Algorithmic randomness over general spaces

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Abstract

The study of Martin-Löf randomness on a computable metric space with a computable measure has had much progress recently. In this paper we study Martin-Löf randomness on a more general space, that is, a computable topological space with a computable measure. On such a space, Martin-Löf randomness may not be a natural notion because there is not a universal test, and Martin-Löf randomness and complexity randomness (defined in this paper) do not coincide in general. We show that SCT_3 is a sufficient condition for the existence and the coincidence and study how much we can weaken the condition.

Keywords: computable analysis, computable topological space, separation axiom, computable measure, Martin-Löf randomness

1 Introduction

What does it mean to say that a binary sequence is "random"? Von Mises [26] tried to give an answer by introducing the concept of a *collective* with the motivation of formalizing probability. A collective is an actual sequence whose limiting frequency exists and remains the same when one replaces the sequence with a subsequence. Although Wald and Church gave a rigorous mathematical definition of this concept using the notion of computability, Ville [25] showed that there exists a collective that does not satisfy the law of the iterated logarithm, which a random sequence should satisfy. Hence a collective is not a natural randomness notion.

About fifty years later, Martin-Löf [17] introduced another definition of randomness that is called *Martin-Löf randomness* now. Martin-Löf randomness has many nice properties. In this paper we forcus on the following two important properties.

(i) Martin-Löf randomness has a universal test, that is, only one test is enough to see whether a sequence is random or not.

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(ii) Martin-Löf randomness has several characterizations by *complexity*. The initial contributors are Levin [14, 15], Schnorr [21] and Chaitin [4].

It should be noted that, unlike a collective, each Martin-Löf random sequence satisfies the strong law of large numbers (SLLN) and the law of the iterated logarithm (LIL) [27, 16]. This is important from the view point of probability theory. As a result, Martin-Löf randomness is now regarded as a natural randomness notion.

Martin-Löf randomness can be generalized to a more general space. Computable analysis [30, 3, 33] defined a *computable metric space* and a *computable topological space*, which are a metric space and a topological space equipped with computability. Martin-Löf randomness on a computable topological space has studied in the literature such as Zvonkin and Levin [36] and Hertling and Weihrauch [10]. However it has not been well developed. The following should be noted the following comparing to Martin-Löf randomness on Cantor space.

(i) They showed the existence of a universal test. However they use stronger computability of the measure than we use in this paper.

(ii) So far no characterization by complexity has been known.

In contrast the study on Martin-Löf randomness on a computable metric space had much progress recently.

(i) Martin-Löf randomness has a universal test.

(ii) A characterization by complexity of Martin-Löf test was partially given by Gács [8] and was completely showed by Hoyrup and Rojas [11]. although they use a stronger condition than usual.

The notion of probability has a strong relation with randomness while the usual probability theory does not have a rigorous definition of randomness. Since the probability theory is developed on a measure space, we would like to have a natural and mathematical notion of randomness on a computable topological space with a computable measure.

An important question is whether Martin-Löf randomness is a natural randomness notion even on a computable topological space with a computable measure. Martin-Löf randomness seems a natural notion on Cantor space and on a computable metric space with a computable measure while it is doubtful that Martin-Löf randomness on a computable topological space with a computable measure is a natural randomness notion.

The main claim of this paper is that Martin-Löf randomness is a natural randomness notion only when the underlying topological space has a somewhat strong topology. We will show that, on a computable topological space with a computable measure,

(i) Martin-Löf randomness does not have a universal test,

(ii) Martin-Löf randomness is not equivalent to a randomness notion induced by a kind of complexity in general.

Another important claim is the following. We define a randomness notion by complexity and call it *complexity randomness*. The definition of this randomness notion uses "closed" sets (not open sets), which is siginificantly different from the attempts up to now. The author believes that a randomness notion defined by complexity should use closed sets and will explain the reason later.

Another question discussed in this paper is what is the right definition of a computable measure on a computable topological space. on a computable metric space with a computable measure, Hoyrup and Rojas [11] gave the complete characterization by complexity of Martin-Löf randomness. The argument depends on the detailed study of a computable measure on a computable metric space. The definition of complexity randomness also highly depends on the argument on the definition of a computable measure on a computable topological space.

We review some works of the study of computability of measures. On the unit interval, Weihrauch [29] gave natural computability of measures. Schröder [23] generalized it to a computable topological space. Bosserhoff [2] also used the definition.

Another way is to consider the space of measures. The space of measures on a computable metric space is another computable metric space, which is proved by Gács [8]. This observation naturally induces a natural computability of measures, that is, a computable measure is a computable point in the space. In this paper we take this approach and generalize it. We will see that the definition coincides with the definition by Schröder [23].

Edalat [7] used regularity to study computability of measures and, because of it, it is not general enough. Weihrauch et al. [35, 12] defined a *computable measurable space* but the requirement of computable measures is too strong from our point of view.

The paper is constructed as follows. In Section 2 we review the needed notions from various areas, namely from topology, measure theory, computable analysis, and about computable topological spaces. In Section 3 we study computability of measures on a computable topological space. In Section 4 we show an unnatural property of Martin-Löf randomness on a computable topological space, taht is, Martin-Löf randomness does not have a universal test in general. In Section 5 we define complexity randomness and study when Martin-Löf randomness and complexity randomness coincide.

2 Preliminaries

2.1 General topology

We review general topology from [34]. A *topology* on a set X is a class τ of subsets of X, called the *open sets*, satisfying the following: closed under union, closed under finite intesection and $\emptyset, X \in \tau$. We say (X, τ) is a *topological space*. A closed set is the complement of an open set. A *base* for τ is a class $\beta \subseteq \tau$

such that each element of τ is the union of elements of β. A *subbase* for τ is a class $\beta' \subseteq \tau$ such that the class of all finite intersections of elements of β' forms a base for τ. A space is *second countable* if it has a countable base.

The following are some separation axioms. Let $\mathcal{A}$ be the class of closed sets.

$$
\begin{aligned}
&T_0 :(\forall x, y \in X, x \neq y)(\exists W \in \tau)((x \in W \wedge y \notin W) \vee (x \notin W \wedge y \in W))),\\
&T_1 :(\forall x, y \in X, x \neq y)(\exists W \in \tau)(x \in W \wedge y \notin W),\\
&T_2 :(\forall x, y \in X, x \neq y)(\exists U, W \in \tau)(U \cap V = \emptyset \wedge x \in U \wedge y \in V),\\
&T_3 :(\forall x \in X, \forall A \in \mathcal{A}, x \notin A)(\exists U, W \in \tau)(U \cap V = \emptyset \wedge x \in U \wedge A \subseteq V).
\end{aligned}
$$

We will speak of T_i-spaces for $i = 0, 1, 2, 3$.

For topological spaces X and Y, a function $f : X \to Y$ is continuous iff for each open set H in Y, $f^{-1}(H)$ is open in X. The class of all continuous mappings from X to Y is denoted by $C(X, Y)$; if $Y = \mathbb{R}$, then this class is denoted by $C(X)$. The set of all bounded functions in $C(X)$ is denoted by $C_b(X)$.

2.2 Measure theory

We review measure theory from [1]. An *algebra* $\mathcal{F}$ of sets is a class of subsets of some fixed set X such that $\emptyset, X \in \mathcal{F}$ and it is closed under union, intersection and complement. An algebra of sets $\mathcal{A}$ is called a *σ-algebra* if it is also closed countable union. A real-valued set function μ on a class of sets $\mathcal{F}$ is called *countably additive* if $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for all pairwise disjoint sets A_n in $\mathcal{F}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$. A countably additive set function defined on an algebra is called a *measure*. A measure is *probabilistic* if $\mu(X) = 1$. In this paper we only consider non-negative probabilistic measures and we use a measure to mean a non-negative probabilistic measure.

On a topological space X, the *Borel σ-algebra* $\mathcal{B}(X)$ is the σ-algebra generated by all open sets. The sets in $\mathcal{B}(X)$ are called the *Borel sets* in the space X. The *Baire σ-algebra* $\mathcal{B}a(X)$ is the σ-algebra generated by all sets of the form $\{x \in X \ : \ f(x) > 0\}$

The sets in $\mathcal{B}a(X)$ is called the *Baire sets* in the space X. The sets of the form $\{x \in X \ : \ f(x) > 0\}$ where $f \in C(X)$ are called *functionally open* and their complements are called *functionally closed*. In a metric space, any closed set is the set of zeros of a continuous function. Hence the Borel and Baire σ-algebras of a metric space coincide.

A measure on the Borel σ-algebra $\mathcal{B}(X)$ is called a *Borel measure* on X and a measure on the Baire σ-algebra $\mathcal{B}a(X)$ is called a *Baire measure*. We use $\mathcal{M}(X)$ to mean the set of all (non-negative probabilistic) Borel measures and $\mathcal{M}_\sigma(X)$ to mean the set of all (non-negative probabilistic) Baire measures. The *weak topology* on the space $\mathcal{M}_\sigma(X)$ of Baire measures on X is the topology with the base of the sets

$$W_{G_1,\ldots,G_n,\epsilon}(\mu) = \{\nu \in \mathcal{M}_\sigma(X) \ : \ \nu(G_i) > \mu(G_i) - \epsilon, i = 1, \ldots, n\},$$

where $G_i = X \backslash f_i^{-1}(0),\ f_i \in C(X),\ \epsilon > 0$. The *A-topology* on the space $\mathcal{M}(X)$ is defined by means of neighborhoods of the form

$$U(\mu, G, \epsilon) = \{\nu \ : \ \mu(G) < \nu(G) + \epsilon\},$$

where $\mu \in \mathcal{M}(X),\ G \in \mathcal{O}(X),\ \epsilon > 0$. In a metric space, the A-topology coincides with the weak topology.

2.3 Computable analysis

We will use essentially the terminology from [33]. We assume that the readers are familiar with computability on Σ^*, which has been well studied [24, 19, 20, 5].

Let Σ be a finite alphabet such that $0, 1 \in \Sigma$. By Σ^* we denote the set of finite words over Σ, and by Σ^ω the set of infinite sequences $p : \mathbb{N} \to \Sigma$ over Σ, $p = (p(0)p(1)\ldots)$. We use the "wrapping function" $\iota : \Sigma^* \to \Sigma^*$, $\iota(a_1a_2\ldots a_k) := 110a_10a_20\ldots a_k011$. For $u \in \Sigma^*$ and $w \in \Sigma^* \cup \Sigma^\omega$ let $u \ll w$ iff $\iota(u)$ is a subword of w. Let $\langle i, j\rangle := (i+j)(i+j+1)/2+j$ be the bijective Cantor pairing function on $\mathbb{N}$. We consider standard functions for finite or countable tupling on Σ^* and Σ^ω denoted by $\langle\cdot\rangle$ in [30, Definition 2.1.7].

A *partial function* is denoted by $f :\subseteq A \to B$ and a *total function* is denoted by $f : A \to B$. Let $Y_0, \ldots, Y_n \in \{\Sigma^*, \Sigma^\omega\}$ and $Y = Y_1 \times \ldots \times Y_n$. A function $f :\subseteq Y \to Y_0$ is computable if for some Type-2 machine M, f is the function f_M computed by M. Informally, a Type-2 machine is a Turing machine, which reads from input tapes with finite or infinite inscription, operates on work tapes and write one-way to an output tape. For $Y_0 = \Sigma^*$, $f_M(y) = w$, if a Turing machine M on input y halts with w on the output tape. For $Y_0 = \Sigma^\omega$, $f_M(y) = q$, if M on input y computes forever and writes writes $q \in \Sigma^\omega$ on the output tape. On Σ^* we consider the discrete topology and on Σ^ω the topology generated by the base $\{w\Sigma^\omega \ : \ w \in \Sigma^*\}$ of open sets. Every computable function is continuous.

A *notation* of a set M is a surjective function $\gamma :\subseteq Y \to M$ where $Y = \Sigma^*$ and a *representation* where $Y = \Sigma^\omega$. An examples is the representation $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$ of the real numbers, which we will define later. A partial function $h :\subseteq Y \to Y_0$ *realizes* a function $f :\subseteq M \to M_0$ if $f(x) = \gamma_0 \circ h(y)$ whenever $x = \gamma(y)$ and $f(x) \downarrow$. This means that $h(y)$ is a name of $f(x)$ if y is a name of $x \in \mathrm{dom}(f)$. The function f is called (γ, γ_0)-computable if it has a computable realization. A point $x \in M_1$ is γ_1*-computable* iff $x = \gamma_1(p)$ for some computable $p \in \mathrm{dom}(\gamma_1)$.

We write $\gamma_1 \leq \gamma_0$ (γ_1 is reducible to γ_0) if $M_1 \subseteq M_0$ and the identity id : $M_1 \to M_0$ is (γ_1, γ_0)-computable. This means that some computable function h translates γ_1-names to γ_0-names, that is, $\gamma_1(p) = \gamma_0 \circ h(p)$. Computable equivalence is defined canonically.

F^{**} is the set of all partial continuous functions $f :\subseteq \Sigma^* \to \Sigma^*$, $F^{*\omega}$ is the set of all partial continuous functions $f :\subseteq \Sigma^* \to \Sigma^\omega$, $F^{\omega *}$ is the set of all partial continuous functions $f :\subseteq \Sigma^\omega \to \Sigma^*$ with open domain and $F^{\omega\omega}$ is the set of all partial continuous functions $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ with G_δ-domain (a G_δ-set is a countable intersection of open sets). For $a, b \in \{*, \omega\}$, let η^{ab} be the standard representations of F^{ab} in [30, Def 2.3.10]. For representations

$\gamma_1 :\subseteq \Sigma^a \to M_1$ and $\gamma_2 :\subseteq \Sigma^b \to M_2$, $a, b \in \{*, \omega\}$, a representation $[\gamma_1 \to_p \gamma_2]$ of the (γ_1, γ_2)-continuous functions $f :\subseteq M_1 \to M_2$ is defined by $f = [\gamma_1 \to_p \gamma_2](q)$ $\iff \eta_q^{ab} := \eta^{ab}(q)$ realizes f w.r.t. (γ_1, γ_2) where The restriction of $[\gamma_1 \to_p \gamma_2]$ to the total (γ_1, γ_2)-continuous functions is denoted by $[\gamma_1 \to \gamma_2]$.

2.4 Computable topological spaces

Definition 2.1 (computable topological spaces). *An* effective topological space *is a 4-tuple* $\mathbf{X} = (X, \tau, \beta, \nu)$ *such that* (X, τ) *is a topological* T_0*-space and* $\nu :\subseteq \Sigma^* \to \beta$ *is a notation of a base* β *of* τ. $\mathbf{X}$ *is a* computable topological space *if* $\mathrm{dom}(\nu)$ *is computable and*

$$\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \ : \ (u, v, w) \in S\} \text{ for all } u, v \in \mathrm{dom}(\nu) \tag{1}$$

for some c.e. set $S \subseteq (\mathrm{dom}(\nu))^3$.

Definition 2.2. *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be an effective topological space.*

Define a representation $\delta :\subseteq \Sigma^\omega \to X$ *of the points as*

$$x = \delta(p) \iff (\forall w \in \Sigma^*)(w \ll p \iff x \in \nu(w))$$

and a representation $\theta :\subseteq \Sigma^\omega \to \tau$ *of the set of open sets as*

$$W = \theta(p) \iff \begin{cases} w \ll p \Rightarrow w \in \mathrm{dom}(\nu) \\ W = \bigcup \{\nu(w) \ : \ w \ll p\}. \end{cases}$$

Define a representation $\delta^- :\subseteq \Sigma^\omega \to X$ *of the points as*

$$\delta^-(p) = x \iff \theta(p) = X \setminus \overline{\{x\}},$$

where $\overline{A}$ *is the closure of the set* $A \subseteq X$ *and define a representation* $\psi^- :\subseteq \Sigma^\omega \to \mathcal{A}$ *of the set of closed sets as*

$$\psi^-(p) = X \setminus \theta(p).$$

A δ-name of a point x is a list of all names of all of its basic neighborhoods and a θ-name of an open set W is a list of base elements exhausting W.

Example 2.3 (computable topological spaces).

(i) *(real line) Define* $\mathbf{R} = (\mathbb{R}, \tau, \beta, \nu)$ *such that* τ *is the real line topology and* ν *is a canonical notation of the set of all open intervals with rational endpoints. The representation* δ *for* $\mathbf{R}$ *is denoted by* ρ.

(ii) *(unit interval) Define* $\mathbf{I} = ([0, 1], \tau', \beta', \nu')$ *as the restriction of* $\mathbf{R}$ *to* $[0, 1]$. *The representation* δ *for* $\mathbf{I}$ *is denoted by* ρ.

(iii) *(extended real line with lower topology) Define* $\overline{\mathbf{R}}^+_< = (\overline{\mathbb{R}}^+, \tau_<, \beta_<, \nu_<)$ *where* $\overline{\mathbb{R}}^+$ *is the set of non-negative reals and* $+\infty$, $\tau_<$ *is the lower topology,* $\beta_<$ *is the set*

$$\{\overline{\mathbb{R}}^+\} \cup \{(q, +\infty] \ : \ q \in \mathbb{Q} \cap [0, +\infty)\},$$

and $\nu_<$ *is a canonical notation of* $\beta_<$*. The representation* δ *of points in* $\overline{\mathbf{R}}^+_<$ *is denoted by* $\overline{\rho}_<$*.*

(iv) *(lower unit interval) Define* $\mathbf{I}_< = ([0,1], \tau'_I, \beta'_<, \nu'_<)$ *as the restriction of* $\overline{\mathbf{R}}^+_<$ *to* $[0,1]$*.*

We sometimes say that a point is *computable* to mean that the point is ρ-computable, that an open set is *c.e.* to mean that the open set is θ-computable and that a closed set is *co-c.e.* to mean that the closed set is ψ^--computable. On $\mathbb{R}$ or $\mathbb{I}$ we say that a real is *c.e.* to mean that the real is $\rho_<$-computable and that a real α is *right-c.e.* if $D - \alpha$ is c.e. for some $D \in \mathbb{N}$. We also say that a function $f :\subseteq X \to X_1$ is *computable* to mean that f is (δ, δ_1)-computable and that a function $f : X \to \overline{\mathbb{R}}^+$ is *lower semicomputable* to mean that f is $(\delta, \overline{\rho}_<)$-computable.

In what follows we assume that any effective topological space is equipped with these representations correspondingly. For example, for an effective topological space $\mathbf{X}_1 = (X_1, \tau_1, \beta_1, \nu_1)$, the representation δ_1 denotes the representation of points in X_1 defined here.

A variety of operations on points, sets and functions are computable w.r.t. the representations from Definition 2.2. We give some additional examples.

Theorem 2.4 ([33])**.**

(i) $\mathrm{eval} : (f, x) \mapsto f(x)$ *is* $([\delta_1 \to_p \delta_2], \delta_1, \delta_2)$*-computable.*

(ii) *For* $f :\subseteq X_1 \to X_2$ *and* $g :\subseteq X_2 \to X_3$, $(f, g) \mapsto g \circ f$ *is* $([\delta_1 \to_p \delta_2], [\delta_2 \to_p \delta_3], [\delta_1 \to_p \delta_3])$*-computable.*

(iii) *There exists a* $([\delta_1 \to_p \delta_2], \theta_2, \theta_1)$*-computable function that maps every continuous function* $f :\subseteq X_1 \to X_2$ *and every open set* $W \subseteq X_2$ *to some open set* $T \subseteq X_1$ *such that* $f^{-1}[W] = T \cap \mathrm{dom}(f)$*.*

Weihrauch [31, 32] studied various computable versions of separation axioms. In this paper we use the following.

Definition 2.5 (axioms of computable separation)**.** *Let* $\mathbf{X}$ *be a computable topological space.*

CT'_2**:** *There is a c.e. set* $H \subseteq \Sigma^* \times \Sigma^*$ *such that*
$(\forall x, y,\ x \neq y)(\exists (u, v) \in H)(x \in \nu(u) \wedge y \in \nu(v))$ *and*
$(\forall (u, v) \in H)(\nu(u) \cap \nu(v) = \emptyset \vee (\exists x)\nu(u) = \{x\} = \nu(v))$*.*

SCT_2**:** *There is a c.e. set* $H \subseteq \Sigma^* \times \Sigma^*$ *such that*
$(\forall x, y,\ x \neq y)(\exists (u, v) \in H)(x \in \nu(u) \wedge y \in \nu(v))$ *and*
$(\forall (u, v) \in H)\nu(u) \cap \nu(v) = \emptyset$*.*

SCT$_3$: *There are a c.e. set* $R \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ *and a computable function* $r :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$ *such that for all* $u, w \in \mathrm{dom}(\nu)$*,*
$\nu(w) = \bigcup\{\nu(u) \ : \ (u, w) \in R\}$*,*
$(u, w) \in R \Rightarrow \nu(u) \subseteq \psi^- \circ r(u, w) \subseteq \nu(w)$*.*

Weihrauch [32] also cosidered the axiom of CT$_2$, which uses a multi-function in the definition. He showed CT$_2 \iff$ CT$'_2$. For simplicity, we prefer to call a space CT$_2$ and to use the definition of CT$'_2$. Note that SCT$_3 \Rightarrow$ CT$_2 \Rightarrow T_2$ and SCT$_3 \Rightarrow T_3$.

3 Computability of measures

In this section we study computability of measures via the representation approach.

In measure theory a lot of connections are known between the properties of X and the corresponding properties of the spaces of measures such as completeness, compactness, metrizability and separability. Gács [8] proved that the space of measures on a computable metric space is another computable metric space. This result is another correspondence between a space and the space of measures on it.

In the result above the space of measures is equipped with the weak topology. On a more general space, however, A-topology is more natural. Recall that the A-topology coincides with the weak topology on a metric space. Similar connections are also known with the A-topology. Refer to [1, 8.10(iv)] for the detail. Here we show a version on a computable topological space of Gács' result.

3.1 A computable topological space of measures

Here we show that the space of measures on a computable topological space is another computable topological space with a natural structure. In what follows a measure is always a Borel measure.

Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space. We consider the space $\mathcal{M}(X)$ of Borel measures. Let τ_A be the A-topology on $\mathcal{M}(X)$. The following sets form a countable subbase of the A-topology:

$$\{\mu \ : \ \mu(G) > q\}$$

where G is a finite union of base sets and $q \in \mathbb{Q} \cap [0, 1]$.

Then $\mathcal{M}(X)$ is second-countable because the above sets form a countable subbase. Furthermore the space $\mathcal{M}(X)$ is always T_0.

Proposition 3.1. *The space* $\mathcal{M}(X)$ *with the A-topology is* T_0*.*

Proof. Let μ_1, μ_2 be measures on X such that $\mu_1 \neq \mu_2$. Then there exists an open set O such that $\mu_1(O) \neq \mu_2(O)$ by [1, Lemma 7.1.2.]. It follows that there exists a finite union G of base sets such that $\mu_1(G) \neq \mu_2(G)$. We can assume that $\mu_1(G) < \mu_2(G)$ without loss of generality. Then there exists $q \in \mathbb{Q} \cap [0,1]$ such that $\mu_1(G) < q < \mu_2(G)$. Let $U = \{\mu \ : \ \mu(G) > q\}$. Note that U is an open set on $\mathcal{M}(X)$ with the A-topology. We also have $\mu_1 \notin U$ and $\mu_2 \in U$. Hence the space is T_0. $\square$

Now we equip $\mathcal{M}(X)$ with a natural structure to be a computable topological space. For a notation $\mu :\subseteq \Sigma^* \to M$ define a notation of finite subsets as

$$\mu^{\mathrm{fs}}(w) = W \iff \begin{cases} (\forall v \ll w) v \in \mathrm{dom}(\mu), \\ W = \{\mu(v) \ : \ v \ll w\}. \end{cases}$$

Then ν^{fs} is a notation of finite unions of base sets. Let μ_A be the notation of the countable subbase of the A-topology such that

$$\mu_A(\langle u, v \rangle) = \{\mu \ : \ \mu(\nu^{\mathrm{fs}}(u)) > \nu_{\mathbb{Q}\cap[0,1]}(v)\}.$$

Let $\nu_A = \bigcap \mu_A^{\mathrm{fs}}$. Then ν_A is a notation of a base. Let $\beta_A = \{\nu_A(w) \ : \ w \in \mathrm{dom}(\nu_A)\}$. Then β_A is the base.

Proposition 3.2. *The 4-tuple* $\mathbf{M}(\mathbf{X}) = (\mathcal{M}(X), \tau_A, \beta_A, \nu_A)$ *is a computable topological space.*

Proof. It is clear that $\mathbf{M}(\mathbf{X})$ is an effective topological space. All we have to do is to see whether (1) holds.

Let $\sigma_i \in \mathrm{dom}(\nu_A)$ for $i = 1, 2$. Then there exists finite sets W_i such that $W_i = \{\tau_i \ : \ \tau_i \ll \sigma_i\}$. Note that $\tau_i \in W_i \Rightarrow \tau_i \in \mathrm{dom}(\mu_A)$. Let $W = W_1 \cup W_2$. Then

$$\bigcap_{\tau \in W} \mu_A(\tau) = \bigcap_{\tau_1 \in W_1} \mu_A(\tau) \cap \bigcap_{\tau_2 \in W_2} \mu_A(\tau) = \nu_A(\sigma_1) \cap \nu_A(\sigma_2).$$

Hence one can computably find σ_3 from σ_1 and σ_2 such that $\nu_A(\sigma_1) \cap \nu_A(\sigma_2) = \nu_A(\sigma_3)$. This ensures the existence of a c.e. set $S_A \subseteq ((\Sigma^*)^3)$ satisfying (1). $\square$

We give an intuitive interpretation of δ_A-name p of $\mu \in \mathbf{M}$. For each finite union G of base sets, we can enumerate all rationals q such that $\mu(G) > q$ from p. We say that $\mu(G)$ is approximated from below by p. Conversely suppose that $\mu(G)$ is approximated from below for each G with help from p'. Then we can construct a representation of μ from p'. In the following we use this argument without giving further details. By this discussion, it is easy to see that δ_A is equivalent to the canonical representation in Schröder [23].

Now the following definition is natural. Note that this definition is a general version of that of a computable measure on a computable metric space in [8, 11].

Definition 3.3. *Let* $\mathbf{X} = (X, \tau, \beta, \nu)$ *be a computable topological space. A measure on* X *is* computable *if it is a computable point on the computable topological space* $\mathbf{M}$.

The following is immediate.

Proposition 3.4. *A measure* μ *is computable iff the meausre of the finite union of base sets is uniformly approximated from below.*

3.2 Some properties

From here we study some computable operations on **M**.

Proposition 3.5. *Let $\mathbb{R}^+$ be the set of non-negative reals and $\overline{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{\infty\}$. We also write $\mathcal{M}$ to mean $\mathcal{M}(X)$.*

(i) *The plus operation $+ : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is $(\delta_A, \delta_A, \delta_A)$-computable.*

(ii) *Let* eval $: \mathcal{M} \times \tau \to \mathbb{I}$ *be such that* $\mathrm{eval}(\mu, G) = \mu(G)$. *Then* eval *is $(\delta_A, \theta, \rho_<)$-computable.*

(iii) *The integral operation $\int : C(X, \mathbb{R}^+) \times \mathcal{M} \to \overline{\mathbb{R}}^+$ is $([\delta \to \rho_<], \delta_A, \overline{\rho}_<)$-computable.*

(iv) *The integral operation $\int : C_b(X, \mathbb{R}^+) \times \mathcal{M} \to \mathbb{R}^+$ is $([\delta \to \rho], \delta_A, \rho)$-computable.*

Proof. (i) Let μ_1, μ_2 be measures in $\mathcal{M}(X)$ and p_1, p_2 be δ_A-names respectively. For each finite union G of base sets, $\mu_1(G)$ and $\mu_2(G)$ are approximated from below by p_1 and p_2. Hence $(\mu_1 + \mu_2)(G)$ is also approximated from below by p_1 and p_2. Then one can construct a name of $\mu_1 + \mu_2$ from p_1 and p_2.

(ii) For each open set G, we can enumerate all finite union of base sets which is contained in G by a θ-name of G. The measure μ of each finite union of base sets is approximated by rationals from below by a δ_A-name of μ. This follows that $\mu(G)$ is approximated by rationals from below.

(iii) Let $f \in \mathcal{C}(X, \mathbb{R}^+)$ and $\mu \in \mathcal{M}(X)$. Note that

$$\int f d\mu = \sup\{\sum_{i=1}^{k} (a_i - a_{i-1}) \cdot \mu(f^{-1}(a_i, \infty)) \ : \ 0 = a_0 < a_1 < \ldots < a_k, \ a_i \in \mathbb{Q}\}.$$

Since (a_i, ∞) is open in $\mathbb{R}^+$ for each i, one obtains θ-name of $f^{-1}(a_i, \infty)$ by the representation of f by Proposition 2.4. Hence $\mu(f^{-1}(a_i, \infty))$ is approximated by rationals from below by the representations of f and μ by (ii).

(iv) Let $f \in \mathcal{C}_b(X, \mathbb{R}^+)$ and $\mu \in \mathcal{M}(X)$. Since f is bounded, there exists D such that $D - f \in \mathcal{C}(X, \mathbb{R}^+)$. Then

$$\int_X f d\mu = D\mu(X) - \int_X (D - f) d\mu (< \infty)$$

is approximated by rationals from above. □

4 Universality

Here we study the existence of a universal Martin-Löf test on a computable topological space with a computable measure. Zvonkin and Levin [36] showed the existence of a universal ML-test with a measure such that the measure of the finite unions of base sets is uniformly *computable*. Hertling and Weihrauch [10] showed the existence with a measure such that the measure of the finite unions of base sets is uniformly *right-c.e.* From now on we consider a computable measure in our definition, that is, a measure such that the measure of the finite unions of base sets is uniformly *c.e.*

4.1 Definition

Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space and μ be a a computable non-negative probabilistic Borel measure.

Definition 4.1 ([36, 10])**.** *A* Martin-Löf test (or ML-test) *on* $\mathbf{X}$ *is a sequence of uniformly c.e. open sets* $\{U_n\}$ *with* $\mu(U_n) \leq 2^{-n}$. *A point* $x \in X$ *is* Martin-Löf random (or ML-random) *if it passes all ML-tests, that is,* $x \notin \bigcap_n U_n$.

Definition 4.2 (universal test)**.** *A ML-test* $\{U_n\}$ *is* universal *if, for each ML-test* $\{V_n\}$*, we have* $\bigcap_n V_n \subseteq \bigcap_n U_n$.

The question of this section is whether a universal test exists. It should be noted that the existence of a universal test is equivalent to the existsnce of a universal integral test.

Definition 4.3. *An* integral test *is a lower semicomputable function* $t : X \to \overline{\mathbb{R}}^+$ *such that* $\int t d\mu \leq 1$. *An integral test is* universal *if, for each integral test* f*, we have* $\{x \mid f(x) = \infty\} \subseteq \{x \mid t(x) = \infty\}$.

Proposition 4.4. *A point* $x \in X$ *is ML-random iff* $t(x) < \infty$ *for all integral tests.*

Proof. Suppose that x is not ML-random. Then there exists a ML-test $\{U_n\}$ with $x \in \bigcap_n U_n$. We can assume that $\{U_n\}$ is decreasing. Let $t(y) = \sup_n \{n \mid y \in U_n\}$. Then t is lower semicomputable and $\int t d\mu < \infty$. Hence $t' = t/c$ is an integral test for some $c \in \mathbb{N}$. Since $\bigcap_n U_n = \{x \mid t(x) = \infty\}$, we have $t'(x) = \infty$.

Suppose that there exists an integral test t such that $t(x) = \infty$. Let $U_n = \{y \mid t(y) > 2^n\}$. Then $\{U_n\}$ is a ML-test. Since $\bigcap_n U_n = \{x \mid t(x) = \infty\}$, we have $x \in \bigcap_n U_n$. □

Proposition 4.5. *There exists a universal ML-test on* $\mathbf{X}$ *with* μ *iff there exists a universal integral test on it.*

Proof. Suppose that there exists a universal test $\{U_n\}$. Let t be the integral test such that $\bigcap_n U_n = \{x \mid t(x) = \infty\}$ as we saw in the proof of Proposition 4.4. Let f be an arbitrary integral test. Then there exists a ML-test $\{V_n\}$ such that $\bigcap_n V_n = \{x \mid f(x) = \infty\}$. Since $\{U_n\}$ is universal, t is also universal. The other direction is proved in the same manner. □

4.2 When a universal test exists

It should be noted that, on a computable metric space with a computable measure, a universal test always exists.

Theorem 4.6 (Hoyrup and Rojas [11], partially by Gács [8])**.** *There exists a universal test on a computable metric space with a computable measure.*

We also know that SCT_3 is a sufficient condition for computable metrization.

Theorem 4.7 ([22, 9, 28])**.** *Suppose that* $\mathbf{X}$ *is* SCT_3*. Then its topology is generated by some computable metric.*

Then we can conclude that SCT_3 is a sufficient condition for the existence of a universal test.

Theorem 4.8. *Suppose that* $\mathbf{X}$ *is* SCT_3*. Then there exists a universal test on* $\mathbf{X}$ *with a computable measure* μ*.*

Here we give a direct proof of this theorem to see how the regularity is used. We prepare some lemmas. We write $U_n \uparrow U$ to mean that $U_n \subseteq U_{n+1}$ for all n and $\lim_n U_n = U$.

Lemma 4.9. *Suppose that* $\mathbf{X}$ *is* SCT_3*. For each c.e. open set* W*, one can construct a sequence* $\{U_n\}$ *of uniformly c.e. open sets and a sequencce* $\{V_n\}$ *of uniformly co-c.e. closed sets such that*

(i) $U_n \uparrow W$, $V_n \uparrow W$,

(ii) $U_n \subseteq V_n \subseteq W$ *for all* n.

Further U_n *can be the finite union of base sets for each* n.

Proof. First we assume that W is a base set. Let R and r be in the definition of SCT_3. Let $w \in \mathrm{dom}(\nu)$ be such that $\nu(w) = W$. Since R is c.e., one can construct a computable sequence $\{u_n\}$ in Σ^* such that

$$\nu(w) = \bigcup\{\nu(u) \mid (u, w) \in R\} = \bigcup_n \{u_n \mid n \in \mathbb{N}\}.$$

Let $U_n = \bigcup_{i \le n} u_i$ and $V_n = \bigcup_{i \le n} \psi^- \circ r(u_i, w_i)$. Then $U_n \uparrow W$ and $V_n \uparrow W$. By the property of r we have $\nu(u_i) \subseteq \psi^- \circ r(u_i, w) \subseteq \nu(w)$. Then $U_n \subseteq V_n \subseteq W$.

Next we prove the lemma in a general case. Since W is c.e. open, one can construct a computable sequence $\{W_m\}$ of base sets such that

$$W = \bigcup_m W_m.$$

Then one can construct a computable double sequence $\{U_n^m\}$ of open sets and a computable double sequence $\{V_n^m\}$ of closed sets such that $U_n^m \uparrow W^m$, $V_n^m \uparrow W^m$ for each m and $U_n^m \subseteq V_n^m \subseteq W^m$ for each n, m. Let

$$\widehat{U}_k = \bigcup_{i \le k} U_k^i \text{ and } \widehat{V}_k = \bigcup_{i \le k} V_k^i$$

for all k. Then $\{\widehat{U}_k\}$ is a computable sequence of finite unions of base sets and $\{\widehat{V}_k\}$ is a sequence of uniformly co-c.e. closed sets. By the construction, $\widehat{U}_k \subseteq \widehat{V}_k \subseteq W$ for all k.

We show that $\widehat{U}_k \uparrow W$. Note that

$$\widehat{U}_k = \bigcup_{i \le k} U_k^i \subseteq \bigcup_{i \le k+1} U_{k+1}^i = \widehat{U}_{k+1}.$$

Let $x \in W$. Then there exists m such that $x \in W_m$. Since $U_n^m \uparrow W_m$, there exists n such that $x \in U_n^m$. Let $k = \max\{m, n\}$. Then

$$x \in U_k^k \subseteq \bigcup_{i \le k} U_k^i = \widehat{U}_k.$$

Hence $W \subseteq \bigcup_k \widehat{U}_k$. It follows that $\widehat{U}_k \uparrow W$.

Finally we show that $\widehat{V}_k \uparrow W$. This is because $\widehat{V}_k$ is increasing and $\widehat{U}_k \subseteq \widehat{V}_k \subseteq W$ for all k. □

Proof of Theorem 4.8. Let $k \in \mathbb{N}$. First we construct a c.e. open set $\widetilde{W}$ from W such that

(i) $\mu(\widetilde{W}) \le 2^{-k}$,

(ii) $\mu(W) \le 2^{-k}$ implies $\widetilde{W} = W$.

For each c.e. open set W, construct $\{U_n\}$ and $\{V_n\}$ as Lemma 4.9. We define $\widetilde{W}$ as

$$\widetilde{W} = \bigcup_n \{U_n \mid \mu(V_n) \le 2^{-k}\}.$$

Since μ is computable and V_n is co-c.e. closed set, $\mu(V_n)$ is right-c.e. Hence $\widetilde{W}$ is c.e. open.

We show $\mu(\widetilde{W}) \le 2^{-k}$. If $\mu(V_n) \le 2^{-k}$, then

$$\mu(\bigcup_{i \le n} U_i) = \mu(U_n) \le \mu(V_n) \le 2^{-k}.$$

Suppose that $\mu(V_{n_0}) > 2^{-k}$ and $\mu(V_{n_0-1}) \le 2^{-k}$ for some n_0. Then $\mu(V_n) \ge \mu(V_{n_0}) > 2^{-k}$ for all $n \ge n_0$. It follows that $\mu(\widetilde{W}) \le \mu(\bigcup_{i \le n_0} U_i) \le 2^{-k}$. Suppose that $\mu(V_n) \le 2^{-k}$ for all n. Then $\mu(W) = \mu(\bigcup_n U_n) = \sup_n \mu(U_n) \le 2^{-k}$.

If $\mu(W) \le 2^{-k}$, then $\mu(V_n) \le 2^{-k}$ for all n and $\widetilde{W} = \bigcup_n U_n = W$.

Since one can computably enumerate all sequences of uniformly c.e. open sets, there exists a double sequence $\{W_n^m\}$ of uniformly c.e. open sets satisfying the following: if $\{A_n\}$ be a sequence of uniformly c.e. open sets, then there exists m such that $A_n = W_n^m$ for all n. For each m, n, let $\widetilde{W}_n^m$ be the c.e. open set such that

(i) $\mu(\widetilde{W}_n^m) \leq 2^{-n}$,

(ii) $\mu(W_n^m) \leq 2^{-n}$ impliew $\widetilde{W}_n^m = W_n^m$.

Then $\{\{\widetilde{W}_n^m\}_n\}_m$ is a computable enumeration of all ML-random tests.

Let $T_n = \bigcup_i \widetilde{W}_{n+i+1}^i$. Then $\{T_n\}$ is a sequence of uniformly c.e. oepn sets. Further

$$\mu(T_n) \leq \sum_i \mu(\widetilde{W}_{n+i+1}^i) \leq \sum_i 2^{-n-i-1} = 2^{-n}.$$

Hence $\{T_n\}$ is a ML-test. For any ML-test $\{A_n\}$, there exists m such that $A_n = \widetilde{W}_n^m$ for all n. Then $A_{n+m+1} \subseteq \widetilde{W}_{n+m+1}^m \subseteq T_n$ for all n. Hence $\{T_n\}$ is universal. □

4.3 A universal test does not exist in general

Here we give negative results. First it should be noted that there does not exist a universal ML-test in general. Although we give a stronger result later, the following example is easy to understand and is worth noting.

Proposition 4.10. *There exists a computable topological space* $\mathbf{X}$ *and a computable measure* μ *on it such that no test is universal.*

Proof. We consider the lower unit interval $\mathbf{I}_<$ in Example 2.3. Let α be a non-computable c.e. real with $\alpha < 1$. Let μ be the Dirac measure at α. In other words the measure μ satisfies the following:

$$\mu(U) = \begin{cases} 1 \text{ if } \alpha \in U \\ 0 \text{ otherwise.} \end{cases}$$

We show that μ is computable. For each rational $q < 1$, the relation $q < \alpha$ is semidecidable. Hence $\mu((q, 1])$ is uniformly c.e. Since each finite union of base sets has the form $(q, 1]$, μ is computable.

Next we show that the set of non-ML-random points on $\mathbf{I}_<$ with μ is

$$\{x \mid \alpha < x \leq 1\}.$$

Let x be a point such that $\alpha < x \leq 1$. Then there exists a rational q such that $\alpha < q < x$. Let $U_n = (q, 1]$ for all n. Since $\mu(U_n) = 0$ for all n, $\{U_n\}$ is a ML-test. Further $x \in (q, 1] = \bigcap_n U_n$. Then x is not ML-random.

Let x be a point such that $0 \leq x \leq \alpha$. Suppose that x is not ML-random for a contradiction. Then there exists a ML-test $\{V_n\}$ with $x \in \bigcap_n V_n$. Since $x \in V_1$, $\alpha \in V_1$ and $\mu(V_1) = 1$. This is a contradiction. Hence x is ML-random.

Let $\{W_n\}$ be a ML-test. Suppose that $\bigcap_n W_n = \{x \mid \alpha < x \leq 1\}$. Since W_1 is c.e. open, there exists a right-c.e. real β such that $W_1 = (\beta, 1]$. Since $\bigcap_n W_n \subseteq W_1$, we have $\beta \leq \alpha$. If $\beta < \alpha$, then $\mu(W_1) = \mu((\beta, 1]) = 1 > 2^{-1}$. Hence $\beta = \alpha$. Since α is lower semicomputable and β is right-c.e., α is a computable real. This is a contradiction. Hence $\{W_n\}$ is not universal. Since $\{W_n\}$ is arbitrary, there does not exist a universal ML-test. □

4.4 When a universal test does not exist

We knew that SCT_3 is a sufficient condition for the existence of a universal test. How much do we weaken the condition. We show that CT_2 is not sufficient.

Theorem 4.11. *There exists a* CT_2 *space* $\mathbf{X}$ *with a computable measure* μ *such that no universal test exists.*

Proof. Let $A \subseteq \mathbb{N}$ be a non-computable c.e. set such that $0 \in A$. Consider the space $X = \{0\} \cup \mathbb{N} \backslash A$ with the discrete topology τ. We define a base β of τ as

$$\beta = \{\{n\} \ : \ n \in X\}$$

and a notation $\nu :\subseteq \Sigma^* \to \beta$ of the base β as

$$\nu(\bar{i}) = \begin{cases} \{0\} & \text{if } i \in A \\ \{i\} & \text{if } i \notin A \end{cases}$$

where $\bar{i}$ is the binary representation of i.

We show that $\mathbf{X} = (X, \tau, \beta, \nu)$ is a computable topological space. Clearly, $\mathrm{dom}(\nu)$ is computable. We define a c.e. set $S \subseteq (\mathrm{dom}(\nu))^3$ as follows. If $i, j \in A$, then $(\bar{i}, \bar{j}, w) \in S \iff w = \bar{i}$. If $i \notin A$ or $j \notin A$, then $(\bar{i}, \bar{j}, w) \in S \iff w = \bar{i} = \bar{j}$. It is not difficult to see that S satisfies (1) in Definitoin 2.1.

We show that $\mathbf{X}$ is CT_2. We define a c.e. set $H \subseteq \Sigma^* \times \Sigma^*$ as

$$H = \{(\bar{i}, \bar{j}) \ : \ i \neq j\}.$$

We prove that H satisfies the property in the definition of CT_2'. Let $(\bar{i}, \bar{j}) \in H$. If $i, j \in A$, then $\nu(\bar{i}) = \nu(\bar{j}) = \{0\}$. If $i \notin A$ or $j \notin A$, then $\nu(\bar{i}) \cap \nu(\bar{j}) = \emptyset$ because $i \neq j$. Let $i, j \in X$ such that $i \neq j$. Then $i \in \nu(\bar{i})$ and $j \in \nu(\bar{j})$.

We define a measure μ as

$$\mu(\{0\}) = 1.$$

We show that μ is computable. Let $\overline{a_1}, \ldots, \overline{a_k} \in \mathrm{dom}(\nu)$. Note that

$$\mu(\nu(\overline{a_1}) \cup \ldots \cup \nu(\overline{a_k})) = \begin{cases} 1 & \text{if } a_j \in A \text{ for some } j \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Then the measure is uniformly lower semicomputable. Hence μ is computable.

We show that the set of non-ML-random points is $\mathbb{N} \backslash A$. Since $\mu(\{0\}) = 1$, the natural number 0 is ML-random. For each $i \notin A$, let $U_n = \{\bar{i}\}$ for all n. Then $\{U_n\}$ is a sequence of uniformly c.e. open sets with $\mu(U_n) = 0$. Hence $\{U_n\}$ is a ML-test. It follows that the natural number i is not ML-random.

We claim that $\mathbb{N} \backslash A$ is open but not c.e. Suppose that $\mathbb{N} \backslash A$ is c.e. Then there exists a computable sequence $\{a_n\}$ of natural numbers such that $\mathbb{N} \backslash A = \bigcup_n \nu(\overline{a_n})$. If $a_n \in A$, then $\{0\} = \nu(\overline{a_n}) \subseteq \mathbb{N} \backslash A$, which is a contradiction. Hence $a_n \notin A$ for all n. It follows that

$$\{a_n \ : \ n \in \mathbb{N}\} = \bigcup_n \nu(\overline{a_n}) = \mathbb{N} \backslash A.$$

Since the complement of A is c.e. and A is c.e. by the definition, A is computable, which is a contradiction.

We prove that each ML-random test is not universal. Let $\{V_n\}$ be a ML-test. Since $\mu(V_1) \leq 2^{-1}$, $0 \notin V_1$. Hence $V_1 \subseteq \mathbb{N}\backslash A$. Since $\mathbb{N}\backslash A$ is not c.e., we have $V_1 \subsetneq \mathbb{N}\backslash A$. Hence $\{V_n\}$ is not universal. □

It should be noted that $\mathbf{X}$ constructed above is not SCT_3 by Theorem 4.8. The space $\mathbf{X}$ is not even SCT_2. Suppose that $\mathbf{X}$ is SCT_2 and H is in the definition of SCT_2. We define a c.e. set $S \subseteq \mathbb{N}$ as

$$S = \{j \ : \ (0^i1, 0^j1) \in H \text{ and } i \in A\} \cup \{i \ : \ (0^i1, 0^j1) \in H \text{ and } j \in A\}.$$

We show that $S \subseteq \mathbb{N}\backslash A$. Consider $j \in S$, $i \in A$ and $(0^i1, 0^j1) \in H$. Then $\mu(0^i1) = \{0\}$ and $\nu(0^i1) \cap \mu(0^j1) = \emptyset$. Hence $j \notin A$. It follows that $S \subseteq \mathbb{N}\backslash A$.

We show that $S \supseteq \mathbb{N}\backslash A$. For each $i \in \mathbb{N}\backslash A$, there exists $(u, v) \in H$ such that $i \in \nu(u)$ and $0 \in \nu(v)$. Since $i \neq 0$, $u = 0^i1$. Since $0 \in \nu(v)$, $v = 0^j1$ and $j \in A$ for some j. Then $i \in S$.

However $S = \mathbb{N}\backslash A$ is impossible, since S is c.e. and $\mathbb{N}\backslash A$ is not c.e. Hence $\mathbf{X}$ is not SCT_2.

The author does not know whether there exists an SCT_2 space with a computable measure on it such that there does not exist a universal ML-test.

5 Complexity randomness

Martin-Löf randomness on Cantor space has a characterization by complexity. In this section we study whether Martin-Löf randomness on a computable topological space with a computable measure has a characterization by complexity.

5.1 Definition

On Cantor space with a computable measure μ, a binary sequence $Z \in 2^\omega$ is not Martin-Löf random iff for all $d \in \mathbb{N}$ there exists n such that

$$K(Z \upharpoonright n) < -\log \mu([\![Z \upharpoonright n]\!]) - d \tag{2}$$

where K is the universal prefix-free Kolmogorov complexity, $Z \upharpoonright n$ is the first n bits of the sequence Z, $\log(0) = -\infty$, $[\![\sigma]\!] = \{X \in 2^\omega \mid \sigma \prec X\}$ and $\prec$ is the prefix relation. To prove the characterization by complexity, it is important that the relation (2) is semidecidable, which is based on the fact that $\mu([\![\sigma]\!])$ is uniformly computable, more precisely, uniformly right-c.e.

On a computable metric space with a computable measure, Hoyrup and Rojas [11] have given a characterization by complexity of Martin-Löf randomness. They devide the whole space into cells whose measures are uniformly computable. However we can not generalize it to a computable topological space in the same manner.

We wish to have a characterization like this: on a computable topological space $\mathbf{X}$ with a computable measure μ, a point $x \in X$ is not Martin-Löf random iff for all $d \in \mathbb{N}$ there exists some set A such that

$$x \in A \text{ and } H(A) < -\log \mu(A) - d$$

where H is some kind of complexity. We require the set A to be *co-c.e. closed* so that $\mu(A)$ is right-c.e. Then the set A can be identified with a computable ψ^--name p. We use the monotone complexity Km so that $H(A) = Km(A)$ can be finite.

A *monotone machine* is a Turing machine, which reads finite strings from an input tape, operates on work taps and write one-way finite strings or infinite sequences to an output tape. Then a monotone machine is similar to a Type-2 machine. We write $M(\sigma)\downarrow$ if M reads exactly σ from its input tape and write some finite string or some infinite sequences to its output tape. Let $M(\sigma)$ denote the string or the sequence.

Another way to define a monotone machine is that a c.e. set of pairs (σ, τ) where $\sigma, \tau \in 2^*$ and for every pair $(\sigma_1, \tau_1), (\sigma_2, \tau_2) \in M$, $\sigma_1 \preceq \sigma_2$ implies $\tau_1 \preceq \tau_2$ or $\tau_2 \preceq \tau_1$.

Definition 5.1 (Monotone complexity; Levin [13, 14])**.** *We define the monotone complexity of $\tau \in 2^* \cup 2^\omega$ with respect to M to be*

$$Km^M(\tau) = \min\{|\sigma| \ : \ \tau \preceq M(\sigma)\downarrow\}$$

or equivalently

$$Km^M(\tau) = \min\{|\sigma| : (\sigma, \rho) \in M \text{ for some } \rho \succeq \tau\}$$

if $\tau \in 2^*$ and

$$Km^M(\tau) = \sup_n Km(\tau \upharpoonright n)$$

if $\tau \in 2^\omega$. One can show that there is a universal monotone machine U and we define $Km(\tau) = Km^U(\tau)$.

The following are well-known results in the theory of algorithmic randomness [6, 18].

Proposition 5.2. *The function Km is monotone, that is, $\sigma \preceq \tau$ implies $Km(\sigma) \le Km(\tau)$ for all $\sigma, \tau \in 2^* \cup 2^\omega$.*

Proposition 5.3. *A sequence $Z \in 2^\omega$ is computable iff there exists $d \in \mathbb{N}$ such that $Km(Z) < d$.*

Theorem 5.4. *A sequence $Z \in 2^\omega$ is Martin-Löf random iff there exists d such that, for all n, $Km(Z \upharpoonright n) > n - d$.*

The following fact is a simple important property of monotone machines.

Proposition 5.5. *Let M, N be monotone machines. Then $M \circ N$ is also a monotone machine.*

Now we give the definition of complexity randomness. We assume that $\Sigma = \{0,1\} = 2$ without loss of generality so that each ψ^--name p of a closed set is in $\{0,1\}^\omega = 2^\omega$. For a closed set A, we define

$$Km(A) = \min_p \{Km(p) \mid \psi^-(p) = A\}.$$

It should be noted that $Km(A)$ is finite iff A is co-c.e. closed, because if A is not co-c.e. closed, then each $\psi^-(p) = A$ implies $Km(p) = \infty$.

Definition 5.6. *Let $\mathbf{X} = (X, \tau, \beta, \nu)$ be a computable topological space and μ be a computable measure on it. A point $x \in X$ is* complexity random *if there exists $d \in \mathbb{N}$ such that*

$$x \in A \Rightarrow Km(A) \geq -\log \mu(A) - d$$

for each closed set A. Here we define $\log(0) = -\infty$.

5.2 Independence from the notation

The monotone complexity Km depends on the representation ψ^-, which also depends on the notation ν. However we can replace the notation with an equivalent one.

Definition 5.7 ([33]). *The computable topological spaces $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$ are equivalent iff $\nu_1 \leq \theta_2$ and $\nu_2 \leq \theta_1$.*

Theorem 5.8 (robustness [33]). *Let $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$ be computable topological spaces. Then $\mathbf{X}_1$ and $\mathbf{X}_2$ are equivalent $\iff$ $\theta_1 \equiv \theta_2$.*

Proposition 5.9. *Let $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$ be computable topological spaces. Let Km_1 and Km_2 be the monotone complexities of closed sets on $\mathbf{X}$ and $\mathbf{X}'$ respectively. If $\theta_2 \leq \theta_1$, then there exists $d \in \mathbb{N}$ such that*

$$Km_1(A) \leq Km_2(A) + d$$

for all closed sets A.

Proof. Since $\theta_2 \leq \theta_1$, there exists a computable function $h :\subseteq 2^\omega \to 2^\omega$ such that $\theta_2(p) = \theta_1 \circ h(p)$ for all $p \in \mathrm{dom}(\theta_1)$. Further h can be seen as a monotone machine, more precisely, there exists a monotone machine N satisfying the following: for each n and $p \in \mathrm{dom}(\theta_1)$, there exists $(\sigma, \tau) \in N$ such that $\sigma \prec p$ and $h(p) \upharpoonright n \prec \tau$. Consider the monotone machine $N \circ M$. By universality of M, there exists d such that $Km(\sigma) \leq Km^{N \circ M}(\sigma) + d$ for all $\sigma \in 2^* \cup 2^\omega$.

For each co-c.e. closed set A, let $p \in 2^\omega$ be a computable sequence such that $Km_2(A) = Km(p)$ and $\psi_2^-(p) = A$. Further let $\sigma \in 2^*$ such that $Km(p) = |\sigma|$ and $p \preceq M(\sigma)$. Then $h(p) \preceq N \circ M(\sigma)$. Hence $Km^{N \circ M}(h(p)) \leq |\sigma|$. It follows that $Km(\sigma) \leq |\sigma| + d$ and

$$Km_1(A) \leq |\sigma| + d = Km(p) + d = Km_2(A) + d.$$

Finally note that $Km_1(A) = Km_2(A) = \infty$ if A is not co-c.e. closed. ∎

Corollary 5.10. *Let $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ and $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$ be equivalent computable topological spaces and μ be a computable measure on it. Then complexity randomness on the spaces coincide.*

It should be noted that the computability of a measure μ does not depend on the notation.

5.3 Some natural properties

Here we rephrase the definition of complexity randomness in some forms and show that the set of complexity random points has measure 1.

In the definition of complexity randomness we can require the set A to be co-c.e. closed.

Proposition 5.11. *A point x is complexity random iff there exists $d \in \mathbb{N}$ such that*

$$x \in \psi^-(p) \Rightarrow Km(p) \geq -\log \mu(\psi^-(p)) - d$$

for each computable sequence $p \in \mathrm{dom}(\psi^-)$.

Proof. The "only if" direction is immediate.

Suppose that x is not complexity random. Then for each $d \in \mathbb{N}$, there exists a closed set A such that $x \in A$ and $Km(A) < -\log \mu(A) - d$. Then $Km(A)$ is finite. Hence A is co-c.e. closed. Let $p \in 2^\omega$ be a sequence such that $Km(A) = Km(p)$ and $\psi^-(p) = A$. Then $x \in \psi^-(p)$ and $Km(p) < -\log \mu(\psi^-(p)) - d$. □

Further we can require the set A to be the complement of the finite union of base sets. Recall that ν^{fs} is a notation of finite unions of base sets. For simplicity, we define $F :\subseteq 2^* \to \mathcal{A}$ be the set function as

$$F(u) = F_\nu(u) = X \setminus \bigcup \nu^{\mathrm{fs}}(u).$$

Proposition 5.12. *A point x is complexity random iff there exists $d \in \mathbb{N}$ such that*

$$x \in F(u) \Rightarrow Km(u) \geq -\log \mu(F(u)) - d$$

for each $u \in \mathrm{dom}(\nu^{\mathrm{fs}})$.

Proof. The "only if" direction is immediate.

Suppose that x is not complexity random. Then for each $d \in \mathbb{N}$ there exists a computable sequence $p \in 2^\omega$ such that $x \in \psi^-(p)$ and $Km(p) < -\log \mu(\psi^-(p)) - 2d$. Then there exists $u \in \mathrm{dom}(\nu^{\mathrm{fs}})$ such that $u \prec p$ and

$$\mu(F(u)) \leq 2^d \cdot \mu(\psi^-(p)).$$

Since $u \prec p$, we have $\bigcup \nu^{\mathrm{fs}}(u) \subseteq \theta(p)$ and $x \notin \bigcup \nu^{\mathrm{fs}}(u)$. Again by $u \prec p$, we have $Km(u) \leq Km(p)$. Hence $Km(u) \leq Km(p) < -\log \mu(F(u)) - d$. □

This proposition says that complexity randomness has always universality.

Proposition 5.13. *The set of complexity random points has measure 1.*

Proof. Let $k \in \mathbb{N}$. We define a c.e. set $U = U_k$ as

$$U = \{(\sigma, u) \ : \ |\sigma| < -\log \mu(F(u)) - k, \ u \in \mathrm{dom}(\nu^{\mathrm{fs}}), \ u \preceq v \text{ and } (\sigma, v) \in M\}.$$

Note that the set of non-complex random sets is $\bigcap_k \bigcup_{\langle p,u\rangle \in U} F(u)$.

We define sets V and W as

$$V = \{u \ : \ (\sigma, u) \in U \text{ for some } \sigma\}, \ W = \{u \ : \ v \notin V \text{ for all } v \prec u\}.$$

For each $u \in W \subseteq V$, let σ_u be such that $(\sigma_u, u) \in U$.

We claim that $\bigcup_{u \in W} F(u) = \bigcup_{(\sigma,u)\in U} F(u)$. The inclusion $\subseteq$ is immediate. We show the other inclusion. Suppose that $x \in F(u)$ for some $(\sigma, u) \in U$. Then $u \in V$. If $u \in W$ then $x \in F(u)$ for this $u \in W$. If $u \notin W$, there exists v such that $v \prec u$ and $v \in W$. Then $x \in F(u) \subseteq F(v)$.

Since W is prefix-free and M is monotone, the set $\{\sigma_u \ : \ u \in W\}$ is prefix-free. Then

$$2^k \mu(\bigcup_{(\sigma,u)\in U} F(u)) \le \sum_{u \in W} 2^k \mu(F(u)) \le \sum_{u\in W} 2^{-|p_u|} \le 1.$$

Hence the set $\bigcap_k \bigcup_{(\sigma,u)\in U_k} F(u)$ has measre 0. □

5.4 K-complexity randomness

Proposition 5.12 says that complexity randomness has a characterization by $Km(u)$ where u is a string and not a sequence. Then we also consider a similar definition.

A *prefix-free machine* is a partical computable function whose domain is prefix-free. There exists a universal prefix-free machine $U :\subseteq 2^* \to 2^*$ and define

$$K(\tau) = \min\{|\sigma| \ : \ U(\sigma) = \tau\}.$$

The following is a basic tool to study complexity K. A *KC set* is a c.e. set $W = \{\langle d_i, \tau_i\rangle \ : \ d_i \in \mathbb{N}, \ \tau_i \in 2^*\}_i$ such that $\sum_i 2^{-d_i} \le 1$.

Theorem 5.14 (KC Theorem; Levin [13], Schnorr [21], Chaitin [4])**.** *For a KC set $\{d_i, \tau_i\}_i$, there ix a prefix-free machine M and strings σ_i of length d_i such that $M(\sigma_i) = \tau_i$ for all i and* $\mathrm{dom}(M) = \{\sigma_i \ : \ i \in \mathbb{N}\}$.

Definition 5.15. *A point x is K-complexity random if there exists $d \in \mathbb{N}$ such that*

$$x \in F(u) \Rightarrow K(u) \ge -\log \mu(F(u)) - d$$

for each $u \in \mathrm{dom}(\nu^{\mathrm{fs}})$.

The notion of K-complexity randomness does not depend on the notation neither.

Proposition 5.16. *Let* $\mathbf{X}_1 = (X, \tau, \beta_1, \nu_1)$ *and* $\mathbf{X}_2 = (X, \tau, \beta_2, \nu_2)$ *be equivalent computable topological spaces. Then* K*-complexity randomness on* $\mathbf{X}_1$ *and* K*-complexity randomness on* $\mathbf{X}_2$ *coincide.*

Proof. Suppose $\nu_2 \leq \theta_1$. Let $F_i = F_{\nu_i}$ for $i = 1, 2$.

We define a partial computable function $h_k :\subseteq 2^* \to 2^*$ as follows. For each $k \in \mathbb{N}$ and $u \in 2^*$, search v such that

$$\nu_1^{\mathrm{fs}}(v) \subseteq \nu_2(u) \text{ and } K(u) < -\log \mu(F_1(v)) - 2k.$$

If found, let $h_k(u) = v$.

We define a KC set L as

$$L = \{\langle |\sigma| + k + 1, h_k(U(\sigma))\rangle \ : \ h_k(U(\sigma)) \downarrow\}$$

where U is the universal prefix-free machine. Then

$$\sum_{\langle n,v\rangle \in L} 2^{-n} \leq \sum_{\sigma \in \mathrm{dom}(U)} 2^{-|\sigma|-k-1} \leq 1.$$

Then there exists $d \in \mathbb{N}$ such that

$$K(h_k(u)) \leq K(u) + k + d + 1$$

for each $u \in \mathrm{dom}(h_k)$.

Suppose that x is not K-complexity random on $\mathbf{X}_2$. For each $k \in \mathbb{N}$, there exists $u \in \mathrm{dom}(\nu_2^{\mathrm{fs}})$ such that $x \in F_2(u)$ and $K(u) < -\log \mu(F_2(u)) - 2k - 1$. Then there exists $v \in \mathrm{dom}(\nu_1^{\mathrm{fs}})$ such that $\nu_1^{\mathrm{fs}}(v) \subseteq \nu_2(u)$ and $\mu(F_1(v)) \leq 2\mu(F_2(u))$. It follows that $F_1(v) \supseteq F_2(u) \ni x$ and

$$K(u) < -\log \mu(F_2(u)) - 2k - 1 \leq -\log \mu(F_1(v)) - 2k.$$

Hence $h_k(u)$ is defined. It follows that

$$K(h_k(u)) \leq K(u) + k + d + 1 < -\log \mu(F_1(h_k(u))) - k + d + 1.$$

Hence x is not K-complexity random on $\mathbf{X}_1$. □

Since $Km(\sigma) \leq K(\sigma) + d$ for all $\sigma \in 2^*$ for some $d \in \mathbb{N}$, complexity randomness implies K-complexity randomness. Hence the following is immediate from Proposition 5.13. Here we also give an easy direct proof.

Proposition 5.17. *The set of* K*-complexity random points has measure 1.*

Proof. For each k, we define a set A_n as

$$A_n = \{u \ : \ K(u) < -\log \mu(F(u)) - k\}.$$

For each $u \in A_k$, we have $2^k \mu(F(u)) \leq 2^{-K(u)}$. Then

$$\mu(\bigcup_{u \in A_k} F(u)) \leq \sum_{u \in A_n} \mu(F(u)) \leq 2^{-k} \sum_{u \in A_n} 2^{-K(u)} \leq 2^{-k}$$

for each k. It follows that $\bigcap_n \bigcup_{u \in A_k} F(u)$ has measure 0. Hence the set of non-K-complexity random points has measure 0. □

5.5 When they coincide

Complexity randomness and K-complexity randomness are randomness notions that can be defined on any computable topological space with any computable measure. However Martin-Löf randomness does not coincide with each of them in general.

Example 5.18. *Consider the lower unit interval* $\mathbf{I}_<$ *in Example 2.3 and the measure* μ *such that* $\mu((q,1]) = 1-q$ *where* $q \in \mathbb{Q}\cap[0,1]$*. Then* μ *is computable. The set of non-ML-random points is* $\{1\}$*. In contrast the set of non-complexity random points is* $\{0\}$*. The set of non-*K*-complexity random points is also* $\{0\}$*.*

We show that SCT_3 is a sufficient condition for the coincidence.

Theorem 5.19. *Let* $\mathbf{X}$ *be an* SCT_3 *space and* μ *be a computable measure on it. Then the following are equivalent for a point* $x \in X$*:*

(i) *x is ML-random.*

(ii) *x is complexity random.*

(iii) *x is K-complexity random.*

The implication (ii)⇒(iii) is immediate.

Proof of (i)⇒(ii) of Theorem 5.19. By Lemma 4.9, there exists a double sequence $\{U^u_n\}$ of uniformly c.e. open sets and a double sequence $\{V^u_n\}$ of uniformly co-c.e. closed sets such that $U^u_n \uparrow \bigcup \nu^{\mathrm{fs}}(u)$ and $V^u_n \uparrow \bigcup \nu^{\mathrm{fs}}(u)$ for all $u \in \mathrm{dom}(\nu^{\mathrm{fs}})$ and $U^u_n \subseteq V^u_n \subseteq \bigcup \nu^{\mathrm{fs}}(u)$ for all $n \in \mathbb{N}$ and $u \in \mathrm{dom}(\nu^{\mathrm{fs}})$. Let $k \in \mathbb{N}$. We define a c.e. set $S_k = S$ of strings as

$$S = \{(\sigma,u,n)\ :\ |\sigma| < -\log\mu(X\setminus V^u_n) - k,\ u \in \mathrm{dom}(\nu^{\mathrm{fs}}),\ u \preceq v,\ (\sigma,v)\in M\}$$

where M is the universal monotone machine.

Let $W_k = \bigcup_{(\sigma,u,n)\in S}(X\setminus V^u_n)$. We will prove that $\mu(W_k) \le 2^{-k}$. The argument is similar to the proof of Proposition 5.13. We define sets A and B as

$$A = \{u\ :\ (\sigma,u,n)\in S \text{ for some } \sigma,n\},\ B = \{u\ :\ v \notin A \text{ for all } v \prec u\}.$$

For each $u \in B$, let n_u be the smallest n such that $(\sigma,u,n) \in S$ for some σ and let σ_u be such that $(\sigma_u,u,n_u)\in S$.

We claim that

$$\bigcup_{u\in W}(X\setminus V^u_{n_u}) = \bigcup_{(\sigma,u,n)\in S}(X\setminus V^u_n).$$

The inclusion $\subseteq$ is immediate. We show the other direction. Suppose that $x \in X\setminus V^u_n$ for some $(\sigma,u,n)\in S$. Then $u \in A$. If $u \in B$, then $V^u_{n_u} \subseteq V^u_n$ and $x \in X\setminus V^u_{n_u}$. If $u \notin B$, then there exists v such that $v \prec u$ and $v \in B$. By the construction in Lemma 4.9, we can assume

$$V^v_n \subseteq V^u_n \text{ for each } v \prec u.$$

It follows that $V^v_{n_v} \subseteq V^v_n \subseteq V^u_n$ and $x \in X \setminus V^v_{n_v}$.

Since B is prefix-free and M is monotone, the set $\{\sigma_u \ : \ u \in B\}$ is prefix-free. Then

$$\begin{aligned}\mu(\bigcup_{(\sigma,u,n)\in S}(X\setminus V^u_n)) =& \mu(\bigcup_{u\in W}(X\setminus V^u_{n_u})) \\ & \leq \sum_{u\in W}\mu(X\setminus V^u_{n_u}) \leq \sum_{u\in W} 2^{-|\sigma_u|-k} \leq 2^{-k}.\end{aligned}$$

For each k, let $T_k = \bigcup_{(p,u,n)\in S_k}(X\setminus V^u_n)$. Then $\{T_k\}$ is ML-test.

Suppose that x is not complexity random. By Proposition 5.12, for each k, there exists $u_0 \in \mathrm{dom}(\nu^{\mathrm{fs}})$ such that $x \in F(u_0)$ and $Km(u_0) < -\log\mu(F(u_0)) - k$. Let σ_0 be such that $Km(u_0) = |\sigma_0|$, $v \preceq u_0$ and $(\sigma_0, v) \in M$ for some v. Since $|\sigma_0| < -\log\mu(F(u_0)) - k$ and $\lim_n \mu(X\setminus V^{u_0}_n) = \mu(F(u_0))$, there exists n_0 such that $|\sigma_0| < -\log\mu(X\setminus V^{u_0}_{n_0}) - k$. Then $(\sigma_0, u_0, n_0) \in S$. Hence

$$x \in F(u_0) \subseteq X\setminus V^{u_0}_{n_0} \subseteq T_k.$$

Since k is arbitrary, x is not ML-random. □

Before giving a proof of the remaining implication, we show a lemma.

Lemma 5.20. *Let* $\mathbf{X}$ *be an* SCT_3 *space and* μ *be a computable measure on it. For each c.e. open set* W *and* $n \in \mathbb{N}$, *one can compute a computable sequence* $\{V_m\}$ *of the finite unions of base sets and a computable sequence* $\{C_m\}$ *of the complements of the finite unions of base sets such that*

(i) $W \subseteq \bigcup_m C_m$,

(ii) $V_m \subseteq C_m$, $\mu(C_m) - \mu(V_m) \leq 2^{-n-m-1}$ *and*

(iii) $\sum_m \mu(C_m) \leq 2\mu(W) + 2^{-n}$ *for each* n.

Proof. By Lemma 4.9, there exists a sequence $\{U_k\}$ oe uniformly c.e. open sets and a sequence $\{A_k\}$ of uniformly co-c.e. closed sets such that $U_k \uparrow W$, $A_k \uparrow W$ and $U_k \subseteq A_k \subseteq W$. Further we assume that U_k is the finite union of base sets.

We claim that, for each m, one can compute $k = k(m)$ such that $\mu(A_k) - \mu(U_k) < 2^{-n-m-3}$. Note that the real $\mu(U_k)$ is approximated from below and $\mu(A_k)$ is approximated from above. Furthere $\mu(W) - \mu(U_k) \to 0$ as $k \to \infty$, we have $\mu(A_k) - \mu(U_k) \to 0$. Hence we can compute such k. We assume that $k(m) < k(m+1)$ for all m.

We define $B_k = B_{k(m)}$ for each m as follows. Since A_k is co-c.e. closed set and $\mu(A_k) - \mu(U_k) < 2^{-n-m-3}$, there exists B_k such that $A_k \subseteq B_k$, B_k is the complement of the finite union of base sets and $\mu(B_k) - \mu(U_k) < 2^{-n-m-3}$. It should be noted that B_k is defined only when $k = k(m)$ for some m.

We define a computable sequence $\{V_m\}$ of the finite unions of base sets and a computable sequence $\{C_m\}$ of the complements of the finite unions of base sets as follows. Let $V_0 = U_{k(0)}$, $C_0 = B_{k(0)}$,

$$V_m = U_{k(m)}\setminus B_{k(m-1)} \text{ and } C_m = B_{k(m)}\setminus U_{k(m-1)}$$

for all $m \geq 1$. It should be noted that $X \backslash C_m = (X \backslash B_{k(m)}) \cup U_{k(m-1)}$ is the finite union of base sets for each m.

We show $W \subseteq \bigcup_m C_m$. Since $U_k \uparrow W$, it suffices to show that

$$U_{k(m)} \subseteq C_0 \cup \cdots \cup C_m \tag{3}$$

by induction over m. The case of $m = 0$ is true by the definition. Suppose that the inclusion (3) is true for $m - 1$. Then

$$U_{k(m)} \subseteq A_{k(m)} \subseteq B_{k(m)} \subseteq C_m \cup U_{k(m-1)} \subseteq C_0 \cup \cdots \cup C_m.$$

Hence the inclusion (3) is true for m.

We prove $V_m \subseteq C_m$ for all m. Suppose that $x \in V_m$ Then $x \in U_{k(m)}$ and $x \notin B_{k(m-1)}$. Since $U_{k(m)} \subseteq B_{k(m)}$, $x \in B_{k(m)}$ and $X \notin U_{k(m-1)}$. Hence $x \in C_m$

Next we show $\mu(C_m) - \mu(V_m) \leq 2^{-n-m-1}$. Since $U_{k(m-1)} \subseteq U_{k(m)} \subseteq B_{k(m)}$, we have

$$\mu(C_m) = \mu(B_{k(m)} \backslash U_{k(m-1)}) = \mu(B_{k(m)}) - \mu(U_{k(m-1)}).$$

We also have

$$\begin{aligned}\mu(V_m) =& \mu(U_{k(m)} \backslash B_{k(m-1)}) = \mu(U_{k(m)}) - \mu(U_{k(m)} \cap B_{k(m-1)}) \\ \geq& \mu(U_{k(m)}) - \mu(B_{k(m-1)}).\end{aligned}$$

It follows that

$$\begin{aligned}\mu(C_m) - \mu(V_m) \leq& \mu(B_{k(m)}) - \mu(U_{k(m-1)}) - \mu(U_{k(m)}) + \mu(B_{k(m-1)}) \\ \leq& 2^{-n-m-3} + 2^{-n-(m-1)-3} < 2^{-n-m-1}.\end{aligned}$$

Finally we prove $\sum_m \mu(B_m) \leq 2\mu(W) + 2^{-n}$. For $m \geq 1$,

$$C_m = B_{k(m)} \backslash U_{k(m-1)} = (B_{k(m)} \backslash A_{k(m)}) \uplus (A_{k(m)} \backslash A_{k(m-1)}) \uplus (A_{k(m-1)} \backslash U_{k(m-1)}$$

and

$$\begin{aligned}\mu(C_m) <& 2^{-n-m-3} + \mu(A_{k(m)} \backslash A_{k(m-1)}) + 2^{-n-(m-1)-3} \\ <& \mu(A_{k(m)} \backslash A_{k(m-1)}) + 2^{-n-m-1}\end{aligned}$$

Then

$$\sum_m \mu(C_m) \leq \mu(B_{k(0)}) + \sum_m \mu(A_{k(m)} \backslash A_{k(m-1)}) + \sum_m 2^{-n-m-1}.$$

The first term is less than or equal to $\mu(W)$. The second term is also less than or equal to $\mu(W)$. The third term is equal to 2^{-n}. □

Proof of (iii)⇒(i) of Theorem 5.19. Let $\{W_n\}$ be a ML-test. By Lemma 5.20, there exists a computable suquence $\{V_{\langle n,m\rangle}\}$ of the finite unions of base sets and a computable sequence $\{C_{\langle n,m\rangle}\}$ of the complements of the finite unions of base sets such that

$$W_{2n+6} \subseteq \bigcup_m C_{\langle n,m\rangle},\ V_{\langle n,m\rangle} \subseteq C_{\langle n,m\rangle},$$
$$\mu(C_{\langle n,m\rangle}) - \mu(V_{\langle n,m\rangle}) \le 2^{-2n-5-m-1} = 2^{-2n-m-6},$$
$$\sum_m \mu(C_{\langle n,m\rangle}) \le 2\mu(W_{2n+6}) + 2^{-2n-5} \le 2^{-2n-5} + 2^{-2n-5} = 2^{-2n-4}$$

for each n.

We construct a KC set as follows. Let

$$L_1 = \{\langle n+m+3, \langle n,m\rangle\rangle \ :\ n,m \in \mathbb{N}\}.$$

Note that $\sum_{n,m} 2^{-n-m-3} = 1/2$. We construct another KC set L_2. Since $V_{\langle n,m\rangle} \subseteq C_{\langle n,m\rangle}$, we have $-\log\mu(C_{\langle n,m\rangle}) \le -\log\mu(V_{\langle n,m\rangle})$. Since $C_{\langle n,m\rangle}$ is c.e. open and $V_{\langle n,m\rangle}$ is co-c.e. closed, $-\log\mu(V_{\langle n,m\rangle})$ is approximated from above and $-\log\mu(C_{\langle n,m\rangle})$ is approximated from below. Then the relation $-\log\mu(V_{\langle n,m\rangle}) + \log\mu(C_{\langle n,m\rangle}) < 1$ is semi-decidable. If the relation holds, there exists an integer b such that $b-1 < -\log\mu(C_{\langle n,m\rangle}) \le -\log\mu(V_{\langle n,m\rangle}) < b+1$, which is equivalent to

$$2^{-b-1} < \mu(V_{\langle n,m\rangle}) \le \mu(C_{\langle n,m\rangle}) < 2^{-b+1}.$$

Note that such b can be found effectively. Let $b(n,m)$ be the integer b if found. Let

$$L_2 = \{\langle b(n,m)-n-1, \langle n,m\rangle\rangle \ :\ b(n,m) \text{ is defined}\}.$$

Then

$$\sum_{b(n,m)\downarrow} 2^{-b(n,m)+n+1} \le \sum_{n,m} 2^{n+2}\mu(C_{\langle n,m\rangle}) \le \sum_n 2^{n+2}\cdot 2^{-2n-4} = 1/2.$$

Hence $L_1 \cup L_2$ is a KC set.

Let $f :\subseteq 2^* \to \mathbb{N}$ be the prefix-free machine constructed from $L_1 \cup L_2$ by KC theorem. Recall that $C_{\langle n,m\rangle}$ is the comlement of the finite union of base sets. Then there exists a computable sequence $\{u_{\langle n,m\rangle}\}$ such that $F(u_{\langle n,m\rangle}) = C_{\langle n,m\rangle}$. Let $h(\sigma) = u_{f(\sigma)}$. Then h is a prefix-free machine.

Suppose that $x \in X$ is not ML-random. Since $x \in \bigcap_n W_n$ and $W_{2n+6} \subseteq \bigcup_m C_{\langle n,m\rangle}$, there exists m such that $x \in C_{\langle n,m\rangle} = F(u_{\langle n,m\rangle})$ for each n.

Suppose that $b(n,m)$ is defined. Then there exists $\sigma \in 2^*$ such that $|\sigma| = b(n,m)-n-1 < -\log\mu(C_{\langle n,m\rangle}) - n$, which implies

$$K_h(u_{\langle n,m\rangle}) \le -\log\mu(F(u_{\langle n,m\rangle})) - n.$$

Suppose that $b(n,m)$ is not defined and $\mu(C_{\langle n,m\rangle}) \le 2^{-2n-m-4}$. Then $-\log\mu(C_{\langle n,m\rangle}) - n - 1 \ge n+m+3$. Note that there exists $\sigma \in 2^*$ such that $|\sigma| = n+m+3$ and $h(\sigma) = u_{\langle n,m\rangle}$. Then

$$K_h(u_{\langle n,m\rangle}) \le n+m+3 \le -\log\mu(F(u_{\langle n,m\rangle})) - n - 1.$$

Suppose that $b(n,m)$ is not defined and $\mu(C_{\langle n,m\rangle}) > 2^{-2n-m-4}$. Then

$$\mu(V_{\langle n,m\rangle}) > \mu(C_{\langle n,m\rangle}) - 2^{-2n-m-6} \geq \mu(C_{\langle n,m\rangle})/2.$$

It follows that

$$-\log \mu(C_{\langle n,m\rangle}) \leq \mu(V_{\langle n,m\rangle}) < -\log \mu(C_{\langle n,m\rangle}) + 1.$$

This is a contradiction.

Hence x is not K-complexity random. □

5.6 When they do not coincide

The condition of SCT_3 is a sufficient condition for the coincidnece between ML-randomness and complexity randomness. We prove that we can not weaken the condition to SCT_2. To prove it, we use the following computable topological space.

Consider the unit interval $\mathbf{I} = ([0,1], \tau, \beta, \nu)$ in Example 2.3. Let a be a real in $[0,1]$. We write $I_a = [0,1]\setminus\{a\}$. Let

$$\nu_a(\langle u,v\rangle) = \begin{cases} \nu(v) & \text{if } u = \lambda \text{ and } v \in \mathrm{dom}(\nu), \\ \nu(v) \cap I_a & \text{if } u \neq \lambda \text{ and } v \in \mathrm{dom}(\nu), \end{cases}$$

where λ is the empty string. Let $\beta_a = \{\nu_a(u) \ : \ u \in \mathrm{dom}(\nu_a)\}$. Since I_a is open on $\mathbf{I}$, the topology generated by the base β_a coincides with τ.

Proposition 5.21. *The 4-tuple* $\mathbf{I}_a = ([0,1], \tau, \beta_a, \nu_a)$ *is* SCT_2*. Further the following are equivalent.*

(i) $\mathbf{I}_a$ *is* SCT_3.

(ii) a *is* δ_a*-computable.*

(iii) a *is* δ*-computable.*

Note that $\mathbf{I}_a$ is T_3 for all $a \in [0,1]$.

Proof. We prove that $\mathbf{I}_a$ is a computable topological space. Clearly $\mathbf{I}_a$ is an effective topological space clearly. Note that $\mathrm{dom}(\nu_a)$ is computable. Let $S \subseteq (\Sigma^*)^3$ be a c.e. set such that

$$\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \ : \ (u,v,w) \in S\}$$

for all $u, v \in \mathrm{dom}(\nu)$. Let $S_a \subseteq (\Sigma^*)^3$ be the c.e. set such that

$$(\langle u_1,u_2\rangle, \langle v_1,v_2\rangle, \langle w_1,w_2\rangle) \in S_a \iff (u_2,v_2,w_2) \in S \wedge w_1 = \lambda$$

if $u_1 = v_1 = \lambda$ and

$$(\langle u_1,u_2\rangle, \langle v_1,v_2\rangle, \langle w_1,w_2\rangle) \in S_a \iff (u_2,v_2,w_2) \in S \wedge w_1 \neq \lambda$$

if $u_1 \neq \lambda$ or $v_1 \neq \lambda$. Then S_a satisfies (1) in Definition 2.1.

Since $\mathbf{I}$ is SCT_2 and $\beta \subseteq \beta_a$, $\mathbf{I}_a$ is also SCT_2.

(ii)⇒(iii) Suppose that the real a is δ_a-computable on $\mathbf{I}_a$. Then the set $\{\langle u, v\rangle \in \Sigma^* \ : \ a \in \nu_a(\langle u, v\rangle)\}$ is c.e. Since $a \notin I_a$, we have

$$a \in \nu_a(\langle u, v\rangle) \iff u = \lambda \text{ and } a \in \nu(v)$$

Hence $\{v \in \Sigma^* \ : \ a \in \nu(v)\}$ is c.e. It follows that a is δ-computable.

(iii)⇒(ii) This is proved by tracking back the proof of (ii)⇒(iii).

(iii)⇒(i) Consider a base set W that has the form $W = (p, q) \cap I_a$ where $p, q \in \mathbb{Q}$. Since a is δ-computable, it is decidable whether $a \in (p, q)$ or not. If not, $W = ((p, a) \cap [0, 1]) \cup ((a, q) \cap [0, 1]$. Now it is easy to construct R and r in Definition 2.5.

(i)⇒(ii) Note that I_a is a base set. By Lemma 4.9, one can construct a computable sequence $\{U_n\}$ of finite unions of base sets and a computable sequence $\{V_n\}$ of closed sets such that $U_n \uparrow I_a$, $V_n \uparrow I_a$ and $U_n \subseteq V_n \subseteq I_a$ for all n. Since I_a is not closed and V_n is closed, $U_n \subsetneq I_a$.

Consider the diameter of a set $A \subseteq [0, 1]$ as

$$D(A) = \sup\{|x - y| \ : \ x, y \in A\}.$$

Since U_n is a finite union of base sets, $D([0, 1]\setminus U_n)$ is computable. Since $U_n \subsetneq I_a$, $[0, 1]\setminus U_n$ has at least two elements and $D([0, 1]\setminus U_n) > 0$. Since $U_n \uparrow I_a$, $D([0, 1]\setminus U_n) \to 0$ as $n \to 0$. It follows that a is a δ-computable real. □

Proposition 5.22. *There exists an* SCT_2 *and* T_3 *space with a computable measure on which ML-randomness and complexity randomness does not coincide.*

Proof. Let a be a ML-random real on $\mathbf{I}$. We prove that a is ML-random but not complexity random on $\mathbf{I}_a$ with the Lebesgue measure μ.

The co-c.e. closed set $\{a\}$ has measure 0 and $a \in \{a\}$. Hence a is not complexity random.

Suppose a is not ML-random on $\mathbf{I}_a$. Then there exists a ML-test $\{U_n\}$ with $a \in \bigcap_n U_n$. Since $\{U_n\}$ is uniformly c.e. open, there exist computable sequences $\{u_i^n\}$ and $\{v_i^n\}$ such that

$$U_n = \bigcup\{\nu_a(\langle u_i^n, v_i^n\rangle) \ : \ i \in \mathbb{N}\}.$$

Let

$$V_n = \bigcup\{\nu_a(\langle u_i^n, v_i^n\rangle) \ : \ u_i^n = \lambda \text{ and } i \in \mathbb{N}\}.$$

Then $V_n \subseteq U_n$. Since $a \in \bigcap_n U_n$, there exists i such that $a \in \nu_a(\langle u_i^n, v_i^n\rangle)$ for each n. If $u_i^n \neq \lambda$, $a \in \nu(v_i^n) \cap I_a$, which is a contradiction. Hence the i should satisfy $u_i^n = \lambda$ for each n. It follows that $a \in \bigcap_n V_n$.

However $\{V_n\}$ is a ML-test on $\mathbf{I}$. Hence this contradicts to the fact that a is ML-random on $\mathbf{I}$. □

Discussion

First we studied computability of measures on a computable topological space. We generalize the result by Gács to that the space of measres on a computable topological space is another computable topological space. Then computability of points on the space concindes with computability defined in Schröder [23]. Hence this is the right definition of computability of measures.

Next we studied Martin-Löf randomness on a computable topological space with a computable measure. We showed that there is not a universal test in general, and ML-randomness and complexity randomness do not coincide in general. A sufficient condition is SCT_3. However we can not weaken the condition of SCT_3 to CT_2 for the existence of a universal test and to SCT_2 for the coincidence between ML-randomness and complexity randomness. Hence a computable metric space is a rather general space on which ML-randomness is a natural notion and may be the best to which we can generalize ML-randomness as a natural randomness notion.

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