Randomness and separation axioms

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13 Dec 2011

Conclusion

Question

Martin-Löf randomness is a randomness notion defined in terms of open sets.
How about complexity randomness?

$$(\forall d)(\exists n)K(X \restriction n) \geq n - d.$$

The string $X \restriction n = \sigma$ means $X \in [\sigma]$.
Mathematically $[\sigma]$ is clopen (closed and open).
If one needs to choose one of them,
which is more appropriate?
In my research this should be closed.
I'll explain why.

General topology

Cantor space

$$\Downarrow$$

a metric space

$$\Downarrow$$

a topological space

How about algorithmic randomness?

What are the finds?

- Complexity randomness is a randomness notion defined by closed sets rather than by open sets.
- Martin-Löf randomness is a natural randomness notion on a regular space but not on a more general space.
- A stronger randomness can be a natural randomness notion even on a general space.

A randomness notion highly depends on the strength of topology.

In this talk

- computable topology
- complexity randomness
- universality
- van Lambalgen's theorem

Computable topology

Unit interval

$X = [0,1]$
$[0,b), (a,b), (b,1]$ are open sets for $a, b \in [0,1]$.
The union of open sets is open.
The following sets form a base:

$$[0,q),\ (p,q),\ (p,1]$$

where $p, q \in \mathbb{Q} \cap [0,1]$.
Martin-Löf randomness on this space is equivalent to Martin-Löf randomness on Cantor space.

Lower unit interval

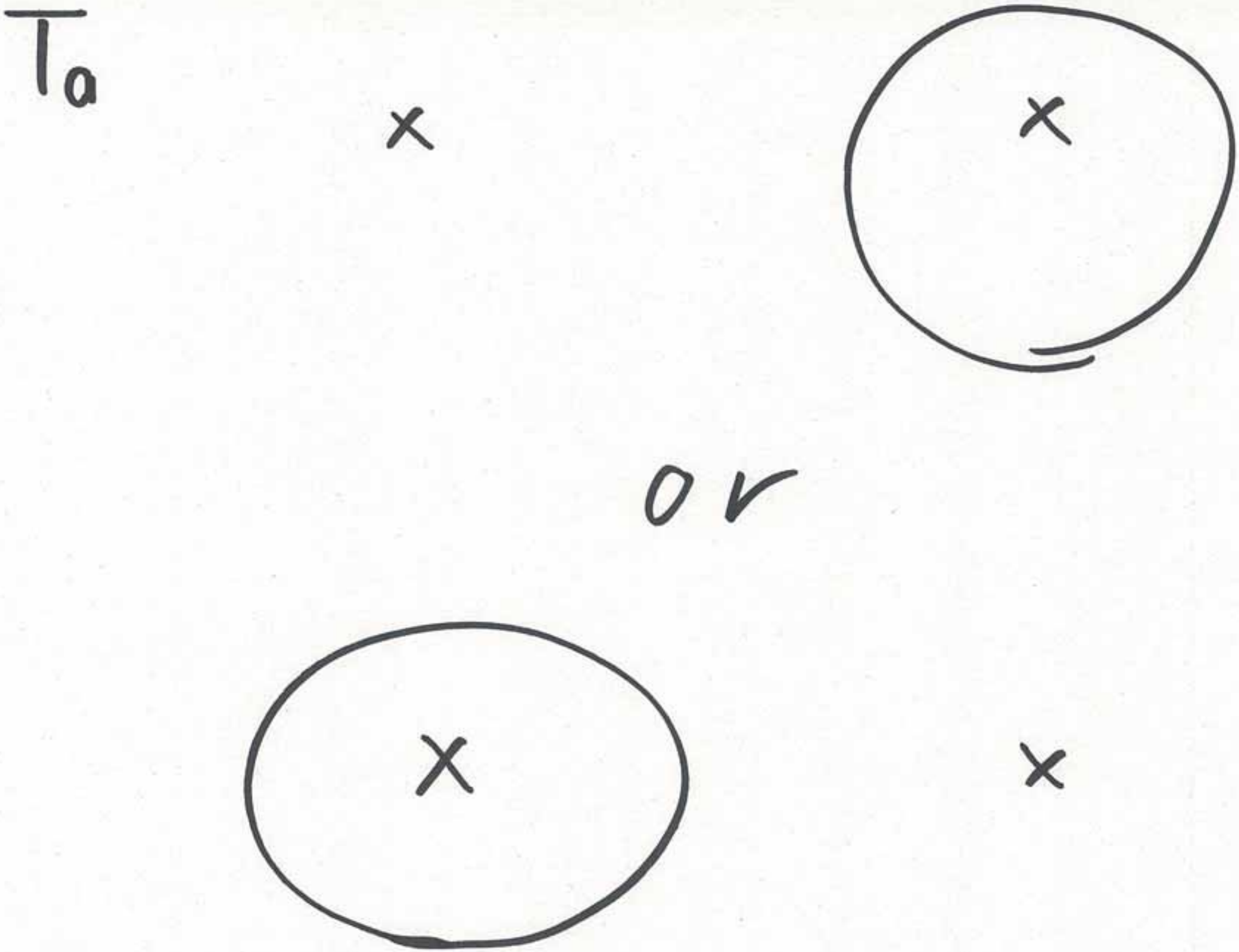
$X = [0, 1]$
$(a, 1]$ is an open set for each $a \in [0, 1]$.
The union of open sets is open.
The following sets form a base:

$$(p, 1]$$

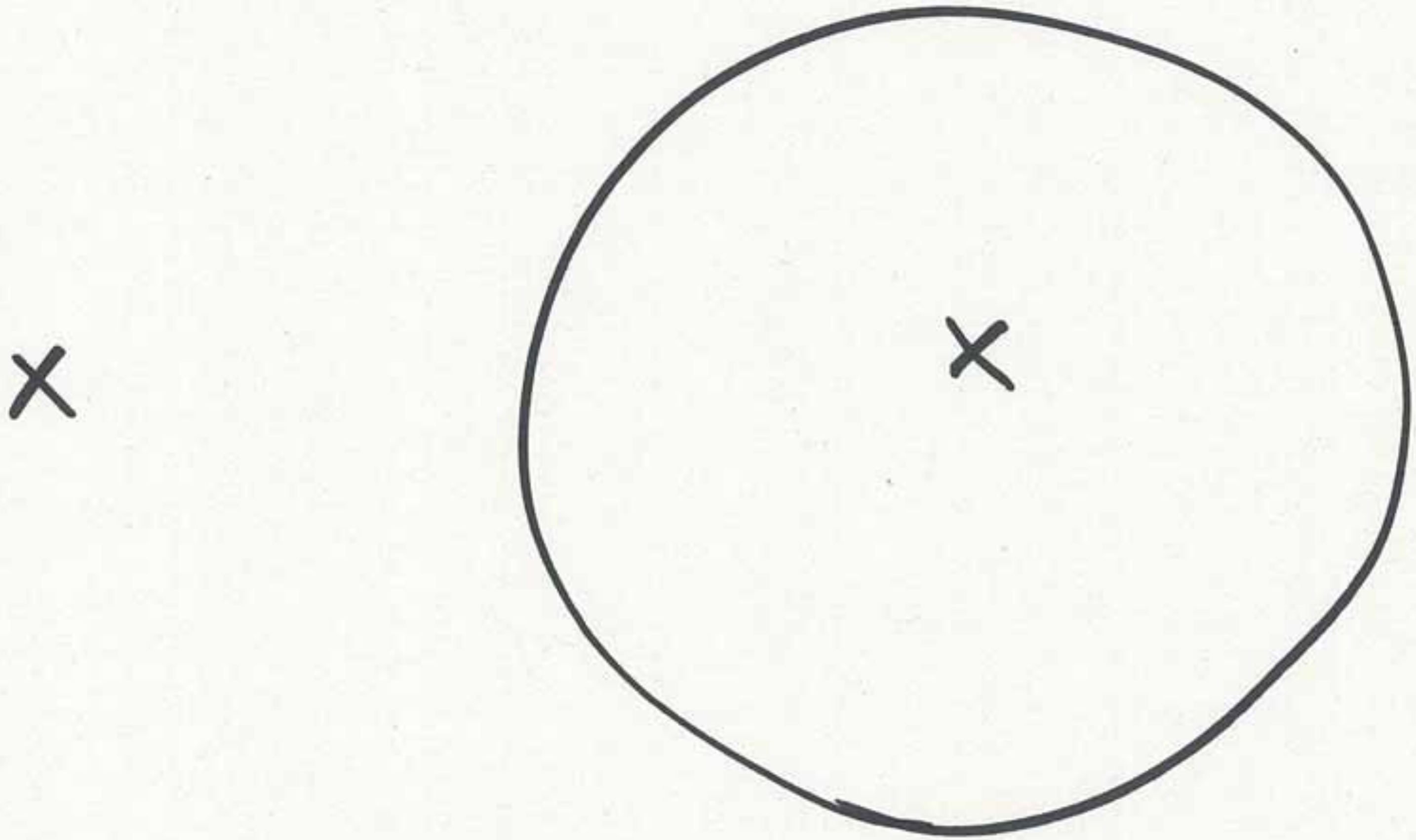
where $p \in \mathbb{Q} \cap [0, 1]$.
Martin-Löf randomness on this space is a natural randomness notion?

Some notions

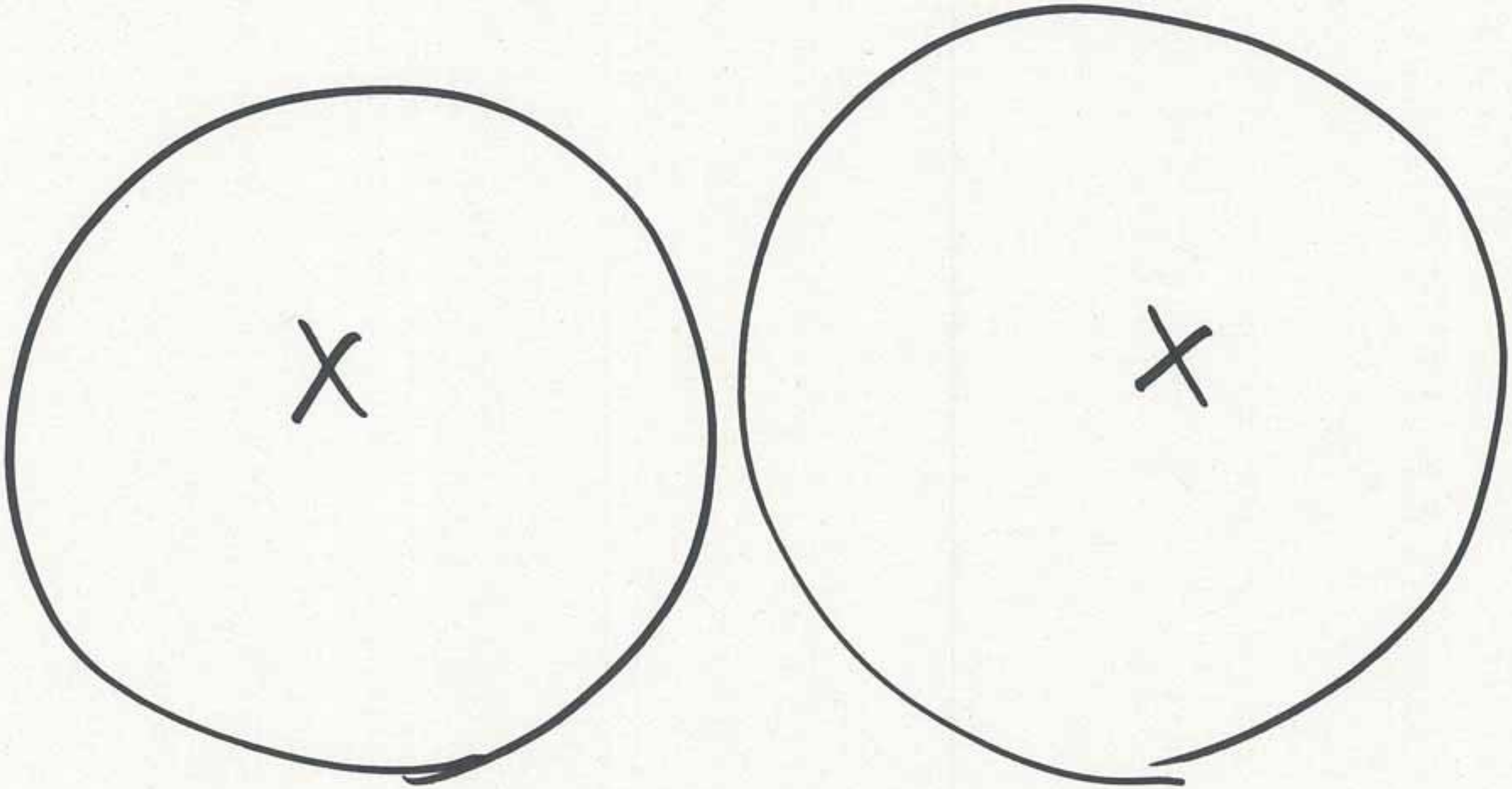
A space is second countable
if it has a countable base.
T_0: for distinct points, an open set containing one and not the other
T_1: for distinct points, an open set containing each not the other
T_2 (Hausdorff): for distinct points x, y, disjoint open U, V s.t. $x \in U$ and $y \in V$
T_3 (regular): for A is closed and $x \notin A$, disjoint open sets U, V s.t. $x \in U$ and $A \subseteq V$.


Ta
or

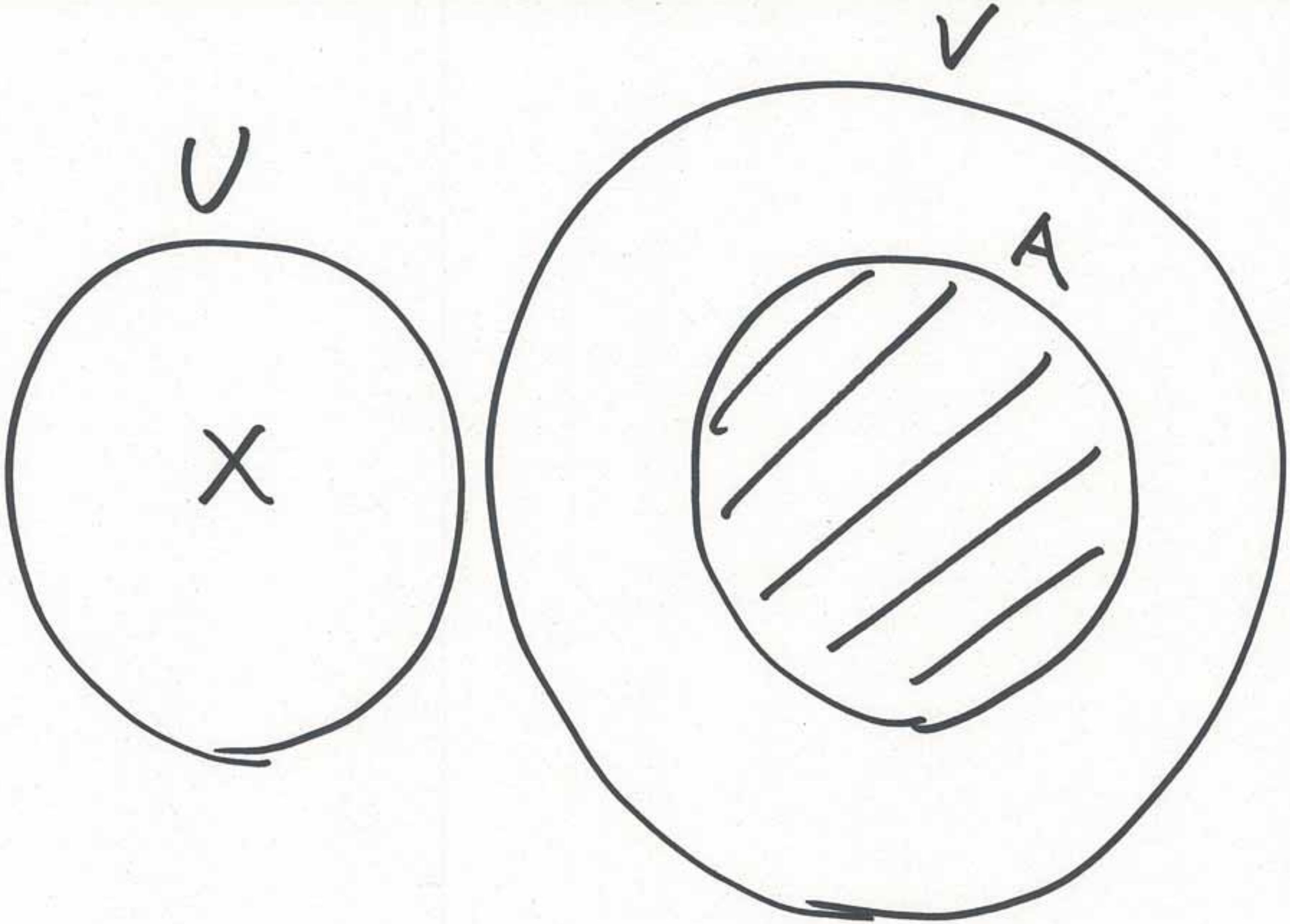
T_1



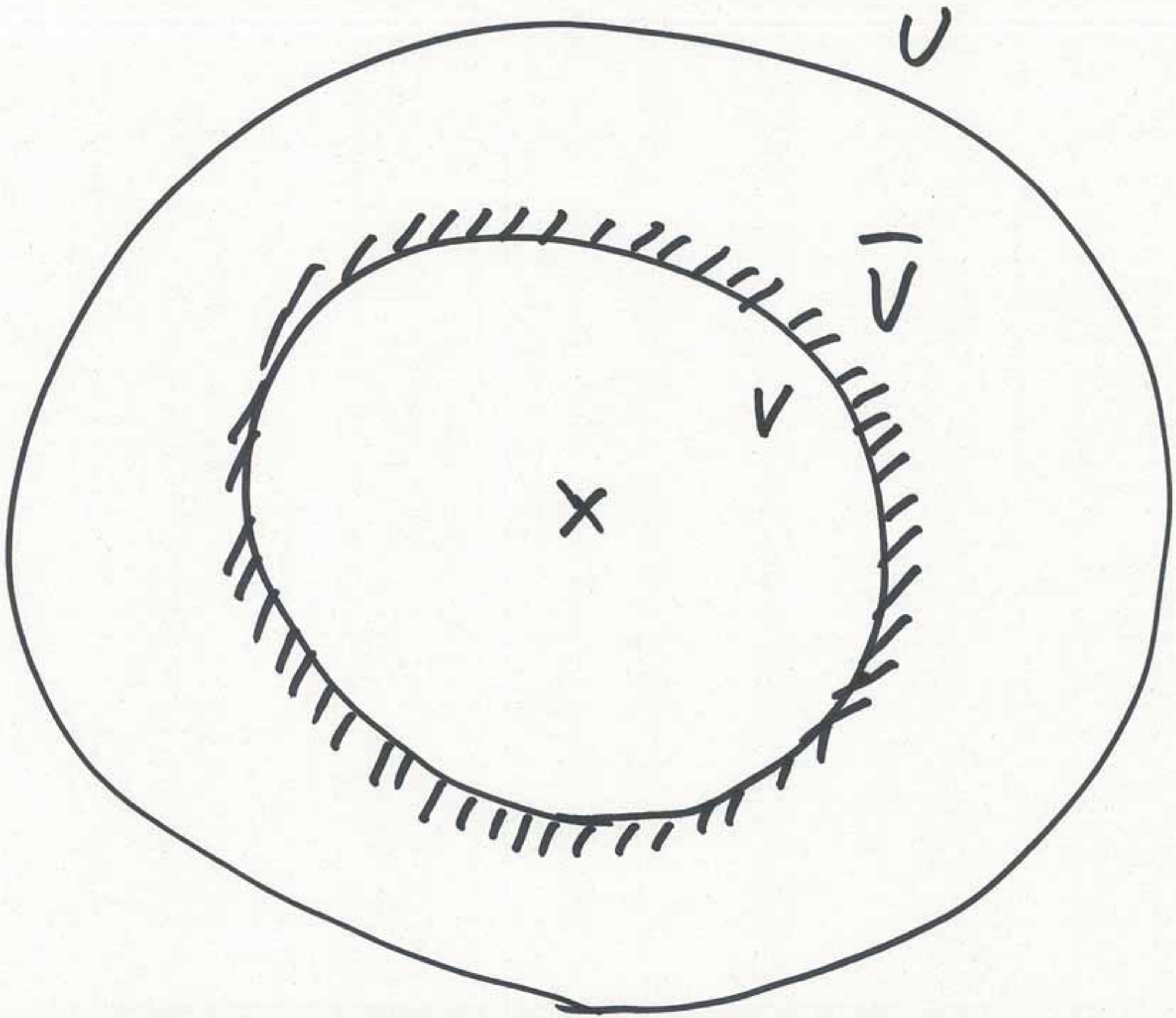
T_2



T_3



T_3



Computable topological space

Def 1 (Weihrauch). *A computable topological space is a 4-tuple* $\mathbf{X} = (X, \tau, \beta, \nu)$ *s.t.*

- (X, τ) *is a topological* T_0*-space,*
- $\nu :\subseteq \Sigma^* \to \beta$ *is a notation of a base* β *of* τ*,*
- $\mathrm{dom}(\nu)$ *is computable,*
- $\nu(u) \cap \nu(v) = \bigcup\{\nu(w) \ : \ (u, v, w) \in S\}$ *for all* $u, v \in \mathrm{dom}(\nu)$ *for some c.e. set* S*.*

Open and closed sets

Def 2.
A open set U is c.e. if U is a computable union of base sets.
A closed set A is co-c.e. if $X \setminus A$ is c.e.

Def 3 (Weihrauch).

SCT_2: *There is a c.e. set* $H \subseteq \Sigma^* \times \Sigma^*$ *s.t.*
$\forall x \neq y \exists (u,v) \in H, x \in \nu(u) \wedge y \in \nu(v)$ *and*
$\forall (u,v) \in H, \nu(u) \cap \nu(v) = \phi$.

SCT_3: *There are a c.e. set* $R \subseteq \mathrm{dom}(\nu) \times \mathrm{dom}(\nu)$ *and a computable function* $r :\subseteq \Sigma^* \times \Sigma^* \to \Sigma^\omega$ *s.t. for all*
$u, w \in \mathrm{dom}(\nu)$,
$\nu(w) = \bigcup \{\nu(u) \;:\; (u,w) \in R\}$,
$(u,w) \in R \Rightarrow \nu(u) \subseteq \psi^- \circ r(u,w) \subseteq \nu(w)$.

$\mathrm{SCT}_3 \approx$ a computable metric space

Computable measure

Def 4 (many)**.**
A probabilistic measure μ on $\mathbf{X}$ is computable if $\mu(G)$ is c.e. uniformly in G where G is a finite union of base sets.

Prop 5. *Let μ be a computable measure on $\mathbf{X}$.*
If U is a c.e. open set, then $\mu(U)$ is c.e.
If A is a co-c.e. closed set, then $\mu(A)$ is right c.e.

Algorithmic randomness

Martin-Löf randomness

Def 6. *Let* $\mathbf{X}$ *be a computable topological space and* μ *be a computable measure on it.*
A ML-test is a sequence $\{U_n\}$ *of uniformly computable open sets with* $\mu(U_n) \le 2^{-n}$.
A point $x \in X$ *is ML-random if* $x \notin \bigcap_n U_n$ *for all ML-tests* $\{U_n\}$.

ML-randomness

Which points are not ML-random on the lower unit interval with Lebesgue measure?

Each base set has the form of $(q, 1]$ where $q \in \mathbb{Q} \cap [0, 1]$.

Each c.e. open set has the form of $(r, 1]$ where r is a right c.e. real.

Then the only one non-ML-random point is 1.

On Cantor space

$A \in 2^\omega$ is not μ-random iff

$$(\forall d)(\exists n)K(A \restriction n) < -\log \mu([A \restriction n]) - d.$$

For $\sigma \in 2^*$,

- $K(\sigma)$ is approximated from above,
- $-\log \mu([\sigma])$ is computable.

The relation should be semidecidable!!
$\Rightarrow -\log \mu([\sigma])$ should be left c.e.
$\Rightarrow \mu([\sigma])$ should be right c.e.

Complexity randomness

Def 7 (M.). *A point x is complexity random if* $\exists d \, \forall A \in \mathcal{A}$

$$x \in A \Rightarrow K(A) \geq -\log \mu(A) - d.$$

Compare to

$$(\exists d)(\forall n) K(X \upharpoonright n) \geq -\log \mu([X \upharpoonright n]) - d.$$

$K(A)$ is the minimal length of the program that produces a representation of A.
If A is not co-c.e. closed, then $K(A) = \infty$.

complexity randomness

Which points are not complexity random
on the lower unit interval with the measure?
Each co-c.e. closed set has the form of $[0, r)$ where r is a right c.e. real.

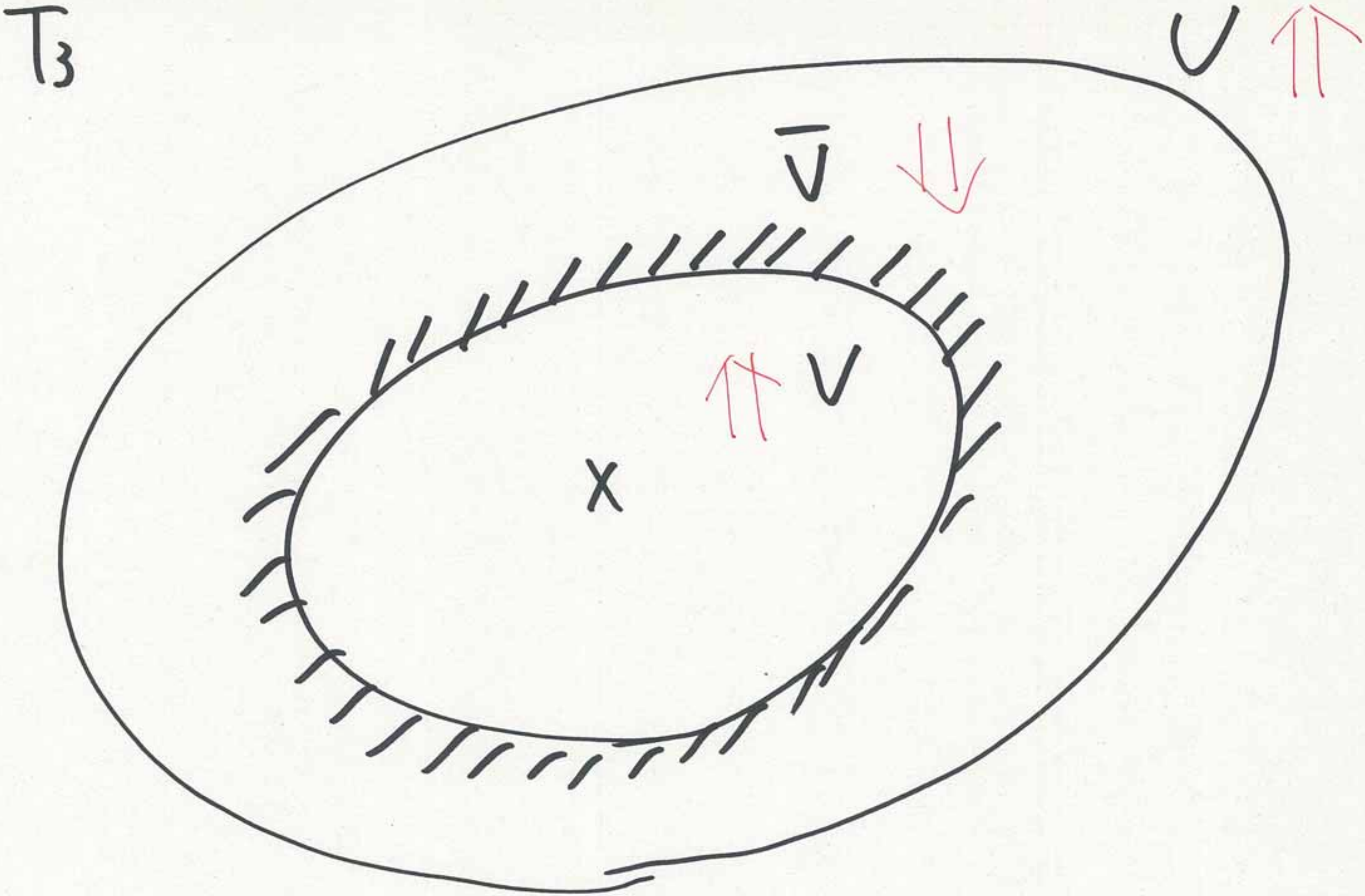
$$(\forall d)(\exists r) x \in [0, r) \wedge K([0, r)) < -\log r - d.$$

The only one non-complexity-random point is 0.
In general ML-randomness and complexity randomness are different.

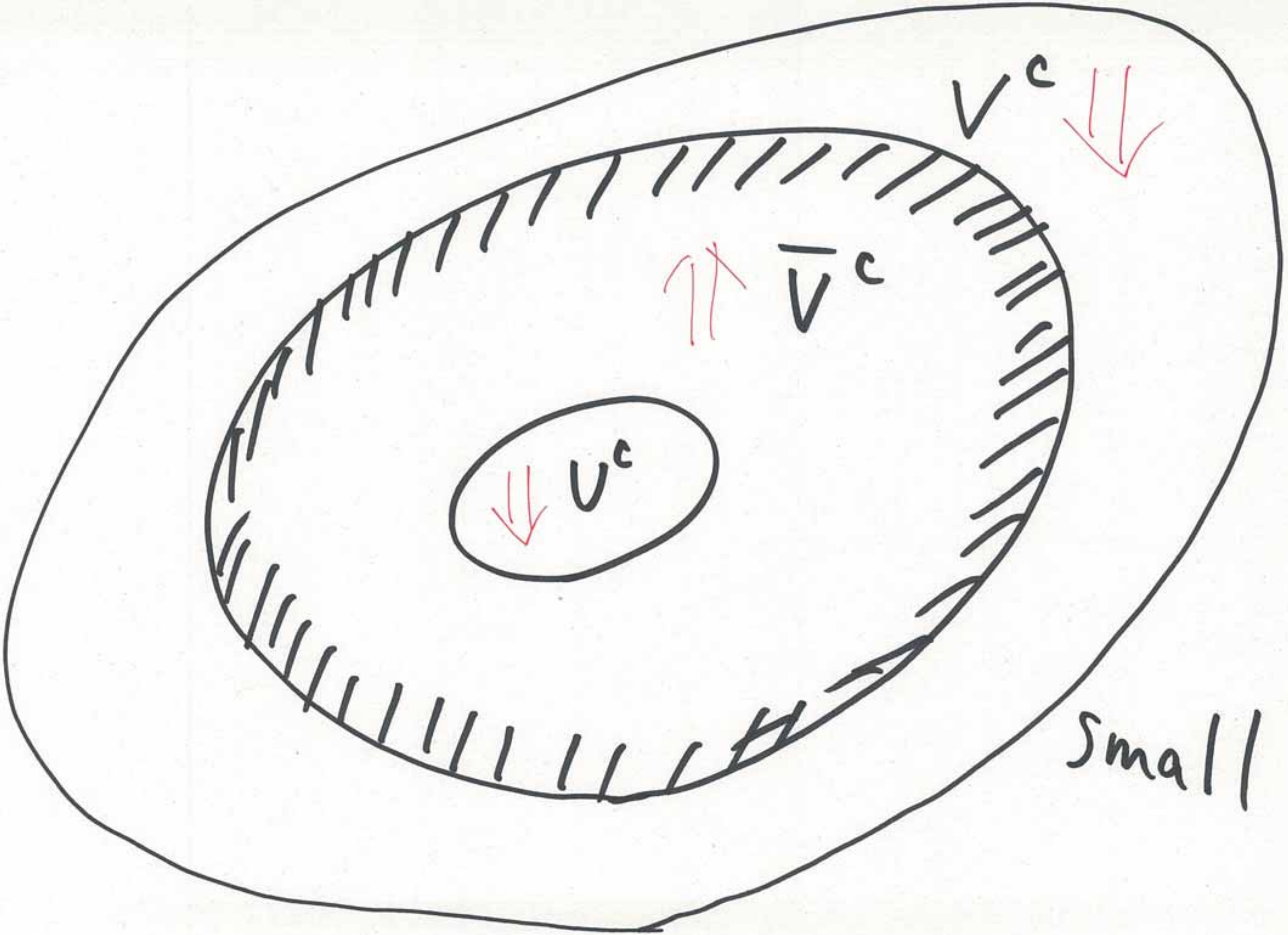
Equivalence

Thm 8 (M.).
Let $\mathbf{X}$ *be an* SCT_3 *space.*
Then
complexity randomness $=$ *Martin-Löf randomness*

Recall that
Cantor space & unit interval
$\Rightarrow$ a computable metric space $\Rightarrow \mathrm{SCT}_3$.



T3
U
V̄
V
x



V^c
V̄^c
U^c
Small

Non-equivalence

Thm 9 (M.)**.**
There exists an SCT_2 *and* T_3 *space*
with a computable measure
on which ML-randomness and complexity randomness does not coincide.

Other results

Universality

There exists a universal ML-test on an SCT_3 space because

- an SCT_3 space has an embedding to a computable metric space,
- complexity K is universal,
- one can prove this directly.

Universality comes from complexity randomness rather than ML-randomness?

Not in general

Consider again the lower unit interval.
Let μ be the measure with $\mu(\{r\}) = 1$
where r is left c.e. and non-computable.

$$\mu((q,1]) = \begin{cases} 1 & \text{if } q < r \\ 0 & \text{otherwise.} \end{cases}$$

Then μ is computable.
Each c.e. open set has the form of $(a, 1]$
where a is right c.e.
If $\mu((a,1]) \leq 2^{-1}$, then $a \geq r$.
Then $a > b > r$ for $b \in \mathbb{Q}$.
No test is universal.

How about SCT_2?

Question 10. *Is there a universal test on an* SCT_2 *space with a computable measure?*

van Lambalgen's Theorem

Thm 11 (M.). *Van Lambalgen's Theorem holds on an* SCT_3 *space.*

Only one implication needs the condition SCT_3!!

Question 12. *How about in general or on an* SCT_2 *space?*

n-randomness

Question 13. *I conjecture that n-ML-randomness and n-complexity randomness are equivalent on a computable topological space with a computable measure if defined appropriately.*

Summary

- Some "good" properties of randomness notions depend on the topology.
- It seems that the properties also depend on computability.

Thank you!