Schnorr Layerwise Computability

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Preliminary

The topic is layerwise computability.

- In which field is the notion used?
  - (mainly) Computable Analysis
- Which field is needed to define the notion?
  - Alagorithmic Randomness

It's puzzling, isn't it?

Cantor space

Computability $\Rightarrow$ Algorithmic Randomness

They interact with each other.

- Far from random iff close to computable.
- If already random, more random iff closer to computable.

It's called a two-way interaction by Nies.

Computability $\Longleftrightarrow$ Algorithmic Randomness

They are deeply connected.
Layerwise computability is the notion in computable analysis that is defined using the notion in algorithmic randomness!!

- Hoyrup, Rojas: An Application of Martin-Löf Randomness to Effective Probability Theory. In: CiE. pp. 260-269 (2009)
- Hoyrup, Rojas: An Application of Effective Probability Theory to Martin-Löf Randomness. In: ICALP (1). pp. 549-561 (2009)

In this talk

I'd like to present you

(i) how related computable analysis and algorithmic randomness are (in my point of view),

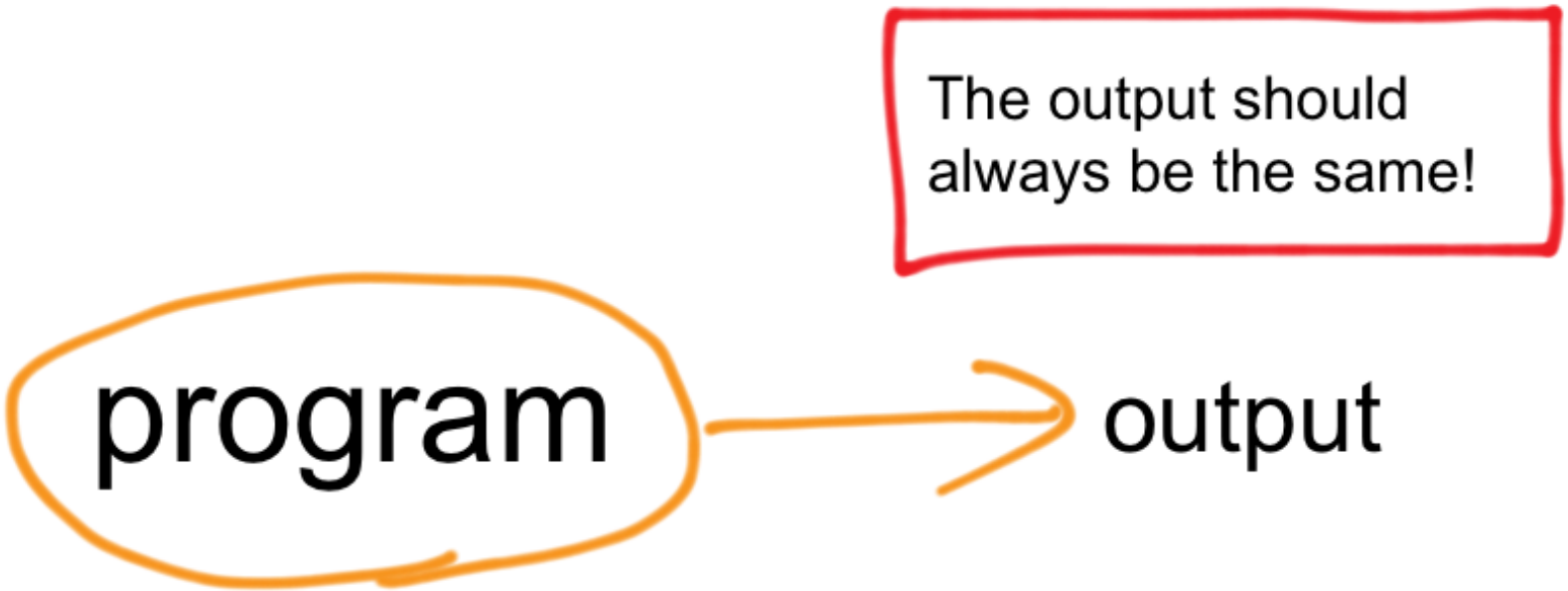
(ii) why (Schnorr) layerwise computability is a natural notion

Overview of this talk

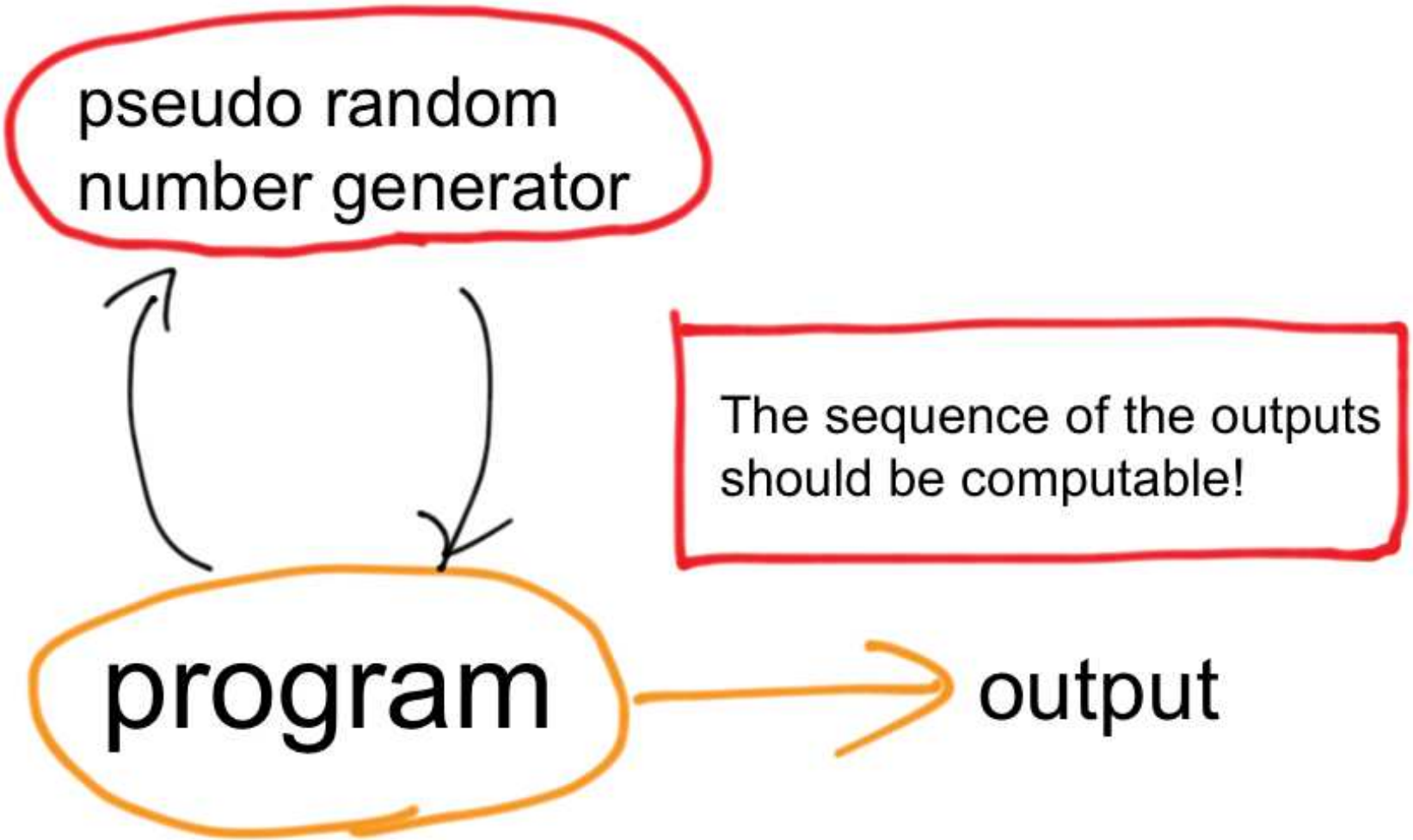
- algorithmic randomness and computable analysis
- layerwise computability
- Schnorr layerwise computability
- The reason layerwise computability may not be a natural notion

Layerwise computability

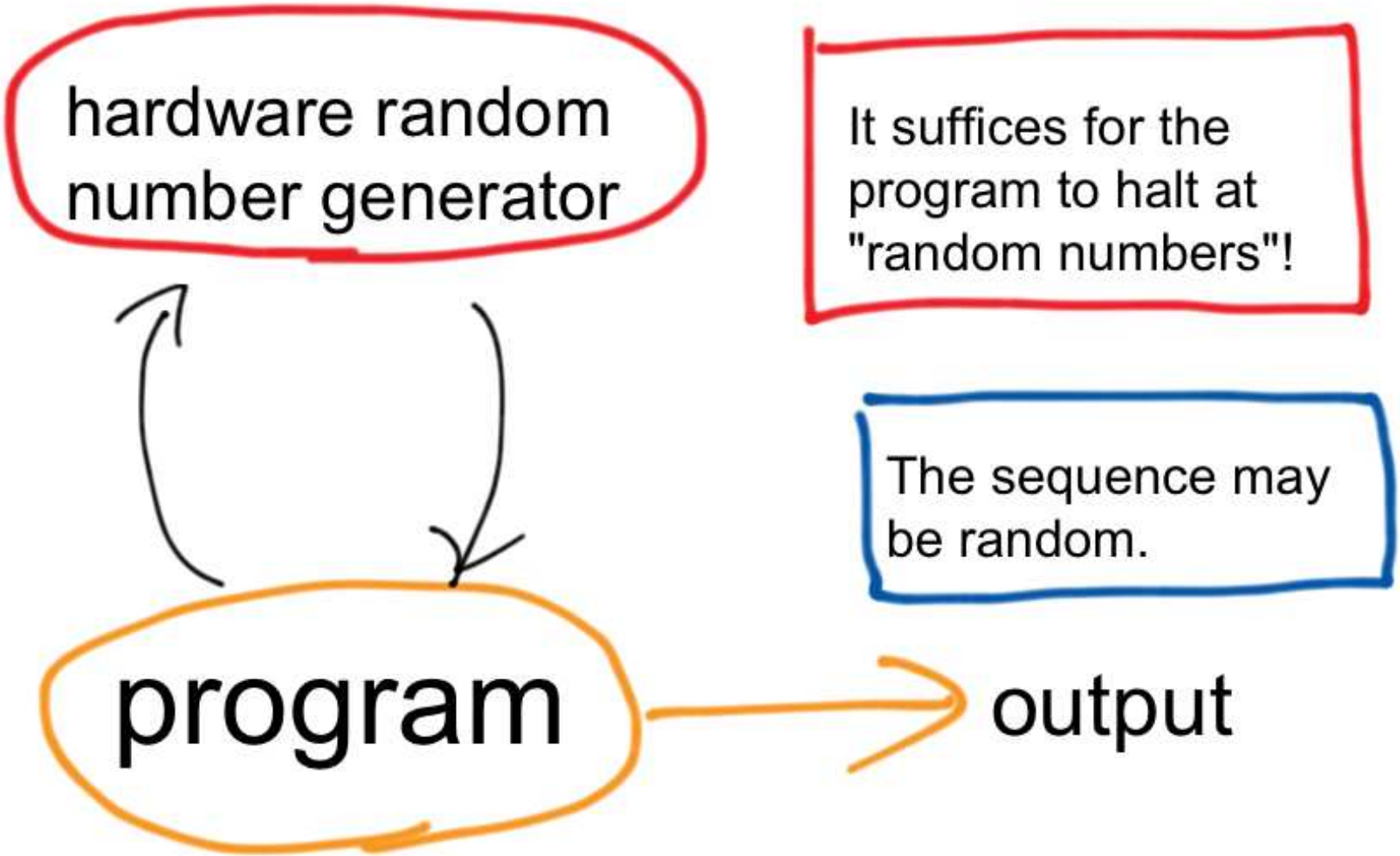
Example (1/3)



Example (2/3)


pseudo random
number generator
The sequence of the outputs
should be computable!
program
output

Example (3/3)


hardware random number generator
It suffices for the program to halt at "random numbers"!
The sequence may be random.
program
output

- $[0,1]$: unit interval
- A basic open set: (p,q), $[0,q)$, $(p,1]$ where $p,q \in \mathbb{Q} \cap [0,1]$
- A open set U is c.e. if a c.e. union of basic open sets.
- Let μ be the Lebesgue measure, that is, $\mu((p,q)) = q - p$.
- A real in the unit interval is identified with its binary expansion.

Martin-Löf randomness

Definition 1 (Martin-Löf (1966)).

- *A Martin-Löf test (or ML-test) is a sequence $\{U_n\}$ of uniformly c.e. open sets with $\mu(U_n) \leq 2^{-n}$.*
- *A real x passes a ML-test $\{U_n\}$ if $x \notin \bigcap_n U_n$.*
- *A real x is Martin-Löf random (or ML-random) if x passes all ML-tests.*

There exists an optimal ML-test $\{U_n\}$, that is, for each ML-test $\{V_n\}$ there exists d such that $V_{n+d} \subseteq U_n$.

Function defined a.e.

Note that $0 = 0^\omega$ is not ML-random.
Consider the function $f :\subseteq [0,1] \to \mathbb{N}$ such that

$$f(A) = \min\{n \mid A(n) = 1\}.$$

f is defined almost everywhere.
In particular f is defined for each ML-random point.
So it's natural to restrict the domain to random elements.
But it's just a partial computable function.
It's more natural to assume that f is not very large.

SLLN

Let $S(n) = S_A(n) = \sum_{k=1}^{n} A(k)$ for $A \in 2^\omega$.
In probability theory

Theorem 2 (Strong Law of Large Numbers)**.**

$$S(n) \to \frac{1}{2} \textit{ with probability } 1.$$

In algorithmic randomness

Theorem 3 (Folklore)**.**

$$S_A(n) \to \frac{1}{2} \textit{ for each ML-random real } A.$$

Convergence speed

Let $c(A)$ be the smallest c for which $A \notin U_c$.
Then there exists a computable function $n(c, \epsilon)$ such that

$$\left| \frac{S(n)}{n} - \frac{1}{2} \right| < \epsilon$$

for every $n > n(c, \epsilon)$.
For given $\lambda > 1$, there exists a computable function $n(c, \epsilon)$ such that

$$S(n) \leq \frac{n}{2} + \lambda \sqrt{\frac{n}{2} \ln \ln n}$$

for every $n > n(c, \epsilon)$. (Davie, 2001)

Interpretation

Roughly speaking

> the convergence speed is computable from its randomness deficiency.

The class of such functions seems important.
Let's define it.

Computable function

A function $f :\subseteq 2^\omega \to 2^\omega$ is computable if it is computable by a Turing machine with one-way output tape.
A real x is identified with a sequence which encodes the set

$$\{(p, q) \mid x \in (p, q),\ p, q \in \mathbb{Q}\}.$$

Such a sequence is called a representation of the real.
A function $f :\subseteq [0, 1] \to \mathbb{R}$ is computable if there exists a computable mapping from a representation of a real x to a representation of $f(x)$.
Computable functions are always continuous!

Lower semicomputability

A fuction $f : [0, 1] \to \mathbb{R}$ is computable iff
$f^{-1}(\text{basic open set})$ is uniformly c.e.
A function $f : [0, 1] \to \mathbb{R}^+$ is lower semicomputable iff
$f^{-1}(> q)$ is uniformly c.e.
Roughly speaking a lower semicomputable function is a function approximated from below.

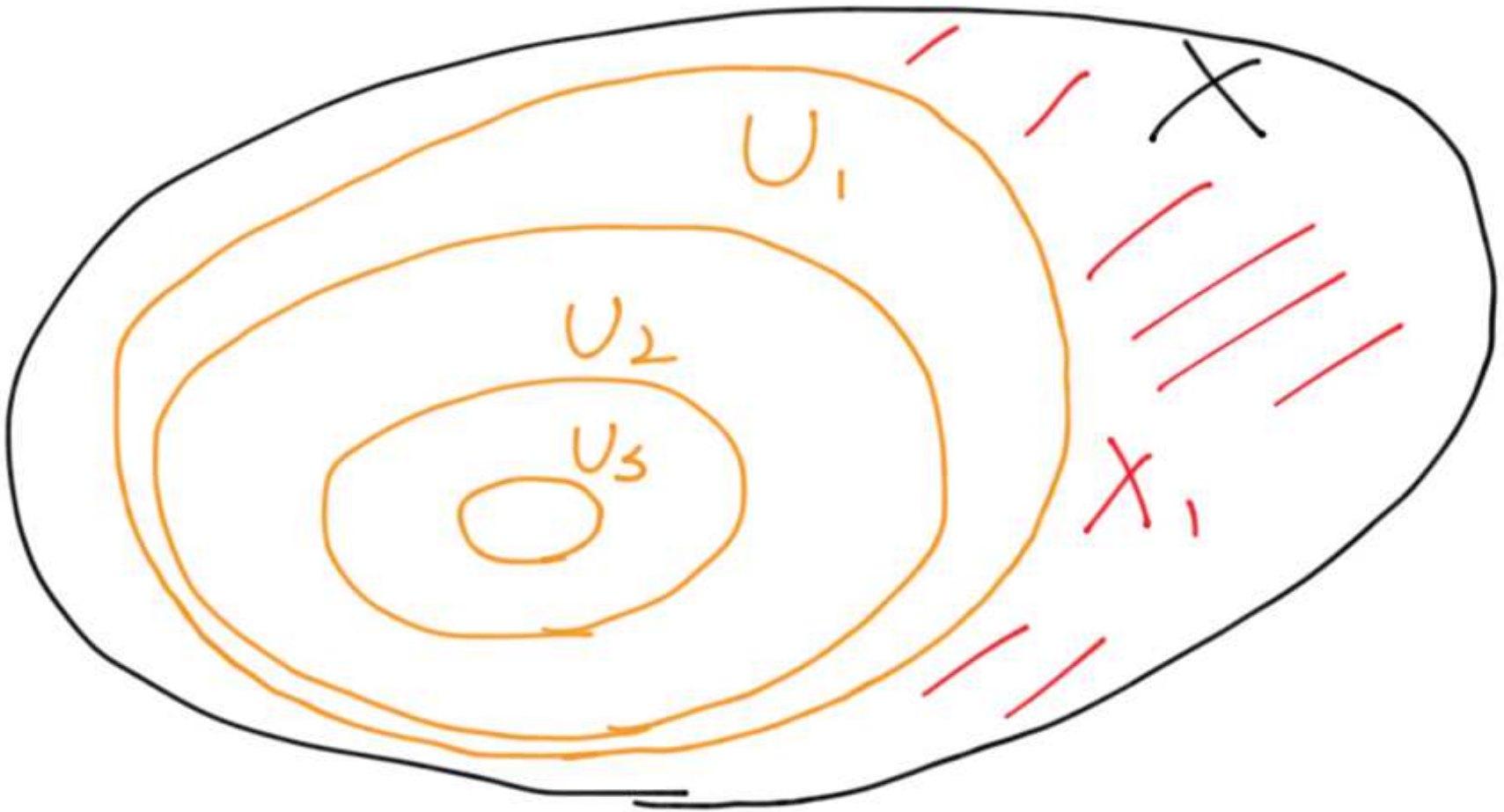
Layerwise computability

Let $\{U_n\}$ be an optimal ML-test.
Let $X_n = X \backslash U_n$ where $X = [0, 1]$.
These sets are called layers.
Note that $\lim_n X_n$ is the set of ML-random points.

Definition 4 (Hoyrup and Rojas (2009a,b))**.**
A function $f :\subseteq X \to \mathbb{R}^+$ is layerwise lower semicomputable if $f|_{X_n} :\subseteq X \to \mathbb{R}$ is uniformly lower semicomputable.
A function $f :\subseteq X \to \mathbb{R}$ is layerwise computable if $f|_{X_n} :\subseteq X \to \mathbb{R}$ is uniformly computable.

$A \mapsto \lim_n \frac{S_A(n)}{n}$ is layerwise computable.

Layer



Basic properties

Proposition 5 (Hoyrup and Rojas (2009a,b)).
If $f : X \to \mathbb{R}^+$ is bounded and layerwise computable, then $\int f d\mu$ is computable.
If $\int f d\mu$ is computable and f is layerwise lower semicomputable, then f is layerwise computable.

In general a layerwise computable function does not have a computable integration.

Which layerwise computable functions have computable integrations?

Schnorr layerwise computability

Integral test

A clue is found in a different research.

Definition 6 (Miyabe). *An integral test for Schnorr randomness is a nonnegative lower semicomputable function that has a computable integration.*

Theorem 7 (Miyabe). *A point x is Schnorr random iff $t(x) < \infty$ for each integral test t for Schnorr randomness.*

Note that an integral test for Schnorr randomness should be layerwise computable.
But it should have a relation with Schnorr randomness!!

Definition 8 (Miyabe).
A function $f :\subseteq X \to \mathbb{R}$ is Schnorr layerwise computable if there exists a Schnorr test $\{U_n\}$ such that $f|_{X\setminus U_n} :\subseteq X \to \mathbb{R}$ is uniformly computable.

Note that ML-randomness $\Rightarrow$ Schnorr randomness.
Schnorr layerwise computability $\Rightarrow$ layerwise computability
but the converse does not hold.

Properties

$A \mapsto \lim_n \frac{S_A(n)}{n}$ is Schnorr layerwise computable.

Theorem 9 (Miyabe)**.**
If a function is bounded and Schnorr layerwise computable, then its integration is computable.

Coincidence

Definition 10 (Miyabe)**.**
Two functions $f, g :\subseteq X \to \mathbb{R}$ are Schnorr equivalent if $f(x) = g(x)$ for each Schnorr random point.

Theorem 11 (Miyabe)**.**
A difference between two integral tests for Schnorr randomness is Schnorr layerwise computable.
A Schnorr layerwise computable function whose L^1-norm is computable is Schnorr equivalent to a difference between two integral tests for Schnorr randomness.

Proof 1

We will show that an integral test f for Schnorr randomness is Schnorr layerwise computable.
Let $f =_{\mathrm{WR}} \sum_n s_n$
where s_n is nonnegative finite rational step functions such that $||s_n||_1 \leq 2^{-2n}$.
Then $U_n = \{x \mid s_n(x) > 2^{-n}\}$ is a Schnorr test.
Note that $V_k = \bigcup\{U_n \mid n > k\}$ is also a Schnorr test.
Suppose $x \in X \backslash V_k$. Then $s_n(x) \leq 2^{-n}$ for each $n > k$.

$$f(x) - \sum_{m=1}^{n} s_m(x) = \sum_{m=n+1}^{\infty} s_n(x) \leq \sum_{m=n+1}^{\infty} 2^{-m} = 2^{-n}$$

for each $n > k$.

Proof 2

- Suppose that f is a Schnorr layerwise computable function whose L^1-norm is computable.
- For each point x, construct a approximation of $f(x)$ from below.
- If one finds that $x \in U_n$, then reset $f(x)$ by subtracting a sufficiently large natural number.
- Be sure that each lower semicomputable function has a computable integration.

Refinement

(i) The difference btw two integral tests for Sch-rd

(ii) The difference btw two layerwise lower semicomp. with comp. integrations

(iii) Schnorr layerwise computability with a comp. L^1-norm

(iv) Layerwise computability

(ii)$\Rightarrow$(iv) by Hoyrup & Rojas
The equivalence among (i)-(iii) by M.

Schnorr equivalence

Theorem 12 (Miyabe)**.**
Let f, g be Schnorr layerwise computable functions whose L^1-norms are computable.
Then f, g are Schnorr equivalent iff $||f - g||_1 = 0$.

Discussion

Randomness notions

The following hierarchy is known:

$$\text{Schnorr rd.} \supsetneq \text{comp. rd.} \supsetneq \text{ML-rd.} \supsetneq \text{weak 2-rd.}$$

Rd. notion	layerwise ver.
weak 2-rd.	limit layerwise comp.?
ML-rd.	continuous layerwise comp.?
?	layerwise comp.
comp. rd.	computably layerwise comp.?
Schnorr rd.	Schnorr layerwise comp.

Another coincidence

Definition 13 (Pathak, Rojas, and Simpson)**.** *For an L_1-computable function $f \in L_1([0,1]^d)$, define*

$$\hat{f}(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if } x \text{ is Schnorr random,} \\ 0 & \text{otherwise} \end{cases}$$

where f_n is a computable approximation of finite rational step functions.

Actually we can show that on a computable metric space the limit of a computable sequence of finite rational step functions is Schnorr equivalent to a Schnorr layerwise computable function and the converse with a condition.

Future works

- Many results are proved related to layerwise computability. Most of them will be reproved with Schnorr layerwise computability.
- Is it interesting to study Schnorr layerwise lower semicomputability?
- Does there exist a natural function that is layerwise computable but not Schnorr layerwise computable?

Thanks!

References

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