L^1-computability and weak L^1-computability

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Introduction

In Weihrauch (2000) approach,

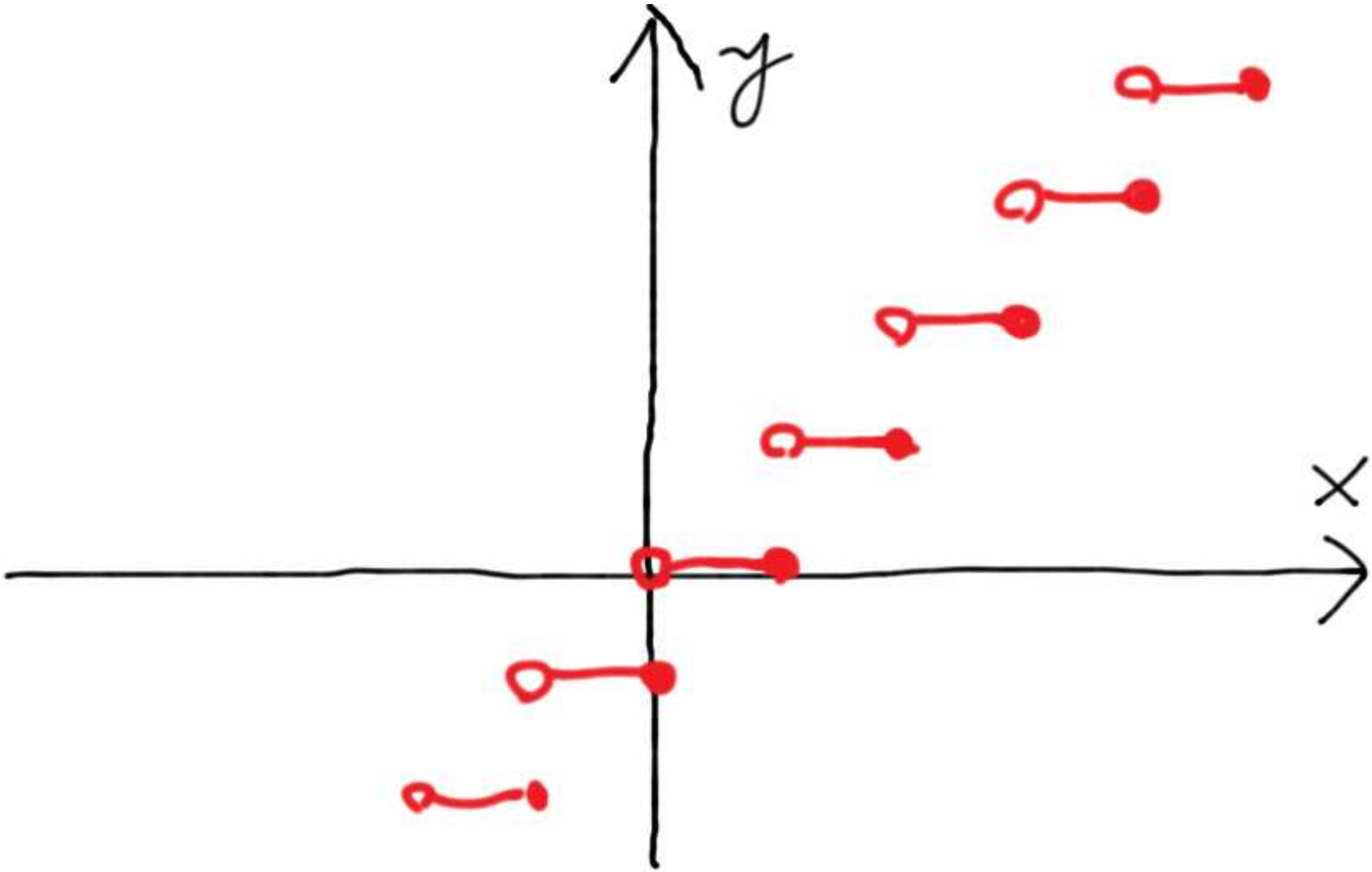
computability $\Rightarrow$ continuity.

This means that the floor function

$$\mathrm{floor}: \mathbb{R} \to \mathbb{R},\ \mathrm{floor}(x) = \lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \le x\}$$

is not computable because it is not continuous.
Such a simple function should be computable in some sense.
Then in what sense?

Example



Defined on random points

The floor function is "almost computable"
because it is computable almost everywhere.
For a given randomness notion, we can consider the class of functions defined only on the random points.

Kurtz rd.	a.e. computability
Schnorr rd.	Schnorr layerwise computability
?	layerwise computability
ML-rd.	?

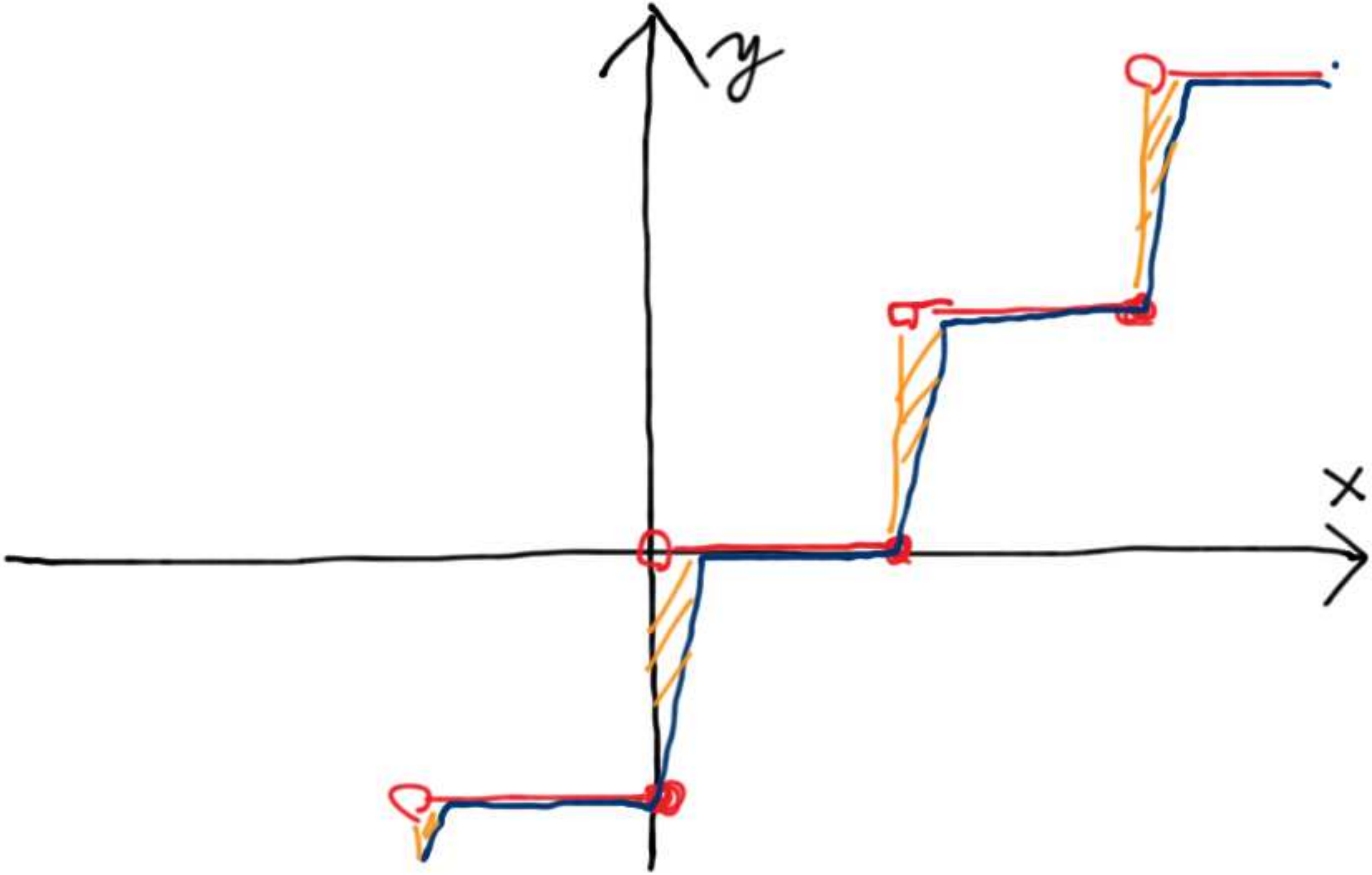
Well approximated

Another explanation is that the floor function "simple" because it can be well approximated.
Acturally there exists a sequence $\{f_n\}$ of uniformly computable functions such that

$$||\text{floor} - f_n||_1 \leq 2^{-n}$$

where $||f||_1 = \int |f| d\mu$.
Such a function is called a L^1-computable function by Pour-El and Richards (1989).

Example



The Goal of this talk

- L^1-computability is essentially equivalent to Schnorr layerwise computability.
- Weak L^1-computability has a strong relation to Martin-Löf randomness.
- Limit L^1-computability to weak 2-randomness.

The overview of this talk

- computable metric space
- lower semicomputable function and its approximations
- L^1-computability
- weak L^1-computability
- limit L^1-computability

Preliminary

Computable metric space

Definition 1 (See Brattka et al. (2008) etc.).
A computable metric space is a triple (X, d, α) such that

- $d : X \times X \to \mathbb{R}$ *is a metric on* X,
- $\alpha : \mathbb{N} \to X$ *is a sequence such that* $\{\alpha_i \mid i \in \mathbb{N}\}$ *is dense in* X,
- $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ *is a computable sequence in* $\mathbb{R}$.

α_i is called an ideal point.
$B(\alpha_i, q_i) = \{x \mid d(\alpha_i, x) < q\}$ is a basic open ball.
$\overline{B}(\alpha_i, q_i) = \{x \mid d(\alpha_i, x) \leq q\}$ is a basic closed ball.

Real line

$(\mathbb{R}, d, \alpha)$ is a computable metric space

where

- $d(a, b) = |a - b|$ and
- α is a canonical notation of all rationals.

But we sometimes assume that $d(a, b) \leq 1$ by changing the distance with another distance that induces the same computable topology.

Extended real line

$(\overline{\mathbb{R}}, d, \alpha)$ is a computable metric space

where

- $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$,
- $d(a, b) = |f(a) - f(b)|$ where $f(x) = \frac{x}{1+|x|}$, $f(\infty) = 1$ and $f(-\infty) = -1$, and
- α is a canonical notation of $\mathbb{Q} \cup \{\pm\infty\}$.

Cauchy representation

- $\{x_i\}$ converges rapidly to x if $d(x, x_i) < 2^{-i}$ for all i.
- The Cauchy representation is the representation $\delta :\subseteq \Sigma^\omega \to X$ such that $\delta(p) = x$ when p encodes a sequence of ideal points which converges rapidly to x.
- A point x is computable if there exists a computable representation p such that $\delta(p) = x$.

Representation ρ

Define repsentations $\rho, \rho_<, \rho_>$ on $\mathbb{R}$ as follows:

- $\rho(p) = x$ when p encodes $\{(a,b) \mid x \in (a,b),\ a,b \in \mathbb{Q}\}$,
- $\rho_<(p) = x$ when p encodes $\{a \mid a < x,\ a \in \mathbb{Q}\}$,
- $\rho_>(p) = x$ when p encodes $\{a \mid x < a,\ a \in \mathbb{Q}\}$.

Define repsentations $\overline{\rho}, \overline{\rho}_<, \overline{\rho}_>$ on $\overline{\mathbb{R}}$ similarly.
A real is computable in the usual sense iff it is ρ-computable.

Computable function

Let δ_1, δ_2 be representations on X_1, X_2 respectively.

- A function $f :\subseteq X_1 \to X_2$ is (δ_1, δ_2)-computable if there exists a computable mapping from a δ_1-representation of a point $x \in X_1$ to a δ_2-representation of $f(x) \in X_2$.
- A function $f :\subseteq X \to \mathbb{R}$ is computable if it is (δ, ρ)-computable.
- A function $f :\subseteq X \to \overline{\mathbb{R}}$ is lower semicomputable if it is $(\delta, \overline{\rho}_<)$-computable.

Computable measure

- A probabilistic measure on the computable metric space is computable if $\mu(B_{i_1} \cup \cdots \cup B_{i_k})$ is uniformly lower semicomputable.
- If f is nonnegative and lower semicomputable and μ is computable, then $\mu(f) = \int f d\mu$ is lower semicomputable.
- If f is bounded and computable and μ is computable, then $\mu(f)$ is computable.

See Hoyrup and Rojas (2009a); Miyabe (c) for the detail.

Example

Let μ be a probabilistic measure on the Cantor space.

μ is computable
$\Rightarrow \mu([\sigma])$ and $\mu(X \setminus [\sigma])$ are lower semicomputable.
$\Rightarrow \mu([\sigma])$ is uniformly computable.

The converse is immediate.

Base set & quasi-base set

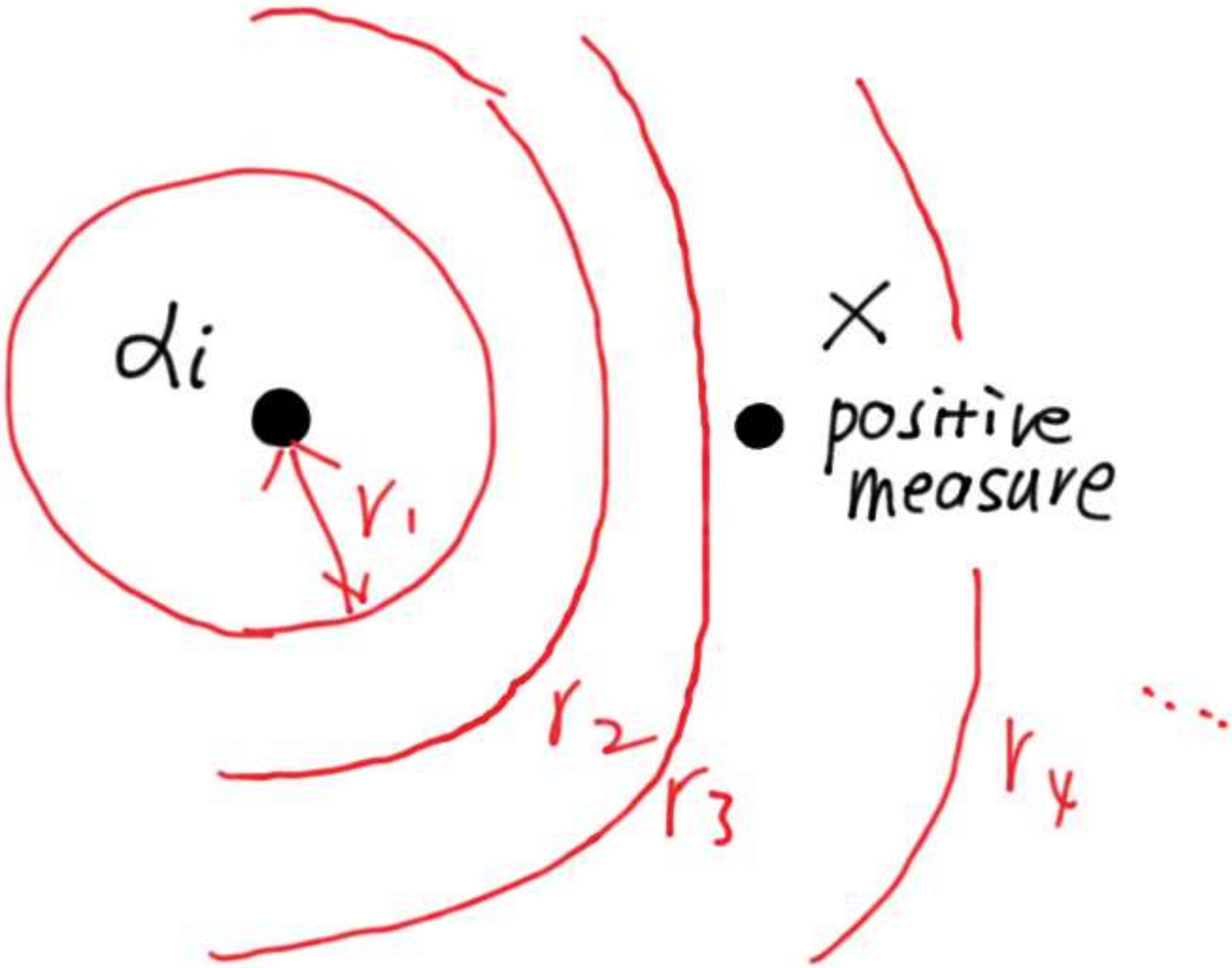
Theorem 2 (Bossehoff (2008); Hoyrup and Rojas (2009a)). *There exists a dense computable sequence $\{r_j\}$ such that*

$$\mu(\overline{B}(\alpha_i, r_j) \setminus B(\alpha_i, r_j)) = 0$$

for all i and j.

We call $B(\alpha_i, r_j)$ a base set and $\overline{B}^c(\alpha_i, r_j)$ a quasi-base set. Note that $\mu(U)$ is uniformly computable where U is a finite intersection of base sets and quasi-base sets.

Example



Cell

Definition 3 (Hoyrup and Rojas (2009a)).
For $\sigma \in 2^*$, the cell $\Gamma(\sigma)$ is defined by induction on $|\sigma|$:

$$\Gamma(\epsilon) = X,\ \Gamma(\sigma 0) = \Gamma(\sigma) \cap B_k, \Gamma(\sigma 1) = \Gamma(\sigma) \cap \overline{B}_k^c$$

where ϵ is the empty string and $k = |\sigma|$.

- The measure of a cell is uniformly computable.
- A Kurtz random point has a binary representation.

Martin-Löf randomness

An open set is c.e. if it is a computably union of basic open sets.

Definition 4.

A Martin-Löf test (or ML-test) is a sequence $\{U_n\}$ of uniformly c.e. open sets with $\mu(U_n) \leq 2^{-n}$.

A point $x \in X$ passes a ML-test $\{U_n\}$ if $x \notin \bigcap_n U_n$.

A point x is Martin-Löf random (or ML-random) if it passes each ML-test.

notion	cond.
weak 2-rd.	$\mu(U_n) \to 0$
ML-rd.	$\mu(U_n) \leq 2^{-n}$
Schnorr rd.	$\mu(U_n)$ is uniformly computable
Kurtz rd.	$\mu(U_n) = 1$

Definition 5 (Miyabe (a)).
Two functions $f, g :\subseteq X \to \mathbb{R}$ *are* *Kurtz equivalent* *if* $f(x) = g(x)$ *for each Kurtz random point* x.

L^1-computability

Classical definition

Definition 6 (Pour-El and Richards (1989)).
A function $f : X \to \mathbb{R}$ *is* *L^1-computable* *if there exists a sequence* $\{f_n\}$ *of uniformly computable functions such that*

$$||f - f_n||_1 \leq 2^{-n}$$

for all n.

- Their definition is on Banach spaces.
- No information is given for each point.

Definition 7 (Miyabe (b)).
A function $f :\subseteq X \to \mathbb{R}$ is L^1-computable with an effective code if there exists a computable sequence $\{s_n\}$ of finite rational step functions such that

$$f = \lim_n s_n \text{ and } ||s_{n+1} - s_n||_1 \leq 2^{-n}$$

for all n.

Let L_c be the set of L^1-computable functions with effective codes.
Pathak, Rojas, and Simpson studied a similar function on $\mathbb{R}^d$.

Results

Theorem 8 (Miyabe (b)).
A point x is Schnorr random iff $f(x)$ is defined for each $f \in L_c$.

Definition 9 (Miyabe (b)).
Two functions f, g are Schnorr equivalent if $f(x) = g(x)$ for each Schnorr random point.

Theorem 10 (Miyabe (b)).
Two functions $f, g \in L_c$ are Schnorr equivalent iff $||f - g||_1 = 0$.

Finite rational step function

Let $\mathcal{I}$ be the set of all finite intersections of base sets and quasi-base sets.

Definition 11.
A finite rational step function is a finite sum

$$s = \sum_{k=1}^{n} q_k \mathbf{1}_{E_k}$$

where $q_k \in \mathbb{Q}$ and $E_k \in \mathcal{I}$.

Integral test

Definition 12 (Miyabe (b)).
An integral test for Schnorr randomness is a nonnegative lower semicomputable function that has a computable integration.

Theorem 13 (Miyabe (b)).
A point is Schnorr random iff $f(x) < \infty$ for each integral test for Schnorr randomness.

One direction immediately follows form this theorem.

The other direction

We show that $f(x)$ is defined for each $f \in L_c$ and each Schnorr random point.
Let

$$f = \sum_n s_n \text{ where } ||s_n||_1 \leq 2^{-n}$$

and $\{s_n\}$ is a sequence of finite rational step functions.
Replace the sets in $\mathcal{I}$ with cells.
(Note that they are Kurtz equivalent.)
Split s_n to the positive part s_n^+ and the negative part s_n^-.
Then $f = (\sum_n s_n^+) - (\sum_n s_n^-) = g - h$ is a difference between two integral tests for Schnorr randomness.
Hence $g(x) < \infty$ and $h(x) < \infty$, which implies $f(x) \downarrow$.

Coincidence

L_c: the set of L^1-computable functions with effective codes
D_c: the set of the differences between two lower semicomputable functions with computable integrations

Theorem 14 (Miyabe (b)).
A $f \in L_c$ is is Kurtz equivalent to a function $g \in D_c$.
A $g \in D_c$ is Kurtz equivalent to a function $f \in L_c$.

Schnorr layerwise comp.

Hence L^1-computability is essentially equivalent to Schnorr layerwise computability.

Definition 15 (Miyabe (b)).
A function $f :\subseteq X \to \mathbb{R}$ is Schnorr layerwise computable if there exists a Schnorr test $\{U_n\}$ such that $f|_{X \setminus U_n} :\subseteq X \to \mathbb{R}$ is uniformly computable.

Theorem 16 (Miyabe (b)).
A function $f \in D_c$ is Schnorr layerwise computable.
A Schnorr layerwise computable function whose L^1-norm is computable is Schnorr equivalent to a function $f \in D_c$.

Weak L^1-computability

ML-randomness version

Definition 17.
An integral test is a nonegative lower semicomputable function $t : X \to \overline{\mathbb{R}}$ *such that* $\mu(t) < \infty$.

Theorem 18.
A point x *is ML-random iff* $f(x) < \infty$ *for each integral test.*

Let's study the set of the differences between two integral tests!!
But the set is larger than the set of L^1-computable functions (with effective codes).

Weak computability (1/2)

- r is computable iff $r = \lim_n q_n$ for a computable sequence $\{q_n\}$ of rationals such that $|r - q_n| \leq 2^{-n}$ for each n.
- r is left-c.e. iff $r = \lim_n q_n$ for a non-decreasing computable sequence $\{q_n\}$ of rationals.
- r is difference left-c.e. iff $r = \lim_n q_n$ for a computable sequence $\{q_n\}$ of rationals such that $\sum_n |q_{n+1} - q_n| < \infty$. (by Ambos-Spies et al. (2000))

They call such a real a weak computable real.

Weak computability (2/2)

notion	real	function
comp.	$\|r - q_n\| \leq 2^{-n}$	$\|\|f - s_n\|\|_1 \leq 2^{-n}$
weak comp.	$\sum_n \|q_{n+1} - q_n\| < \infty$	$\sum_n \|\|s_{n+1} - s_n\|\|_1 < \infty$

Weak L^1-computability (1/3)

Definition 19 (M.).
A function $f :\subseteq X \to \mathbb{R}$ is weakly L^1-computable if there exists a computable sequence $\{s_n\}$ of finite rational step functions such that

$$f = \lim_n s_n \text{ and } \sum_n ||s_{n+1} - s_n||_1 < \infty.$$

Weak L^1-computability (2/3)

L_w: the set of weakly L^1-computable functions
D_w: the set of the differences between two integral tests

Theorem 20 (M.).
A function $f \in L_w$ is Kurtz equivalent to a function $g \in D_w$.
A function $g \in D_w$ is Kurtz equivalent to a function $f \in L_w$.

Corollary 21.
A point x is ML-random iff
$f(x)$ is defined for each weakly L^1-computable function f.

Weak L^1-computability (3/3)

Definition 22 (M.).
Two functions f, g are ML-equivalent if $f(x) = g(x)$ for each ML-random point.

Corollary 23 (M.).
Let f, g be weakly L^1-computable functions.
Then f, g are ML-equivalent iff $||f - g||_1 = 0$.

Definition 24.
A real r is Δ^0_2 if $\{q \in \mathbb{Q} \mid q < r\}$ is Δ^0_2.

Proposition 25.
A real r is Δ^0_2 iff
$$r = \lim_n q_n \text{ for a computable sequence } \{q_n\} \text{ of rationals.}$$

notion	real	function
comp.	$\lvert r - q_n\rvert \le 2^{-n}$	$\lvert\lvert f - s_n\rvert\rvert_1 \le 2^{-n}$
weak comp.	$\sum_n \lvert q_{n+1} - q_n\rvert < \infty$	$\sum_n \lvert\lvert s_{n+1} - s_n\rvert\rvert < \infty$
limit comp.	defined	defined

Which randomness notion does it correspond?

Weak 2-randomness

Definition 26 (M.).
An integral test for weak 2-randomness is a nonnegative lower semicomputable function $t : X \to \overline{\mathbb{R}}$ *such that* $t(x) < \infty$ *almost everywhere.*

Theorem 27 (M.).
A point x *is weakly 2-random iff* $f(x) < \infty$ *for each integral test for weak 2-randomness.*

Limit L^1-computability (1/3)

Definition 28 (M.).
A function $f :\subseteq X \to \mathbb{R}$ is limit L^1-computable if f is defined a.e. and there exists a computable sequence $\{s_n\}$ of finite rational step functions such that

$$f = \lim_n s_n.$$

Note that a limit L^1-computable function may not be L^1.

Limit L^1-computability (2/3)

L_l: the set of limit L^1-computable functions
D_l: the set of the differences between two integral tests for weak 2-randomness

Theorem 29 (M.).
A function $f \in L_l$ is Kurtz equivalent to a function $g \in D_l$.
A function $g \in D_l$ is Kurtz equivalent to a function $f \in L_l$.

Corollary 30.
A point x is weakly 2-random iff
$f(x)$ is defined for each limit L^1-computable function f.

Limit L^1-computability (3/3)

Definition 31 (M.).
Two functions f, g are weakly 2-equivalent if $f(x) = g(x)$ for each weakly 2-random point.

Corollary 32 (M.).
Let f, g be limit L^1-computable functions.
Then f, g are weakly 2-equivalent iff $||f - g||_1 = 0$.

Summary so far

notion	integral test	L^1-space	layerwise comp.
weak 2-rd.	✓	✓	!
ML-rd.	✓	✓	!
comp. rd.	Rute?	?	?
Schnorr rd.	✓	✓	✓
Kurtz rd.	✓	?	?

Solovay reducibility

Observation (2/2)

Let's begin from the following observation.

Let f be a Schnorr layerwise computable function.
Then $f|_{X\setminus U_n} \subseteq X \to \mathbb{R}$ is uniformly computable.
Hence $f(x)$ is computable from x for each $x \in X$.

Let g be the function such that $g(x) = \Omega$.
Then g is an integral test.
Obviously $g(x)$ is not computable from x for each rational x.

We do not have a straightforward correspondence!!
We need more precise information.

Observation (1/2)

What kind of information?
Roughly speaking, we need how close a finite rational step function is to the target function for each point.
This reminds me of Solovay reducibility!!

Solovay reducibility

Let α, β be left-c.e. reals.

Definition 33 (Solovay). *$\alpha \leq_S \beta$ if $\exists c$ and $\exists f :\subseteq \mathbb{Q} \to \mathbb{Q}$ computable s.t.*

$$\beta > q \in \mathbb{Q} \Rightarrow \alpha - f(q) \downarrow < c(\beta - q).$$

Theorem 34 (Downey, Hirschfeldt, and Nies (2002)).

$$\alpha \leq_S \beta \iff \exists d \exists \gamma \text{ s.t. } d\beta = \alpha + \gamma$$

Reducibility for functions

Definition 35 (M.).
Let f and g be nonnegative lower semicomputable.
We define Solovy reducibility $\leq_S$ by

$$f \leq_S g \iff \exists d \exists h \text{ s.t. } d \cdot g =_{\mathrm{WR}} f + h$$

where h is lower semicomputable.

Basic properties

Proposition 36.
There exists an integral test such that f is integrable iff

$$f \leq_S t.$$

Proposition 37.
If $f \leq_S g$ and g has a computable integration, then f also has a computable integration.

Another formulation

Theorem 38 (M.).
f has a computable integration iff there exist an a.e. computable function g with $\mu(g) = 1$, and a Schnorr test $\{U_n\}$ such that

$$f \leq_S g + \sum_n 2^n \cdot \mathbf{1}_{U_n}$$

and $\sum_n 2^n \mu(U_n)$ is computable.

Proposition 39. *For any reals α and β,*

$$\alpha \leq_S \beta \Rightarrow \alpha \leq_T \beta$$

Theorem 40 (M.).
If $f \leq_S g$ and $g(x) < \infty$,
then $f(x)$ is computable from x and $g(x)$.

Corollary 41 (M.).
If f has a computable integration and x is Schnorr random,
then $f(x)$ is computable from x.
If f is integrable and x is ML-random,
then $f(x)$ is computable from x and $t(x)$.

Discussion

Summary

- Schnorr randomness, ML-randomness, weak 2-randomness
- characterization by integral tests
- L^1-computability, weak L^1-computability, limit L^1-computability
- Solovay reducibility for functions generalizes the layerwise computability.

Future works

- Study Solovay reducibility for functions more.
- Radon-Nikodym Theorem for other randomness notions.
- Effectivize many other classical theorems in analysis.

Thanks!

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