Characterization of Kurtz randomness by a differentiation theorem

Kenshi Miyabe

19 Dec 2011

Motivation

Motivation

$x \in [0, 1]$ is Martin-Löf random

if and only if

each computable function of bounded variation is differentiable at x

by Demuth (1975) and Brattka-Miller-Nies.

Known results

So far known

- Martin-Löf randomness,
- weak 2-randomness,
- computable randomness,
- Schnorr randomness

Question

WHY

In general

Randomness

??

Analysis

Integral test

Randomness $\iff$ integral test $\iff$ Analysis

On differentiability

- Pass a test
- Finite for an integral test
- Defined for difference between two integal tests
- A Lebesgue point for the function (?)

In this talk

- integral test
- characterization by integral tests
- characterization via differentiation theorem
- their difference version
- versions of other randomness notions

Integral test

ML-randomness

$[0,1]$: unit interval

base set: $[0,q),(p,q),(p,1]$ where $p,q \in \mathbb{Q} \cap [0,1]$

c.e. open set: computable union of base sets

μ: Lebesgue measure

Def 1 (Martin-Löf 1966).

Martin-Löf test *is* $\{U_n\}$ *uniformly c.e. open sets with* $\mu(U_n) \leq 2^{-n}$

$z \in [0,1]$ passes *the ML-test if* $z \notin \bigcap_n U_n$.

z *is* Martin-Löf random *if it passes all Martin-Löf tests.*

Lower semi-computable

Def 2.

$f : [0,1] \to \overline{\mathbb{R}}$ *is lower semi-computable*
$\iff f^{-1}(> q)$ *is uniformly c.e.*

f can be approximated from below.

Integral test

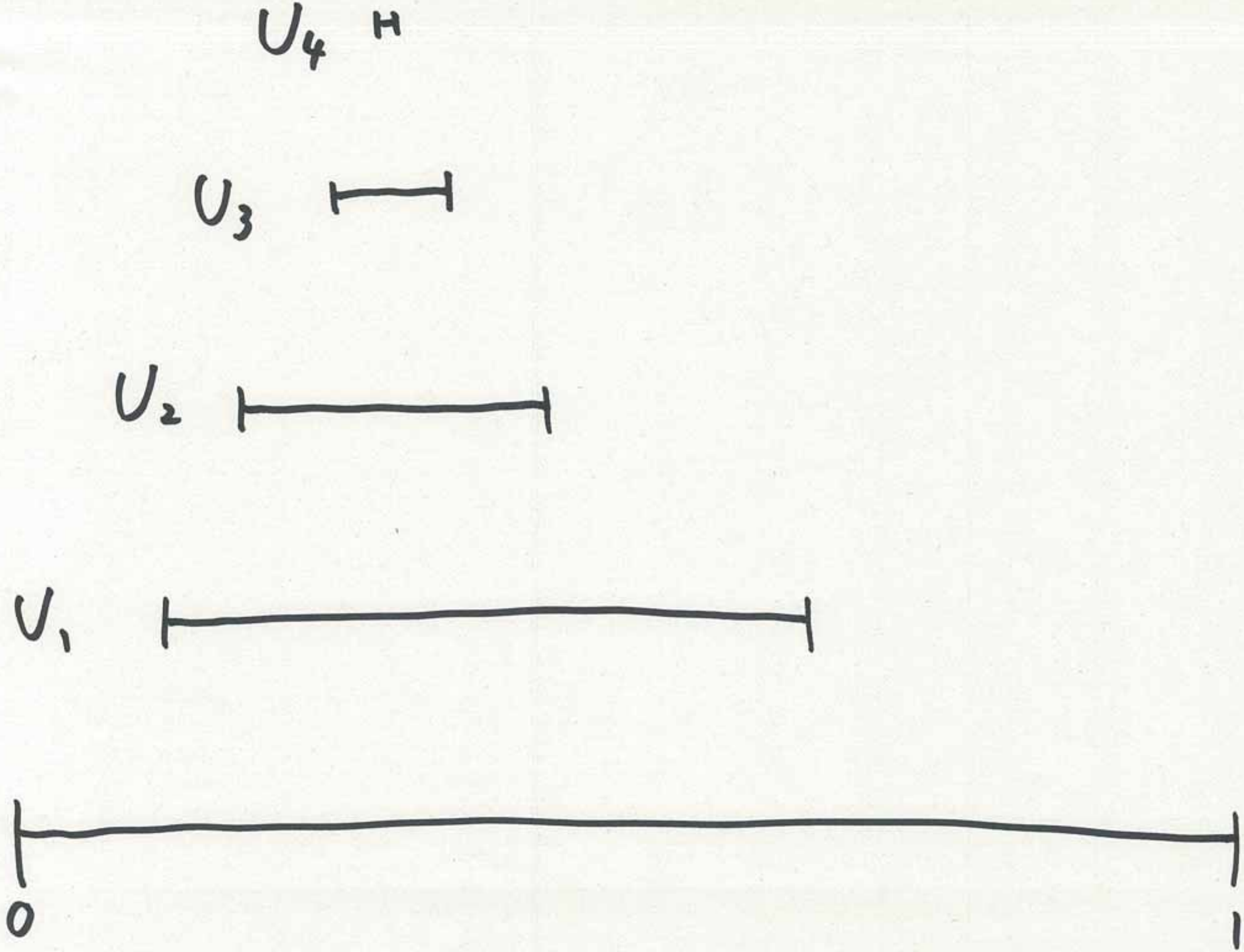
Def 3.

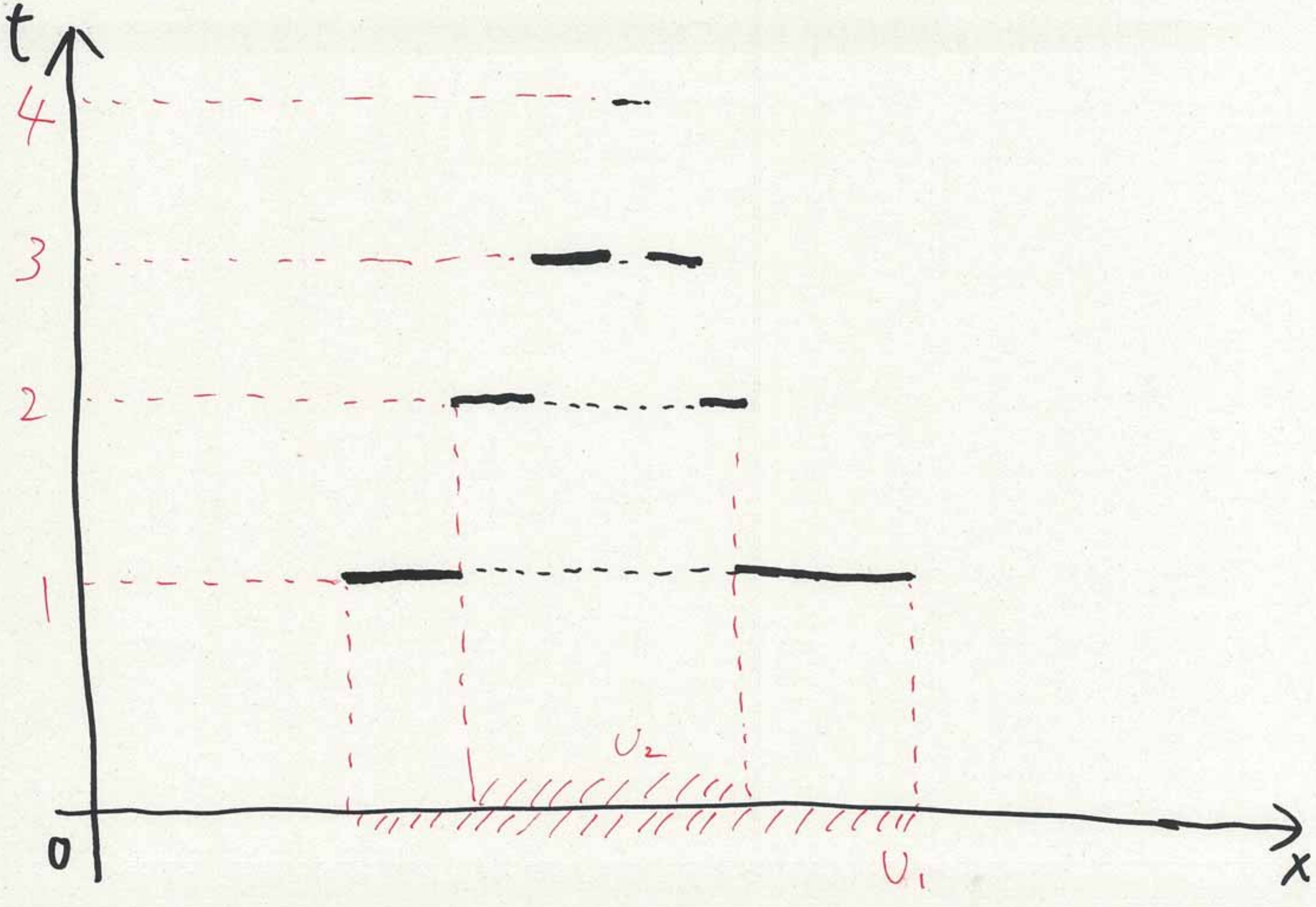
An ***integral test*** *is a lower semi-computable function* $t : X \to \overline{\mathbb{R}}$ *with* $\int t d\mu < \infty$.

Prop 4.

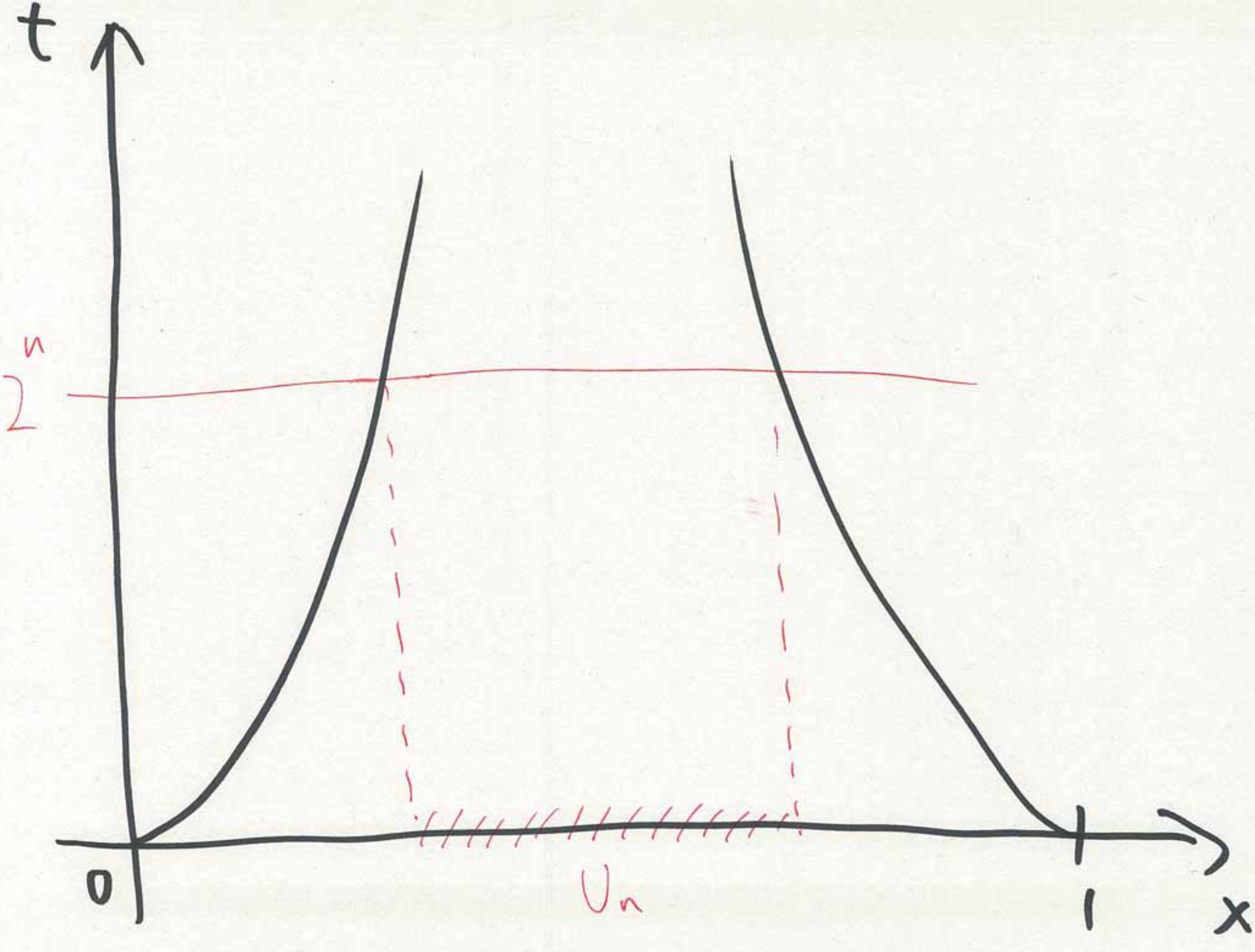
$z \in X$ *is ML-random*

$$\iff t(z) < \infty \text{ for each integral test } t.$$


U_4
U_3
U_2
U_1
0
1



t
4
3
2
1
0
x
U_2
U_1


t
2^n
0
U_n
1
x

Kurtz randomness

Kurtz randomness

Def 5.
x is Kurtz random
if x is contained in all c.e. open sets with measure 1.

This is a weaker notion than ML-randomness
Is there a characterization by integral tests?

Computability

$f : [0, 1] \to \mathbb{R}$ is computable

$\iff f^{-1}((p, q))$ is uniformly c.e.

$f : [0, 1] \to \overline{\mathbb{R}}$ is extended computable $\iff$

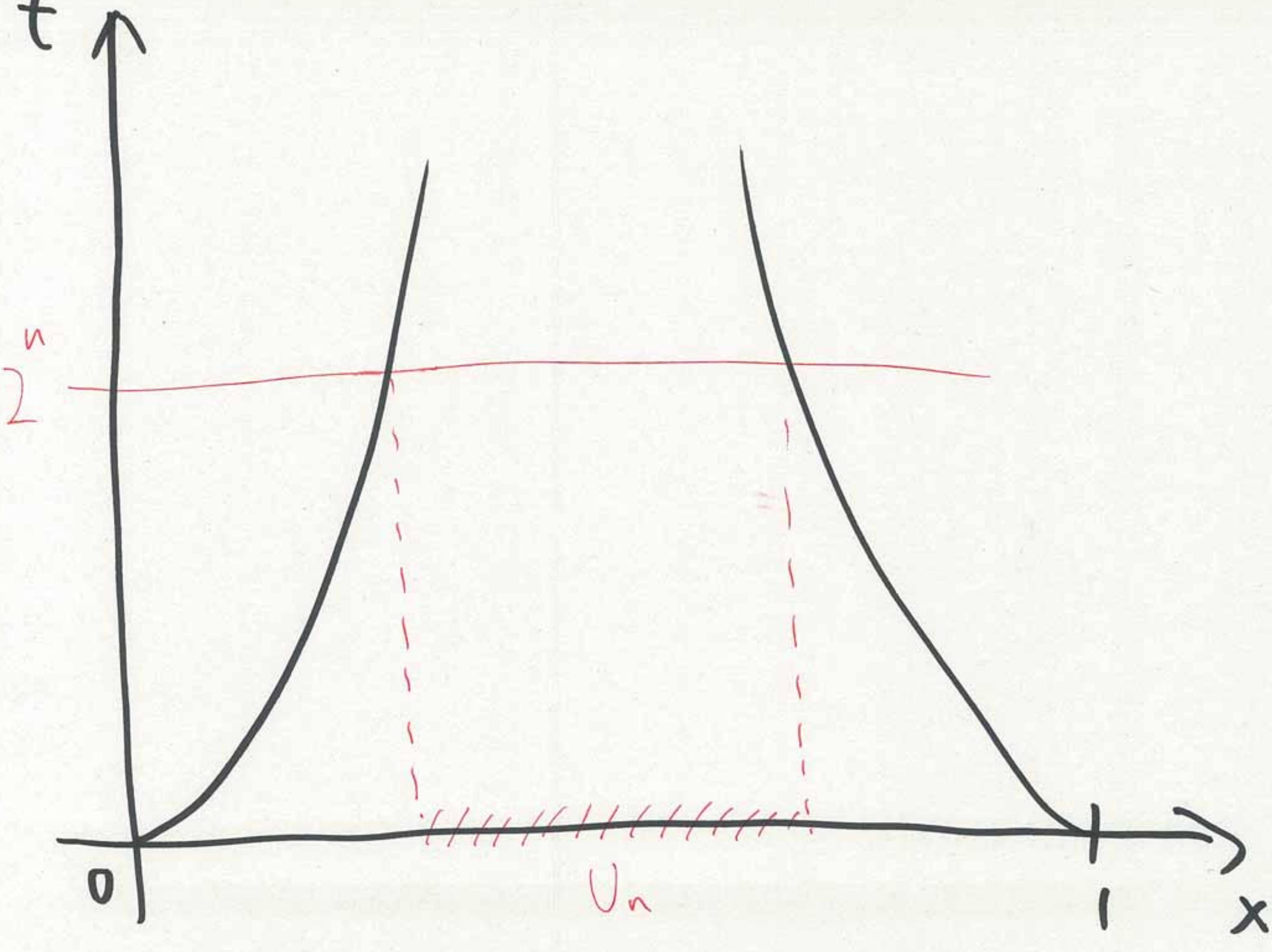
$f^{-1}((p, q))$, $f^{-1}(> p)$, $f^{-1}(< q)$ are uniformly c.e.

Kurtz test if $\mu(U_n) = 1$

Thm 6 (M.). *z is Kurtz random*
$\iff$ $t(z) < \infty$ *for each non-negative extended computable* t *s.t. that* $t(x) < \infty$ *almost everywhere*
$\iff$ $t(z) < \infty$ *for each non-negative extended computable* t *s.t. that* $\int t d\mu < \infty$
$\iff$ $t(z) < \infty$ *for each non-negative extended computable* t *s.t.* $\int t d\mu$ *is computable.*

Finite

$$\{x \ : \ t(x) < \infty\} = \bigcup_n \{x \ : \ t(x) < n\}.$$

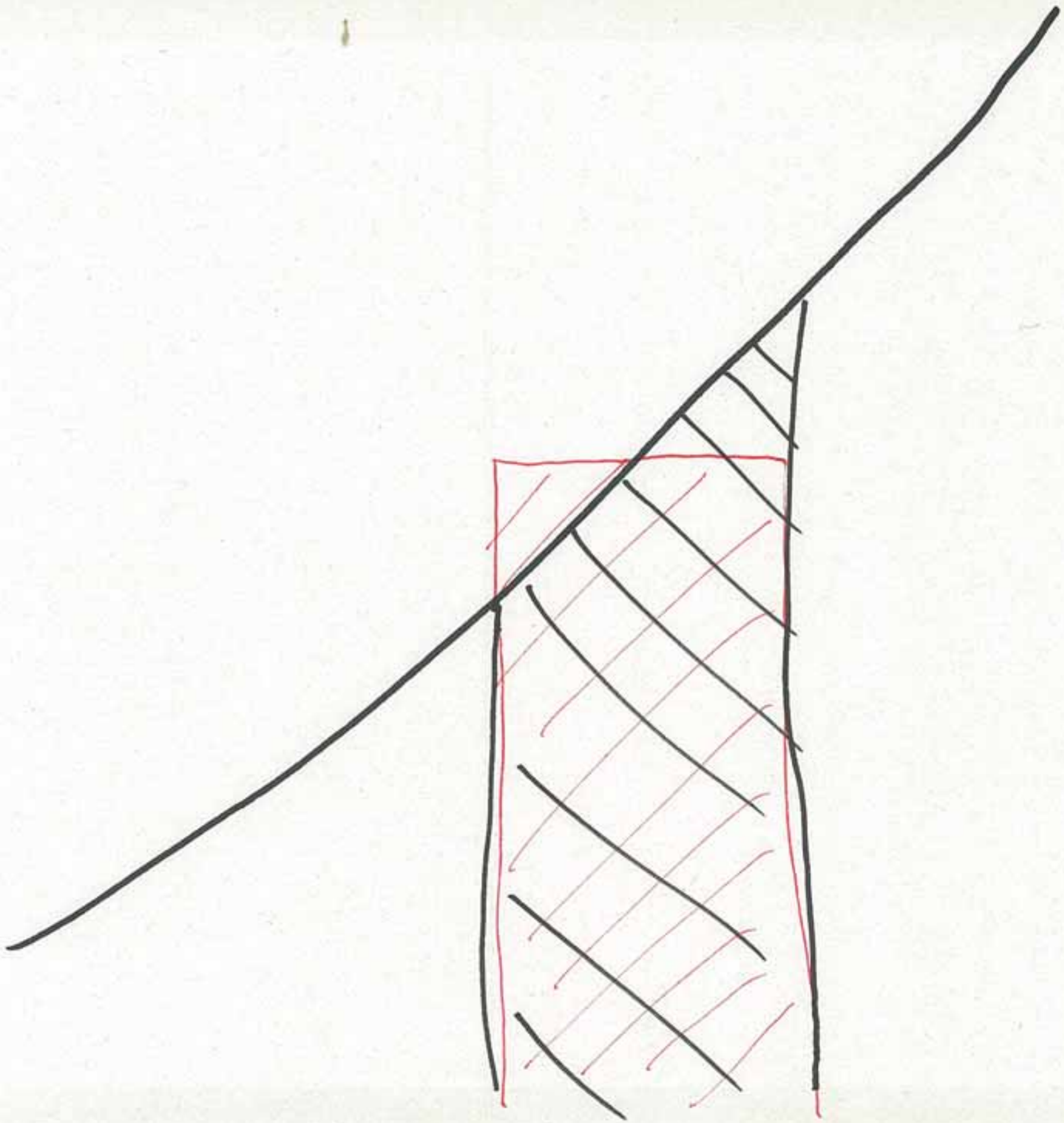

t
2^n
0
U_n
1
x

Note

Def 7. *A point x is a Lebesgue point for f if*

$$\lim_{r \to 0+} \frac{1}{2r} \int_{x-r}^{x+r} f \, d\mu = f(x).$$

- A finite continuous point is a Lebesgue point.
- A computable function is continuous.



Thm 8 (M.). *z is Kurtz random $\iff$ z is a Lebesgue point for each non-neggative extended computable function t s.t. $\int t d\mu$ is computable.*

Cor 9 (M.). *z is Kurtz random $\iff$ each comp. $F : [0,1] \to \mathbb{R}$ whose derivative is non-negative and extended computable is differentiable at z.*

Difference version

Notation

$\mathcal{K}$: the set of non-negative extended computable function f s.t. $f(x) < \infty$ a.e.

$\mathcal{K}_{\mathrm{fin}}$: restriction with $\int f d\mu < \infty$

$\mathcal{K}_{\mathrm{comp}}$: restriction with $\int f d\mu$ is computable

Then

z is Kurtz random

$\iff$ z is a Lebesgue point for $f \in \mathcal{K}$

$\iff$ z is a Lebesgue point for $f \in \mathcal{K}_{\mathrm{fin}}$

$\iff$ z is a Lebesgue point for $f \in \mathcal{K}_{\mathrm{comp}}$.

Def 10.

$$\mathcal{D} = \{f - g \ : \ f, g \in \mathcal{K}\}.$$

$\mathcal{D}_{\mathrm{fin}}, \mathcal{D}_{\mathrm{comp}}$ are similarly defined.

Difference version

z is Kurtz random

$\iff$ z is a Lebesgue point for $f \in \mathcal{D}$

$\iff$ z is a Lebesgue point for $f \in \mathcal{D}_{\mathrm{fin}}$

$\iff$ z is a Lebesgue point for $f \in \mathcal{D}_{\mathrm{comp}}$.

Any simple characterization of $\mathcal{D}$?

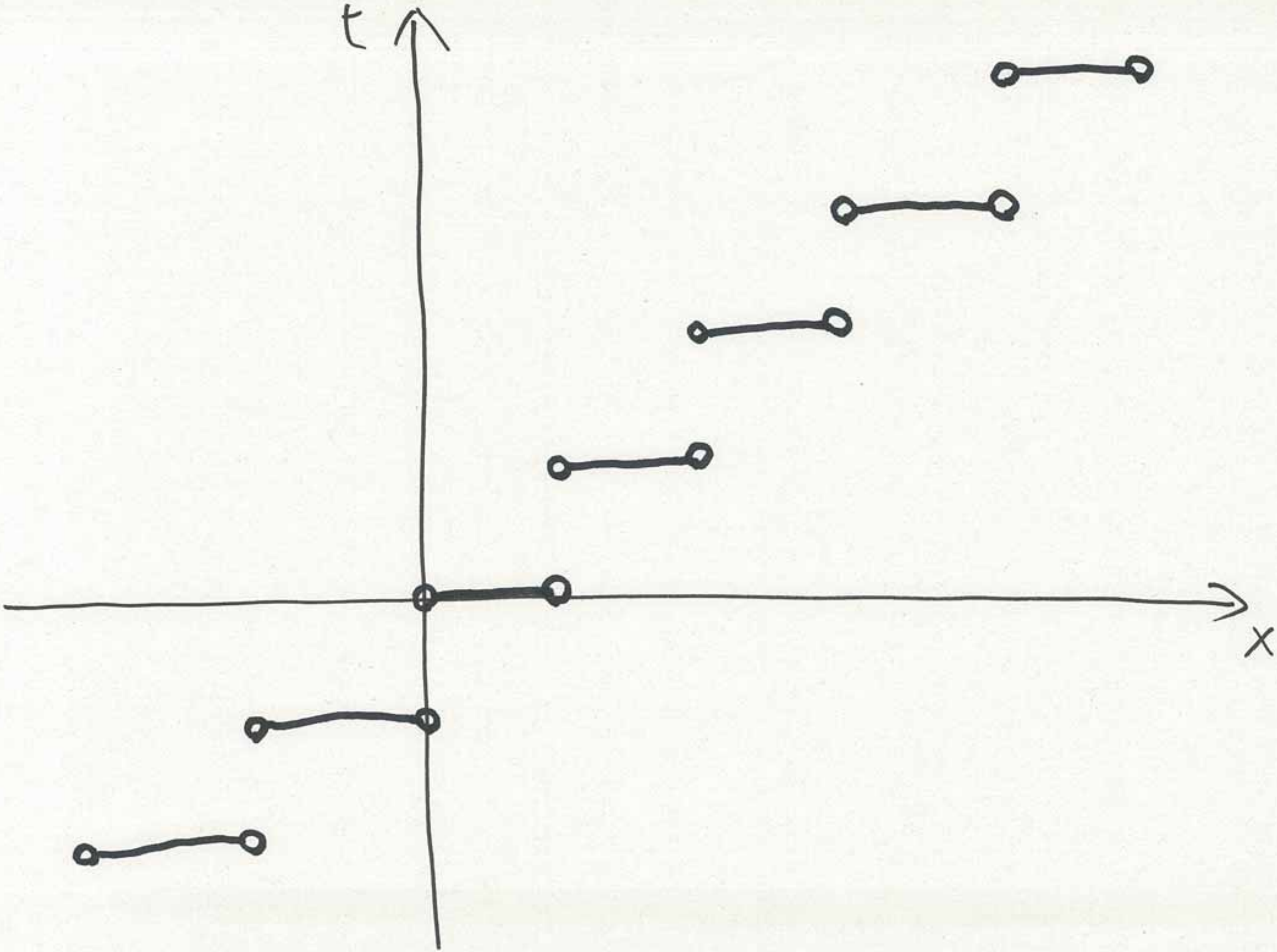
a.e. computability

Def 11.

f *is* ***almost everywhere computable***

or ***a.e. computable***

if it is a computable function defined a.e.


t
x

Notation

$\mathcal{A}$: the set of all a.e. computable functions

$\mathcal{A}_{\rm fin}$: with $\int |f| d\mu < \infty$.

$\mathcal{A}_{\rm comp}$: s.t. $\int |f| d\mu$ is computable

$$\mathcal{D}_{\rm comp} \subseteq \mathcal{A}_{\rm comp}, \ \mathcal{D}_{\rm fin} \subseteq \mathcal{A}_{\rm fin}, \ \mathcal{D} \subseteq \mathcal{A}$$

Prop 12.
$\forall h \in \mathcal{A}$, $n \in \mathbb{N}$ $\exists g \in \mathcal{D}$ *s.t.*

$$|h(x) - g(x)| \leq 2^{-n}$$

on Kurtz random points.

If $h \in \mathcal{A}_{\text{fin}}$, *then* $g \in \mathcal{D}_{\text{fin}}$.

Thm 13. *The following are equivalent.*

(i) z *is Kurtz random.*

(ii) $f(z)$ *is defined for each a.e. computable function.*

(iii) z *is a Lebesgue point for each a.e. computable function.*

The equivalence between (i) & (ii) is by Rute.

Kurtz equivalence

Def 14.
f, g are Kurtz equivalent if $f(x) = g(x)$ on Kurtz random points.

Thm 15 (M.)**.**
Let f, g be a.e. computable functions.
f, g are Kurtz equivalent $\iff$ $\int |f - g| d\mu = 0$.

Other randomness notions

Def 16. *f is weakly L^1-computable if $f = \sum_n f_n$ for uniformly computable functions f_n s.t. $\sum \|f_n\|_1 < \infty$.*

Thm 17 (M.). *Each weakly L^1-computable function is the difference between two integral tests.*
x is Martin-Löf random iff $f(x)$ is defined for each weakly L^1-computable function.

Thm 18 (M.)**.**
Let f, g be weakly L^1-computable functions.
$f(x) = g(x)$ *for each Martin-Löf random point* x
$\iff \int |f - g| d\mu = 0$

Weak 2-randomness

Def 19. *A generalized ML-test is a sequence* $\{U_n\}$ *of uniformly c.e. open sets s.t.* $\mu(U_n) \to 0$.
A point x *is weakly 2-random if* $x \notin \bigcap_n U_n$ *for all generalized ML-tests.*

This is a stronger notion than ML-randomness.

Generalized ML-test if $\mu(U_n) \to 0$

Thm 20 (M.)**.**
z is weakly 2-random
$\iff t(z) < \infty$ for each non-negative lower semi-computable function t s.t. $t(x) < \infty$ a.e.

Defined

Def 21. *f is limit computable if $f = \sum_n f_n$ for uniformly computable functions f_n.*

Thm 22 (M.)**.** *Each limit computable function defined a.e. is the difference between two lower semi-computable functions which are finite a.e. x is weakly 2-random iff $f(x)$ is defined for each limit computable function defined a.e.*

Thm 23 (M.).

Let f, g *be limit computable functions defined a.e.*

$f(x) = g(x)$ *for each weakly 2-random point* x

$\iff \int |f - g| d\mu = 0$

Schnorr randomness

Def 24.
A Schnorr test is a ML-test s.t. $\mu(U_n)$ *is uniformly computable.*
A point x *is Schnorr random if* $x \notin \bigcap_n U_n$ *for all Schnorr tests.*

weaker than ML-randomness

Schnorr test if

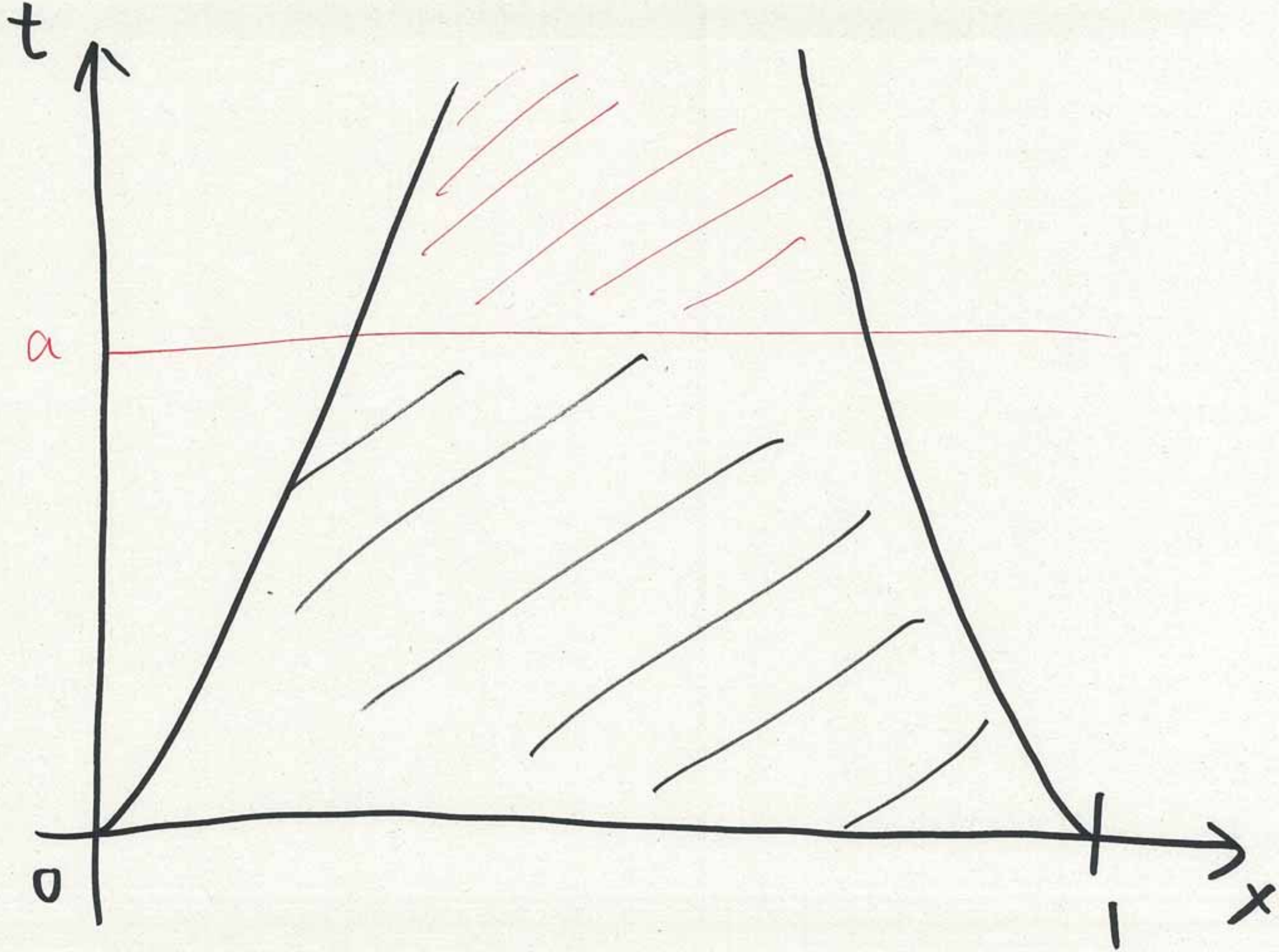
$\mu(U_n)$ is uniformly computable, decreasing to 0

Thm 25 (M.)**.**

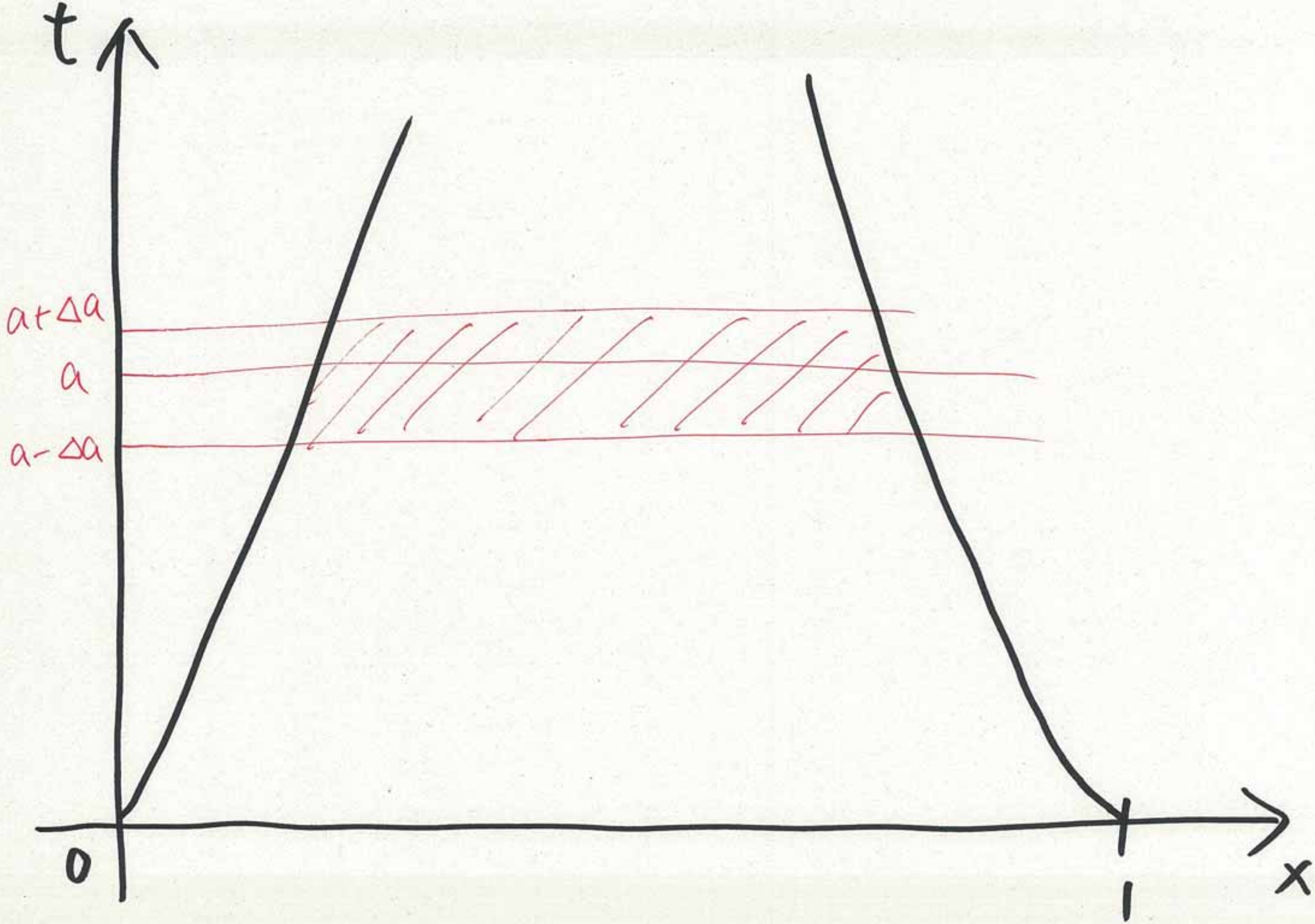
z is Schnorr random

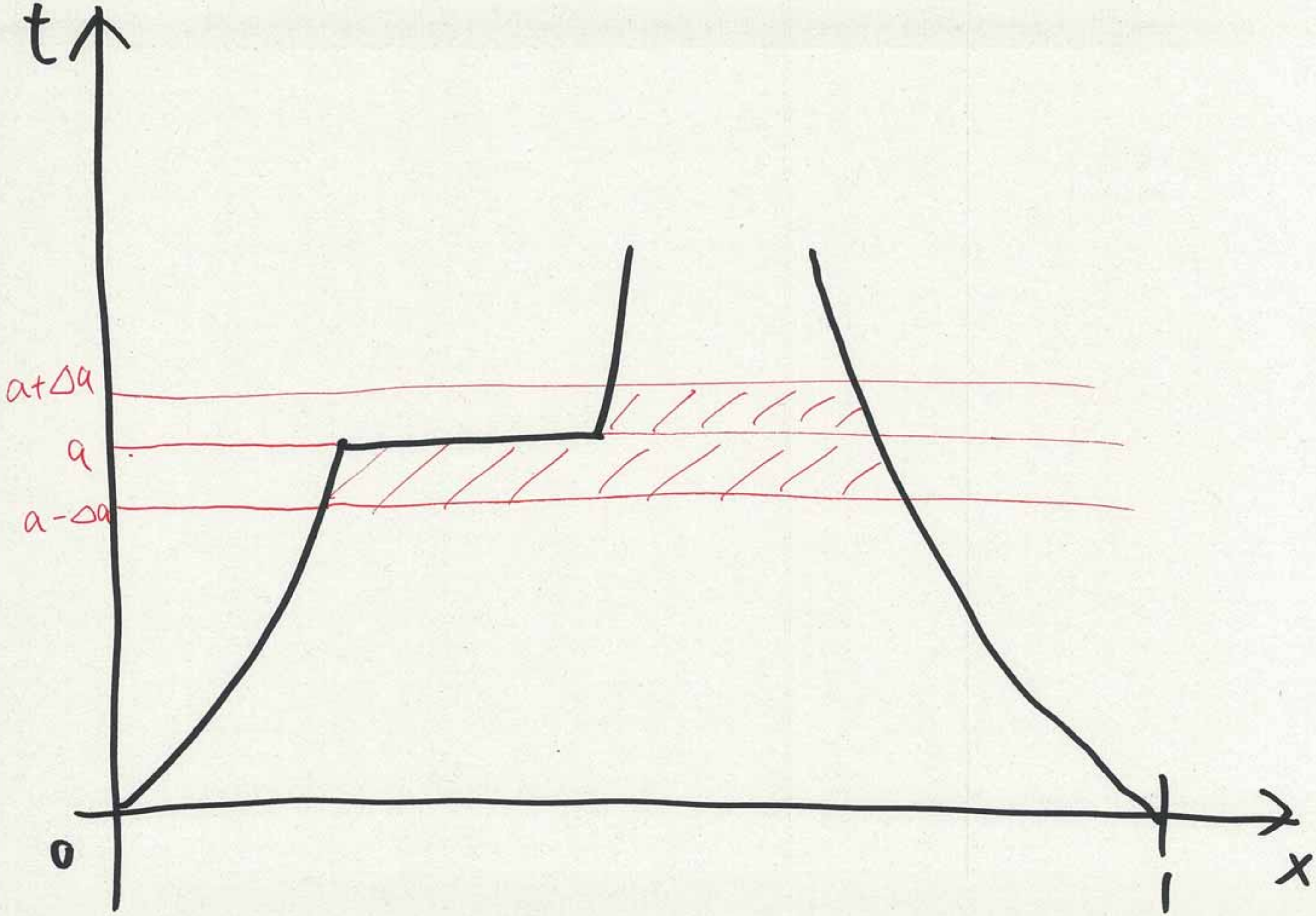
$\iff t(z) < \infty$ for each non-negative

lower semi-comp. t s.t. $\int t d\mu$ is computable.



t
a
0
1
x


t
a+Δa
a
a−Δa
0
1
x


t
a+Δa
a
a-Δa
0
1
x

Defined

Def 26. *f is L^1-computable (with a code) if $f = \sum_n f_n$ for uniformly computable functions f_n s.t. $||f_n||_1 \leq 2^{-n}$.*

Thm 27 (M.). *Each L^1-computable function is the difference between two lower semi-computable functions whose integrals are computable.*
x is Schnorr random iff $f(x)$ is defined for each L^1-computable function.

Equivalence

Thm 28 (M.)**.**

Let f, g be L^1-computable functions.

$f(x) = g(x)$ for each Schnorr random point x

$\iff \int |f - g| d\mu = 0$

Versions of integral tests

randomness	integral test
weak 2-	lower semi-comp. + finite a.e.
Martin-Löf	lower semi-comp. + integrable
Schnorr	lower semi-comp. + computable
Kurtz	extended computable + (*)

Difference versions

randomness	diff btw integral tests
weak 2-	limit comp. defined a.e.
Martin-Löf	weakly L^1-comp.
Schnorr	L^1-comp.
Kurtz	a.e. comp.

Future work

- Demuth randomness by integral tests?
- computably randomness by integral tests?
- Martin-Löf equivalence and Layerwise comp.?
- closer look on Radon-Nikodym derivative
- related with martingale convergence theorem?