$ *is* $([\delta \to \rho_<], \delta_A, \overline{\rho}_<)$*-computable.*
The integral operation $\int : C_b(X, \mathbb{R}^+) \times \mathcal{M}(X) \to \mathbb{R}^+$ *is* $([\delta \to \rho], \delta_A, \rho)$*-computable.*

Measure constructed by integration

Proposition 28. *The operation* $m : C_b(X, \mathbb{R}^+) \times \mathcal{M}(X) \to \mathcal{M}(X)$ *such that*

$$m(f, \mu)(G) = \int_G f d\mu$$

for all $G \in \mathcal{O}(X)$ *is* $([\delta \to \rho], \delta_A, \delta_A)$*-computable.*

Note that Radon-Nikodym operator is not computable.

Computable regularity

Regular space

Definition 29. *A space is regular (or T_3) if for a closed A and a point $x \notin A$ there are disjoint open U, V s.t. $x \in U$ and $A \subseteq V$.*

Proposition 30. *TFAE:*

- *The space is regular.*
- *For an open U and $x \in U$ there is an open V containing x s.t. $\overline{V} \subseteq U$.*

Regular measure

Definition 31. *A measure on $\mathcal{A}$ is regular if for every A and $\epsilon > 0$ there exists a closed set F_ϵ s.t. $F_\epsilon \subseteq A$, $A \backslash F_\epsilon \in \mathcal{A}$ and $\mu(A \backslash F_\epsilon) < \epsilon$.*

Proposition 32. *Every Baire measure μ on a topological space X is regular.*

Recall that Baire measure and Borel measure coincide on a second-countable regular space.

Corollary 33. *Every Borel measure on a second-countable regular space is regular.*

Computably regular space

Definition 34 (essentially by Grubba and Weihrauch 2007). *A* computably regular space *is a CTS for which there is a comp. func.* $t3 :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$ *s.t.*

- $(\forall v \in \mathrm{dom}(\nu))\nu(v) = \bigcup_{(u,v)\in\mathrm{dom}(t3)} \nu(u)$,
- $(\forall (u,v) \in \mathrm{dom}(t3))\nu(u) \subseteq \psi^-(t3(u,v)) \subseteq \nu(v)$.

Every computably regular space is regular.
Every computably regular space is computably normal (when it is defined appropriately).

Computable measure on a computably regular space

How do we define a computably regular measure?
A computable measure on a computably regular space should be computably regular.

Question 35. *Let* $\mathbf{X}$ *be a computably regular space. Is* $\mathbf{M}(X)$ *a computably regular?*

Computably regular measure

Definition 36. *Suppose that the base is complete. A measure on a CTS is* computably regular *if for all* $v \in \mathrm{dom}(\nu)$ *and* $n \in \mathbb{N}$ *there is uniformly comp.* u_n^v *and* $p(v,n)$ *s.t.*

- $\mu\left(\bigcup_n \nu(u_n^v)\right) \to \mu(\nu(v))$,
- $\nu(u_n^v) \subseteq \psi^-(p(v,n)) \subseteq \nu(v)$,
- $\mu(\psi^-(p(v,n)) \setminus \nu(u_n^v)) \leq 2^{-n}$.

Proposition 37. *If a measure is computably regular, then it is regular.*

End

Thank you!