Randomness and differentiability

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Main thesis

Main thesis

Randomness

is equivalent to

Differentiability

by Demuth, Pathak, Nies et al.
Then WHY?
The Lebesgue Differentiation Theorem (LDT) is a part of an explanation.

My suggestion

A bridge between randomness and differentiability

- Pass a test
- Finite for an integral test
- LDT holds for an integral test
- Differentiable for f s.t. f' is an integral test
- Their "difference" versions

Summary

	w2	ML	Sch	CMP	KRT
test	✓	✓	✓	✓	✓
integral test	✓	✓	✓	?	✓
diff i-tests	?	?	✓	?	✓
LDT non-neg	?	?	✓	✓	✓
LDT	?	✓	✓	?	✓
Diff non-neg	?	?	✓	✓	✓
Diff	✓	✓	✓	?	✓

- Algorithmic randomness notions
- Characterizations by differentiability
- Characterizations by integral tests
- Characterization of Kurtz randomness

Preliminary

Algorithmic randomness notions

$[0,1]$: unit interval
base set: open interval with rational endpoints
c.e. open set: computable union of base sets
μ: Lebesgue measure

Def 1. Martin-Löf test *(or ML-test)* $\{U_n\}$*: uniformly c.e. open sets with* $\mu(U_n) \leq 2^{-n}$
$z \in [0,1]$ passes *the ML-test if* $z \notin \bigcap_n U_n$.
z *is* Martin-Löf random *if it passes all Martin-Löf tests.*
Generalized ML-test *if* $\mu(U_n) \to 0$
Schnorr test *if* $\mu(U_n) = 2^{-n}$
Kurtz test *if* $\mu(U_n) = 1$

weakly 2-rnd $\subseteq$ ML-rnd $\subseteq$ Sch-rnd $\subseteq$ Kurtz-rnd

Characterizations by differentiability

f is of *bounded variation* if

$$\sup \sum_{i=1}^{n} |f(t_{i+1}) - f(t_i)| < \infty$$

where $t_1 < t_2 < \ldots < t_n$ in $[0, 1]$.

Thm 2 (essentially by Lebesgue 1904)**.**
$f : [0, 1] \to \mathbb{R}$: bounded variation
$f'(z)$ exists for a.e. z

Thm 3 (Demuth; Nies, Brattka and Miller)**.**
$z \in [0, 1]$ is ML-random
$\iff$ each comp. f of bounded variation is differentiable at z

Other randomness notions

Thm 4 (Nies, Brattka and Miller)**.**
z is weakly 2-random
$\iff$ each a.e. differentiable comp. f is differentiable at z.

Thm 5 (Freer, Kjos-Hanssen, Nies)**.**
z is Schnorr random
$\iff$ each Lipschitz f that is comp. in the variation norm is differentiable at z.

How about Kurtz randomness?
Why is the correspondence?

Computable metric space

Computable metric space

Def 6.
computable metric space $\mathbf{X} = (X, d, S)$

- (X, d): *separable complete metric space*
- $S = \{s_i \ : \ i \in \mathbb{N}\}$: *a countable dense subset*
- $d(s_i, s_j)$: *uniformly computable*

$B(x, r)$: the metric ball $\{y \in X \ : \ d(x, y) < r\}$
base set: $B(s_i, q_i)$
computable point x: $\exists$ comp. $\{i_j\}$ s.t. $x \in \bigcap_j B(s_{i_j}, 2^{-j})$
We write $\rho(p) = x$.
computable function f: $\exists$ comp. g s.t. $\rho(g(p)) = f(\rho(p))$

Computability

Prop 7.
$\mathbf{R} = (\mathbb{R}, d, \mathbb{Q})$ *is a computable metric space.*

Prop 8.
$f : X \to \mathbb{R}$ *is computable*
iff $f^{-1}(B(q_i, r_j))$ *is uniformly c.e. open*

Def 9.
A non-negative $f : X \to \mathbb{R}$ *is* lower semi-computable
if $f^{-1}(> q)$ *is c.e. open uniformly in* q.

Extended computability

Prop 10.
$\overline{\mathbf{R}} = (\mathbb{R} \cup \{\pm\infty\}, d, \mathbb{Q})$ is a computable metric space.

We say $f : X \to \overline{\mathbb{R}}$ is *extended computable* if f is computable in the usual sense.

Prop 11.
$f : X \to \overline{\mathbb{R}}$ is ext-comp.
iff $f^{-1}((p_i, q_i))$, $f^{-1}(> p)$, $f^{-1}(< q)$ are uniformly c.e. open.

Characterizations by integral tests

Integral test

Def 12.
A probabilistic measure μ is computable if $\mu(B_i)$ is uniformly c.e.

Def 13.
An integral test *is lower semi-comp.* $t : X \to \overline{\mathbb{R}}$ *with* $\int t d\mu \leq 1$.

Prop 14.
$z \in X$ *is ML-random iff* $t(z) < \infty$ *for all integral tests.*

Rem 15. *We can replace* $\int t d\mu \leq 1$ *with* $\int t d\mu < \infty$.

Weak 2-randomness

Thm 16.
z is weakly 2-random
iff $t(z) < \infty$ for all lower semi-comp. t s.t. $t(x) < \infty$ a.e.
iff $t(z) < \infty$ for all lower semi-comp. t s.t. $\int f \circ t d\mu < \infty$ for some order function f.

Proof. $\{U_n\}$: a decreasing generalized ML-test
Let $t(x) = \sup_n \{n : x \in U_n\}$.
Let $U_n = \{x : t(x) > n\}$. □

Schnorr and Kurtz randomness

Thm 17.
z is Schnorr random
iff $t(z) < \infty$ for each lower semi-comp. t s.t. $\int t d\mu$ is comp.

Thm 18. *z is Kurtz random*
iff $t(z) < \infty$ for each non-neg. ext-comp. t s.t. $\int t d\mu < \infty$
iff $t(z) < \infty$ for each non-neg. ext-comp. t s.t. $\int t d\mu$ is comp.

Ques 19. *Does this hold on a computable topological space?*

Characterizations of Kurtz randomness

Remove "non-negative"

Cor 20 (essentially by Jason Rute)**.**
z is Kurtz random
iff $f(z)$ converges for each a.e. comp. func. f.

Rem 21*.* *f, g: ext-comp. integral tests*
$f - g$ is an a.e. comp. func.

Lebesgue Differentiation Theorem

Thm 22. *TFAE:*

- z *is Kurtz random.*
- $\lim_{B\downarrow x} \frac{\int_B f d\mu}{\mu(B)}$ *converges to* $f(z)$ *for each ext-comp. integral test* f.
- $\lim_{B\downarrow x} \frac{\int_B f d\mu}{\mu(B)}$ *converges to* $f(z)$ *for each a.e. comp.* L^1*-func.*

Ques 23. *Can we replace* B *with another family of open sets?*

Differentiability

Consider the unit interval.

Cor 24. *TFAE:*

- *A real z is Kurtz random.*
- *f is differentiable at z for each non-dec. comp. func. whose derivative is ext-comp.*
- *f is differentiable at z for each comp. f s.t. f' is a.e. comp.*

Summary for Kurtz randomness

- z is Kurtz random.
- $t(z) < \infty$ for all ext-comp. integral test.
- $f(z)$ converges for each a.e. comp. func f.
- $\lim_{B \downarrow x} \frac{\int_B f d\mu}{\mu(B)}$ converges to $f(z)$ for each ext-comp. integral test f.
- $\lim_{B \downarrow x} \frac{\int_B f d\mu}{\mu(B)}$ converges to $f(z)$ for each a.e. comp. L^1-func.
- f is differentiable at z for each non-dec. comp. func. whose derivative is ext-comp.
- f is differentiable at z for each comp. f s.t. f' is a.e. comp.

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LDT non-neg	?	?	✓	✓	✓
LDT	?	✓	✓	?	✓
Diff non-neg	?	?	✓	✓	✓
Diff	✓	✓	✓	?	✓

Future work

- Characterization of computably randomness by integral tests
- Versions of Schnorr randomness, Martin-Löf randomness and weak 2-randomness
- Computable integration in the characterization of Kurtz randomness
- Radon-Nikodym derivative

Thank you!

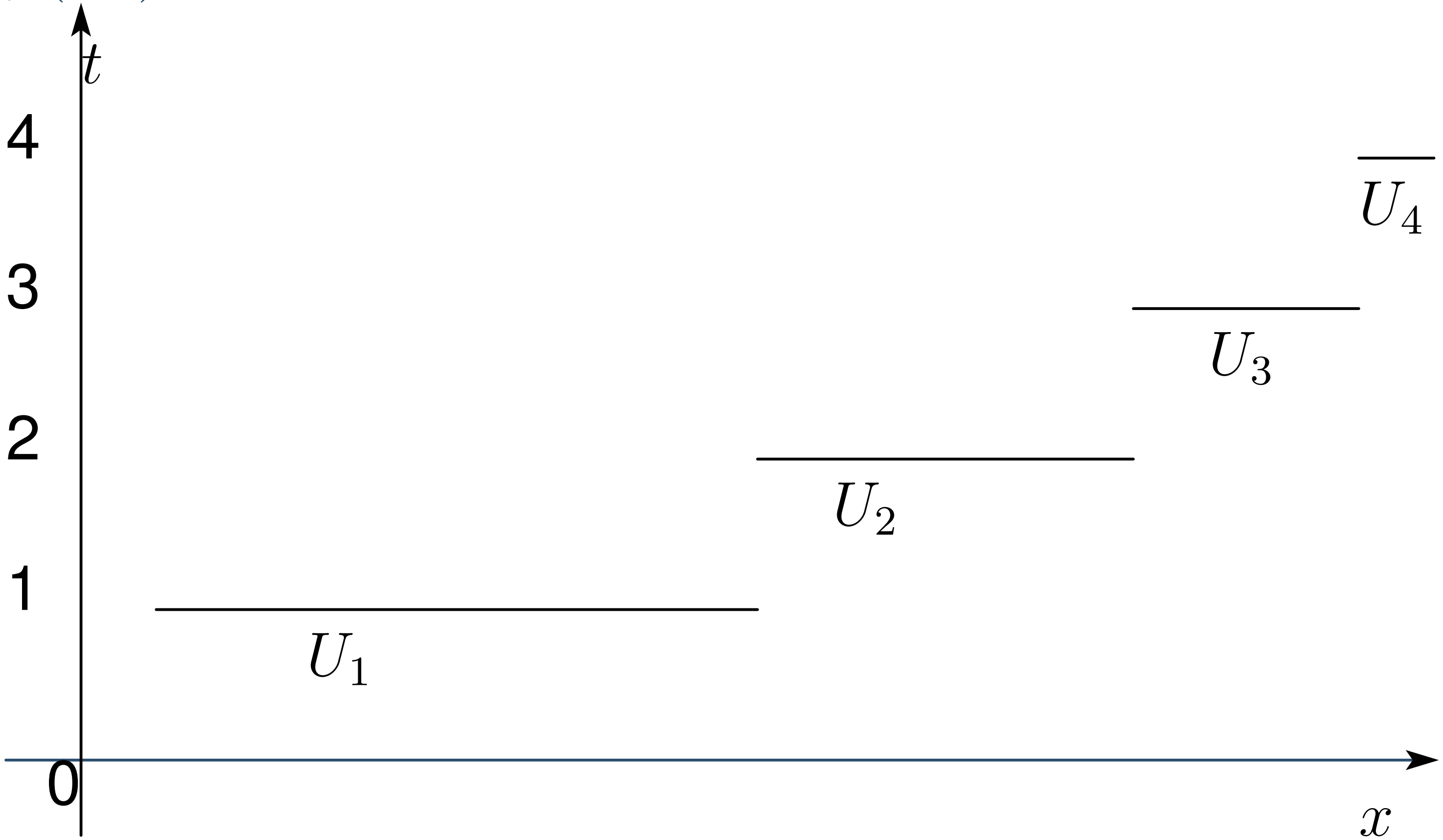
Proof

One implication

Proof. (function $\Rightarrow$ test)
$\bigcup_n \{x : t(x) < n\}$ is a c.e. open set with $\mu(U) = 1$. □

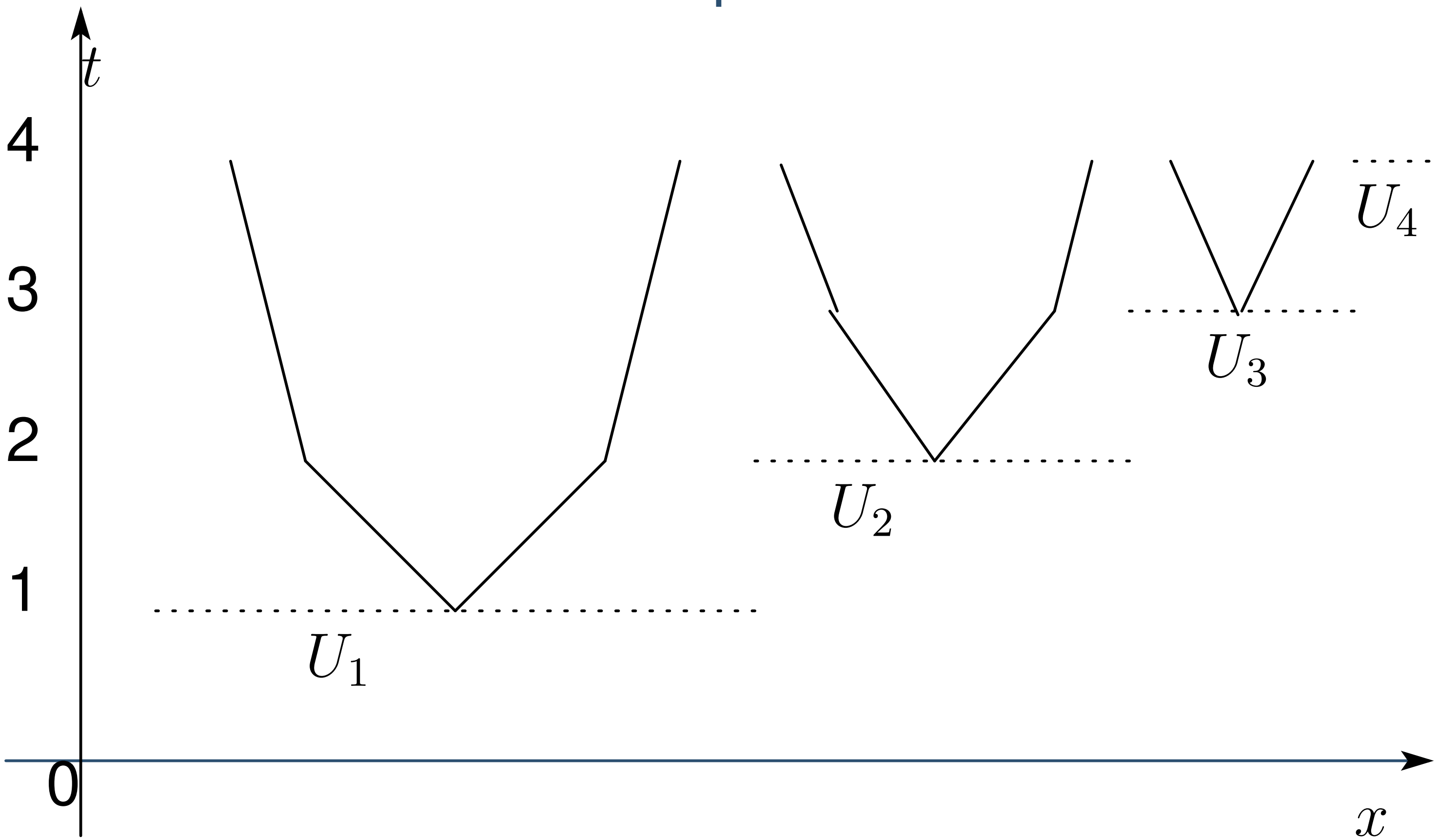
Proof idea

(test $\Rightarrow$ function)
U: a c.e. open set with $\mu(U) = 1$.
Divide U into uniformly c.e. open sets $\{U_n\}_{n\geq 1}$ s.t. $\mu(U_n) = 2^{-n}$.



Proof idea 2

To make the function computable and so continuous ...



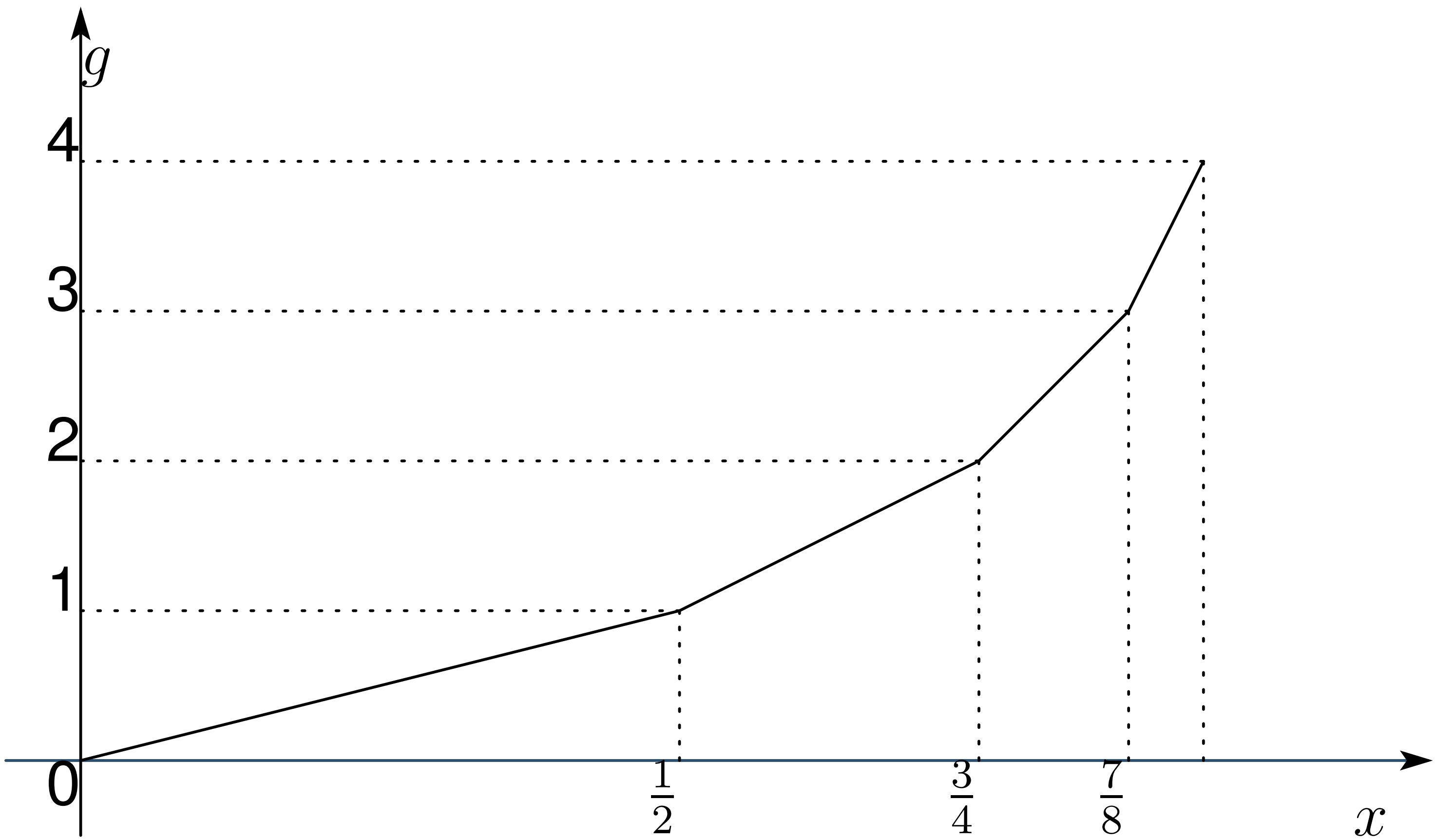
Proof

$g : [0, 1] \to [0, +\infty]$: the polyline s.t.

- the set of endpoints is $\{1 - 2^{-n} : n \geq 0\}$,
- $g(1 - 2^{-n}) = n$,
- $g(1) = \infty$

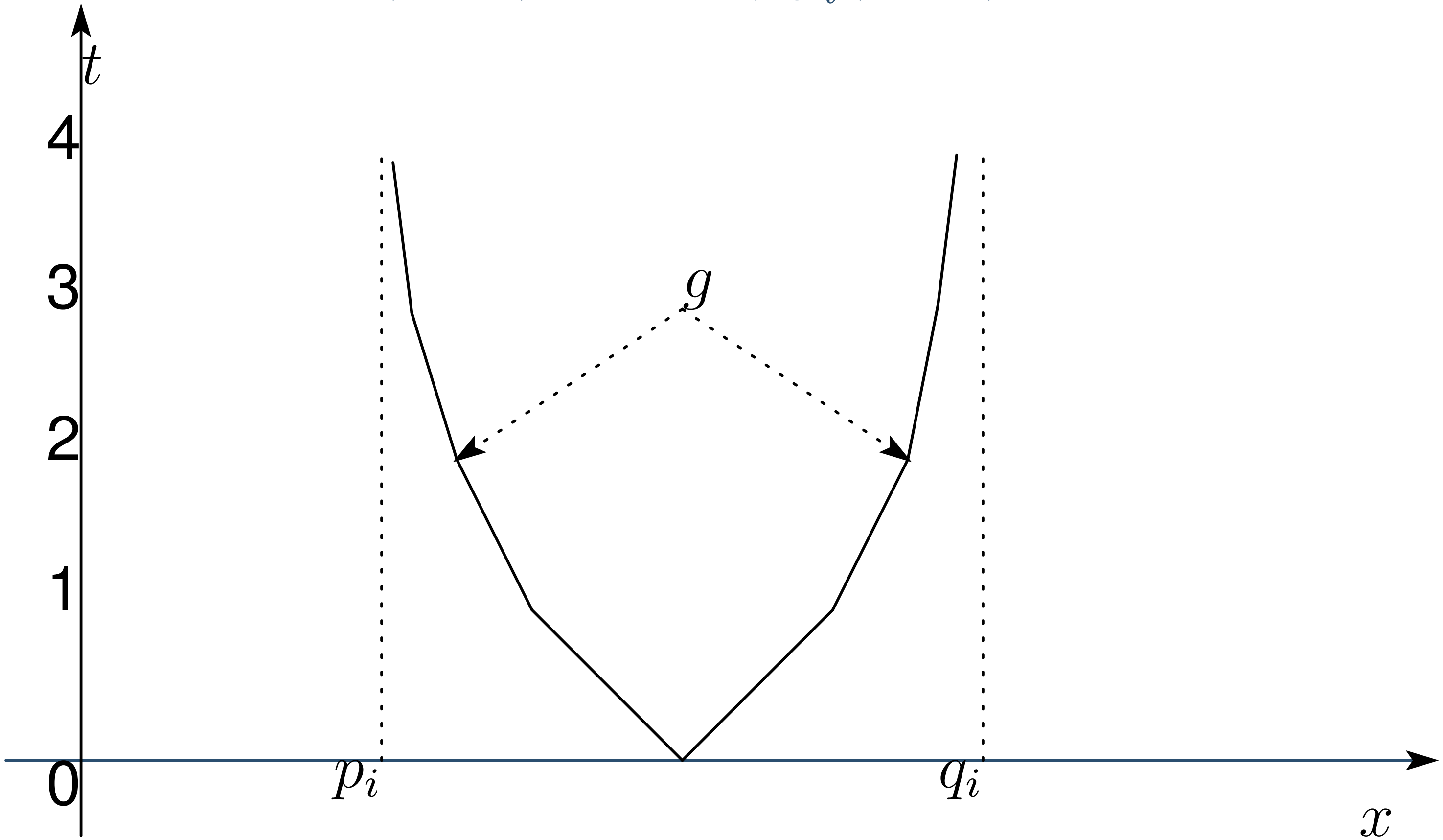
g and the integration are computable.

Proof 2



Proof 3

FINITE pairs (p_i, q_i) s.t. $U_n \setminus \bigcup_i (p_i, q_i)$ contains only rationals



$\int t(x)dx$ is computable.

Proof 4

Is t computable?

Are $t^{-1}([0,q)), t^{-1}((p,q)), t^{-1}((p,+\infty])$ uniformly c.e.?

The set of pairs (p_i, q_i) is finite for each n

$\Rightarrow t^{-1}([0,n))$

$\Rightarrow t^{-1}([0,q))$ and $t^{-1}((p,q))$

Proof 5

How to compute $t^{-1}((p, +\infty])$?
Pick up $n \geq p$ and enumerate all pairs (p_i, q_i) until n.
t maps the complement more than n.