preamble

Characterizing randomness by integral tests

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June 16, 2011

Main thesis

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Randomness

is equivalent to

Differentiability

by Demuth, Nies et al.
Then why?

My suggestion

A test

is equivalent to

an integral test.

Integration is closed related with Differentiation.
In this talk we will see the former relation.
The latter relation needs to be studied further in the point of view of computability.

Some results

random	integral test	differentiability
weak 2-rd.	a.e. finite	a.e. differentiable
Martin-Löf	c.e.	bounded variation
computably	?	non-dec. or Lipschitz
Schnorr	c.e. & comp. int.	Lipschitz & comp. in variation norm
Kurtz	comp	non-dec. & comp. deriv.

In this talk

- We recall algorithmic randomness notions.
- We review computable functions from $[0, 1]$ to $[0, +\infty]$.
- We characterize the algorithmic randomness notions in terms of integral tests.

Martin-Löf randomness

Martin-Löf randomness

On Cantor space 2^ω we consider the product topology and the uniform (Lebesgue) measure μ.
The cylinders $[\sigma] = \{A \in 2^\omega \ : \ \sigma \preceq A\}$ are the base for the topology.
A *c.e.* open set is a union of a c.e. set of cylinders.

Definition 1. *A* Martin-Löf test *(or ML-test) is a sequence $\{U_n\}$ of uniformly c.e. open sets with $\mu(U_n) \leq 2^{-n}$.*
A sequence A passes a ML-test if $A \notin \bigcap_n U_n$.
A sequence is Martin-Löf random *if it passes all Martin-Löf tests.*

Integral test

We identify 2^ω with $[0,1]$.

Definition 2. *A function $t : [0,1] \to [0,+\infty]$ is c.e. if the sets*

$$\{x \;:\; q < t(x)\}$$

are c.e. open uniformly in $q \in \mathbb{Q}$.
A c.e. function is an integral test *if $\int_{[0,1]} t(x)dx \leq 1$.*

Remark 3. *We can replace $\int t(x)dx \leq 1$ with $\int t(x)dx < \infty$.*

Proposition 4. *TFAE:*

- *A real z is ML-random.*
- *$t(z) < \infty$ for all integral tests.*

Differentiability

Recall that f is of *bounded variation* if

$$\sup \sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)| < \infty$$

where the sup is taken over all collections $t_1 < t_2 < \ldots < t_n$ in $[0, 1]$.

Theorem 5 (Demuth; Nies, Brattka and Miller)**.** *A real $z \in [0, 1]$ is ML-random $\iff$ every comp. func. f of bounded variation is differentiable at z.*

Are there any relation between integral tests and differentiability?
I believe so, but we start from a simpler case.

Versions of integral tests

Weak 2-randomness

Definition 6. *A* generalized ML-test *is a sequence* $\{U_n\}$ *of uniformly c.e. open sets with* $\lim_n \mu(U_n) = 0$*.*
A real is weakly 2-random *if it passes all generalized ML-tests.*

Theorem 7. *TFAE:*

- *A real* z *is weakly 2-random.*
- $t(z) < \infty$ *for all c.e. functions* t *such that* $t(x) < \infty$ *a.e.*
- $t(z) < \infty$ *for all c.e. functions* t *such that* $\int f \circ t(x)dx < \infty$ *for some order function* f*.*

Proof

Theorem 8. *A real z is weakly 2-random iff $t(z) < \infty$ for all c.e. functions t such that $t(x) < \infty$ a.e.*

Proof. For a decreasing generalized ML-test $\{U_n\}$, let $t(x) = \sup_n \{n : x \in U_n\}$.
For the converse, let $U_n = \{x : t(x) > n\}$. □

Schnorr randomness

Definition 9. *A* Schnorr test *is a ML-test such that $\mu(U_n)$ is computable uniformly in n, and $\lim_n \mu(U_n) = 0$.*
A real is Schnorr random *if it passes all Schnorr tests.*

Theorem 10. *TFAE:*

- *A real z is Schnorr random.*
- *$t(z) < \infty$ for all c.e. functions t such that $\int t(x)dx$ is computable.*

Computable function

The following is a base for the topology of $[0, +\infty]$:

$$[0, q),\ (p, q),\ (p, +\infty]$$

where $p, q \in \mathbb{Q}^+$.
Let U_i be a computable enumeration of base sets.
A function $t : [0, 1] \to [0, +\infty]$ is *computable* if $t^{-1}(U_i) = \{x\ :\ t(x) \in U_i\}$ are c.e. open uniformly in i.

Remark 11. *The same definition is obtained by considering two computable topological spaces:* $\mathbf{I} = ([0, 1], \tau, \beta, \nu)$ *and* $\overline{\mathbf{R}}^+ = (\overline{\mathbb{R}}^+, \overline{\tau}, \overline{\beta}, \overline{\nu})$.

Kurtz randomness

Definition 12. *A real z is* Kurtz random *(or* weakly 1-random*) if it is contained in every c.e. open set with measure 1.*

Theorem 13. *TFAE:*

- *A real z is Kurtz random.*
- $t(z) < \infty$ *for all computable functions t such that* $\int t(x)dx < \infty$.
- $t(z) < \infty$ *for all computable functions t such that* $\int t(x)dx$ *is computable.*

Differentiability

Corollary 14. *TFAE:*

- *A real z is Kurtz random.*
- *Each non-decreasing computable function f whose derivative is also computable is differentiable at z.*

Proof. Suppose $t(z) = \infty$ for such a t.
Then $f(x) = \int_0^x t(y)dy$ is non-dec. comp.
The derivative of f is t and computable.
The f is not differentiable at z.
Suppose there exists such an f not differentiable at z.
Let t be the derivative of f. Then $t(z) = \infty$.
The t is non-negative and computable, and $\int t(x)dx = f(1)$ is computable. □

One implication

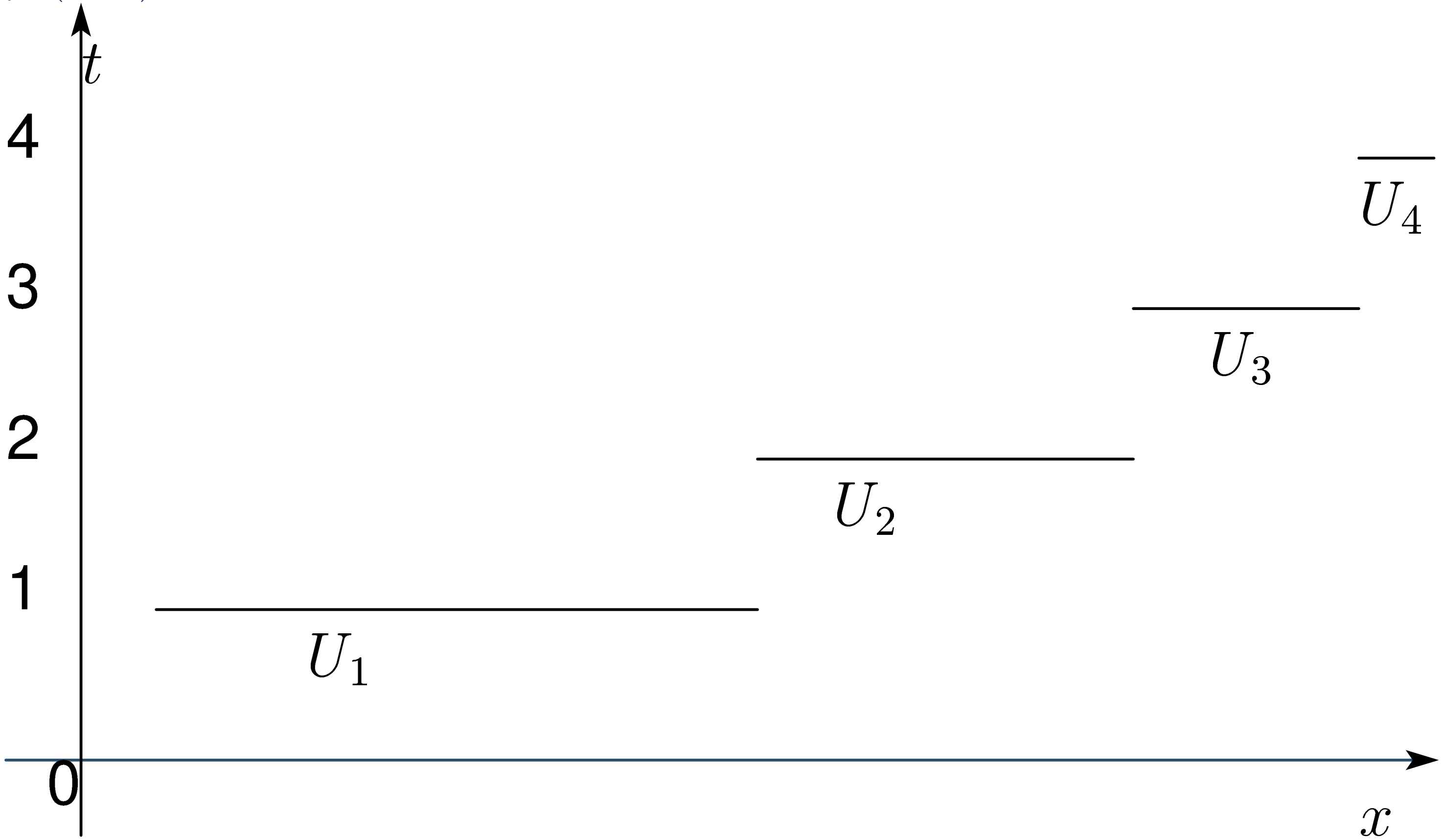
Proof. (function $\Rightarrow$ test)

$\bigcup_n \{x \ : \ t(x) < n\}$ is a c.e. open set with measure 1. □

Proof

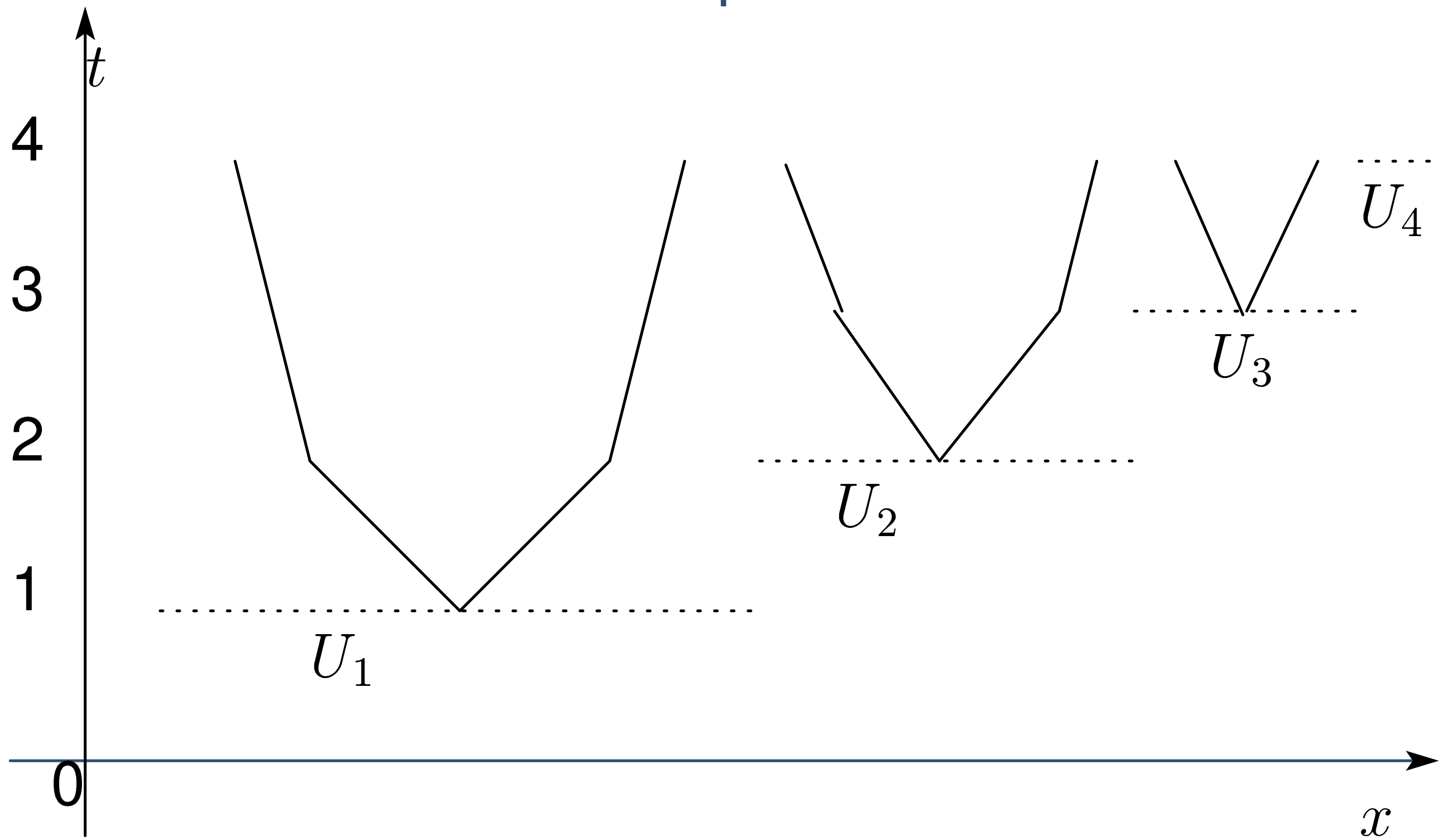
Proof idea

(test ⇒ function)
Let U be a c.e. open set with measure 1.
We divide U into uniformly c.e. open sets $\{U_n\}_{n \geq 1}$ such that $\mu(U_n) = 2^{-n}$.



Proof idea 2

To make the function computable and so continuous ...



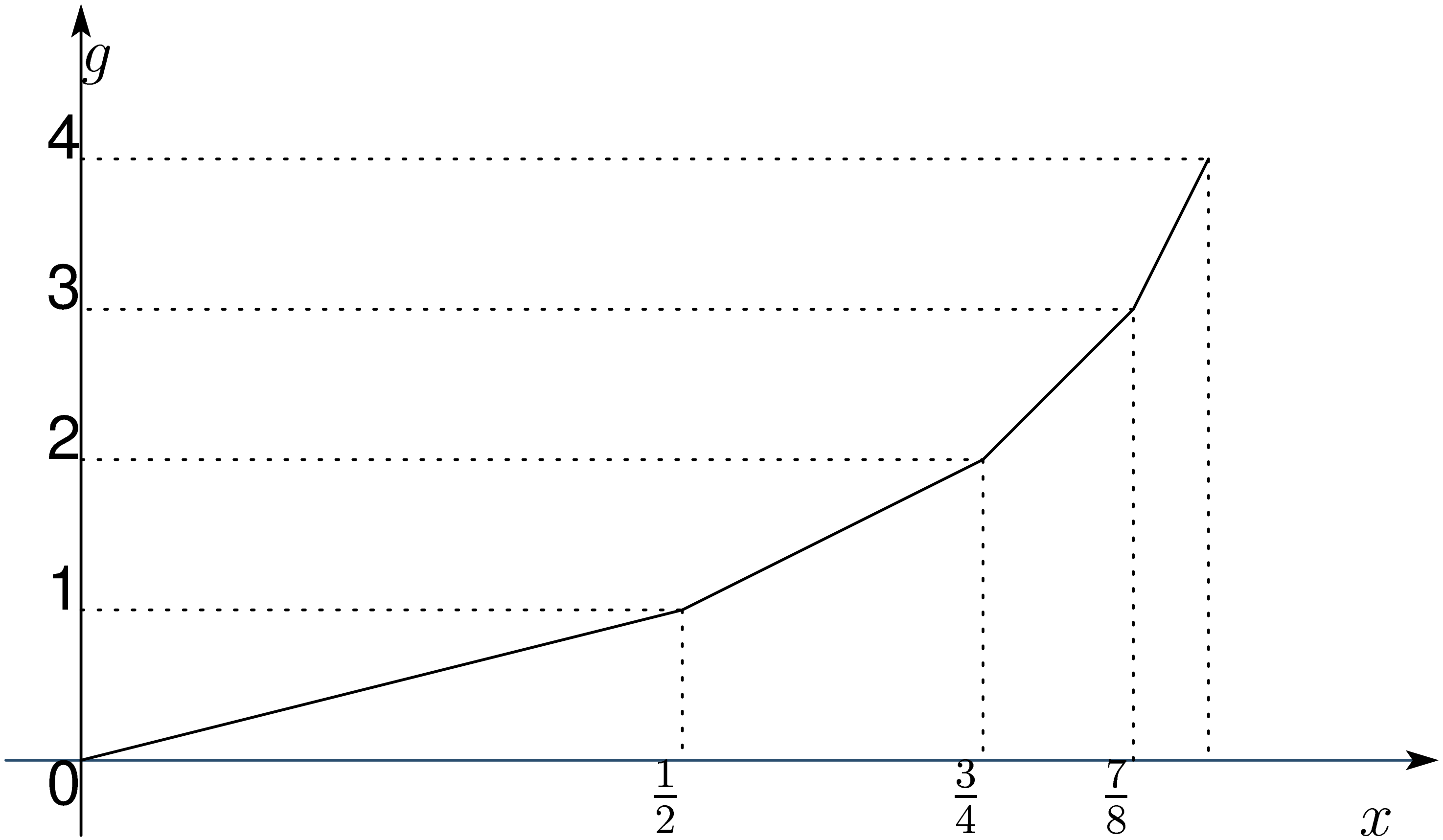
Proof

Let $g : [0, 1] \to [0, +\infty]$ be the polyline satisfying the following:

- the set of endpoints is $\{1 - 2^{-n} : n \geq 0\}$,
- $g(1 - 2^{-n}) = n$,
- $g(1) = \infty$

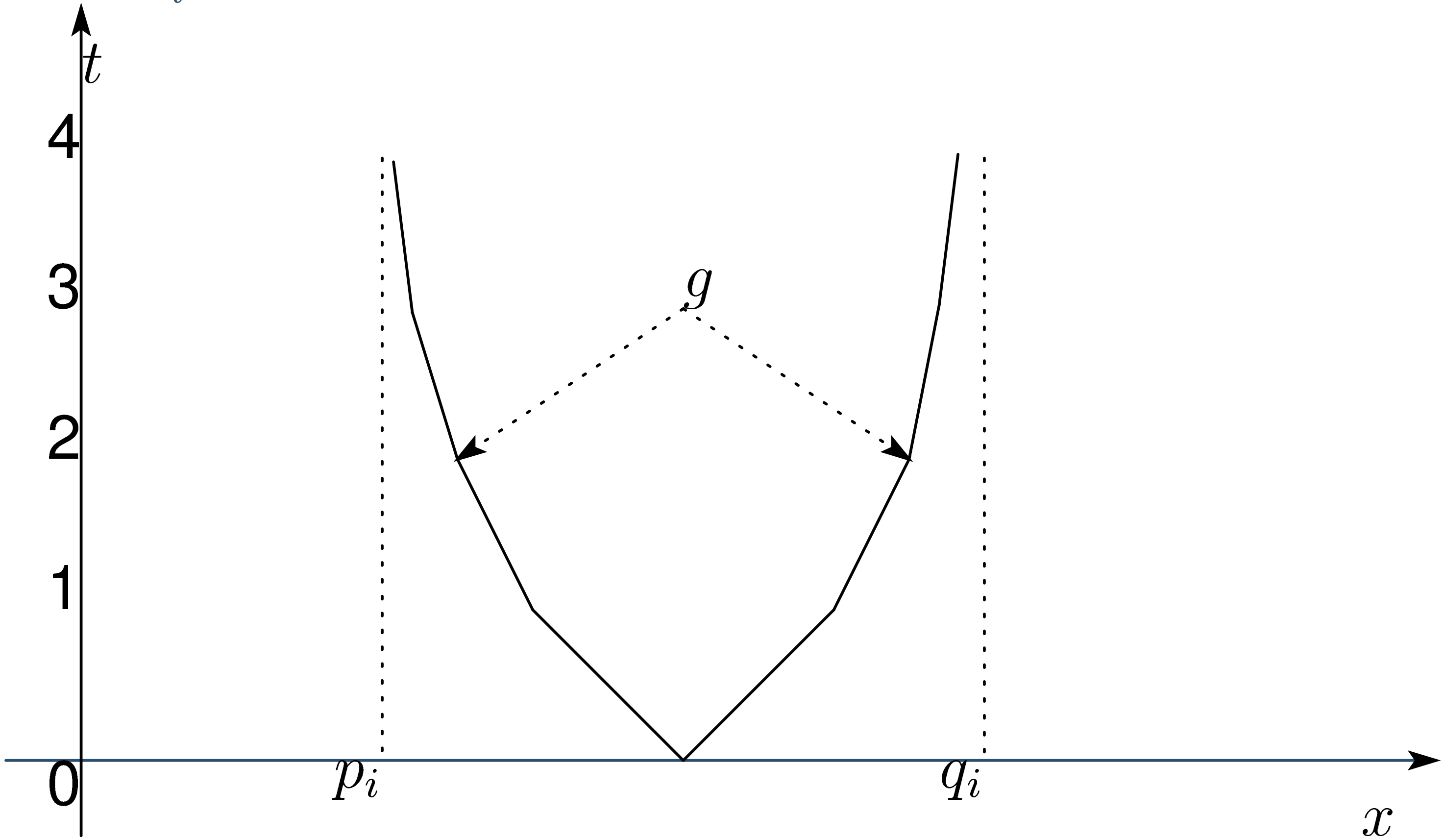
Then g is computable.
Furthermore the integration is also computable.

Proof 2



Proof 3

For each n there uniformly exists FINITE pairs (p_i, q_i) s.t. $U_n \setminus \bigcup_i (p_i, q_i)$ contains only rationals.



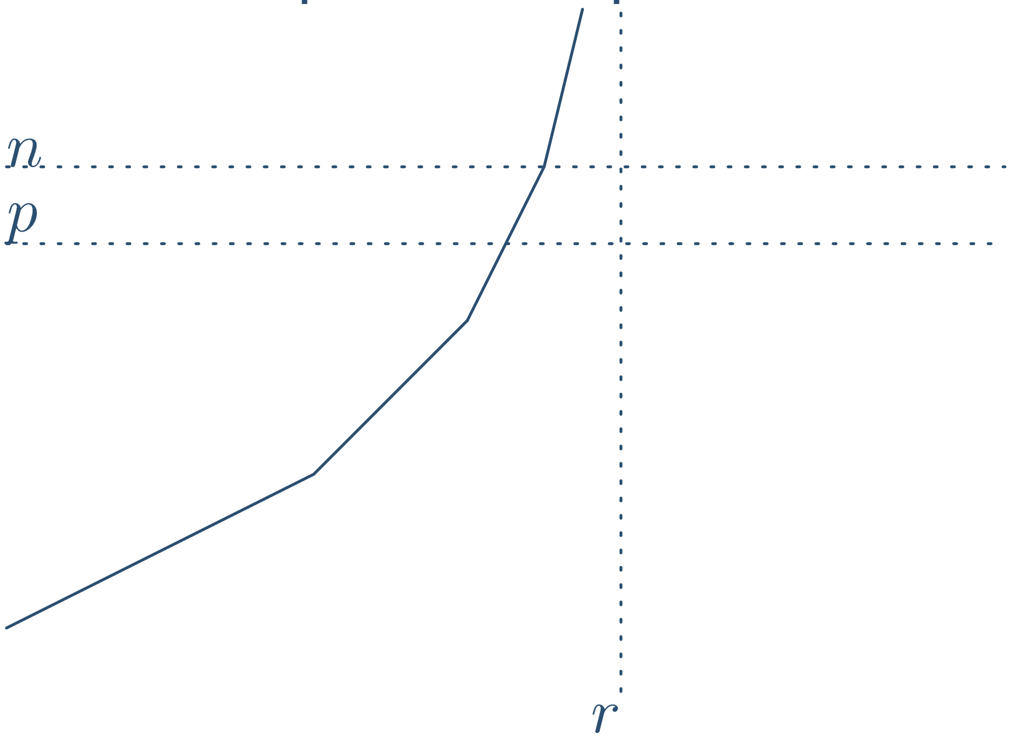
Then the integration $\int t(x)dx$ is computable.

Proof 4

Is the t really computable?
It suffices to show that $t^{-1}([0,q)), t^{-1}((p,q)), t^{-1}((p,+\infty])$ are uniformly c.e.
The pairs (p_i, q_i) are at most finite for each n.
Hence we can completely determine $t^{-1}([0,n))$ for each n.
Then $t^{-1}([0,q))$ and $t^{-1}((p,q))$ are uniformly c.e.

Proof 5

To compute $t^{-1}((p, +\infty])$, pick up $n \geq p$ and enumerate all pairs (p_i, q_i) until n.
Then t maps the complement more than n.



Summary

What we have done

random	integral test	differentiability
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- We characterized some randomness notions in terms of integral tests.
- We gave a characterization of Kurtz randomness in terms of differentiability.

Future works

- Can we drop ”non-decreasing” or replace it with ”Lipschitz” in the characterization of Kurtz randomness?
- Are there any relation between ”c.e.” and ”computable in variation norm” in the characterizations of Schnorr randomness?
- Can we release computability in the characterization of ML-randomness or weak 2-randomness?

End

Thank you!