An integral test for Schnorr randomness and its applications

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Abstract. The author proposed in the previous paper that a characterization of a randomness notion by integral tests is a useful tool to study the relation between algorithmic randomness and computable analysis. In this paper we give a version of Schnorr randomness. With this result we show the connection between L^1-computability and Schnorr layerwise computability. Finally we apply them to study the points on which two Radon-Nikodym derivatives are equal.

1 Introduction

The theory of algorithmic randomness gives definitions of which points in a measure space are random. Roughly speaking, a point is random if it does not contain in any effective set with measure 0. The most famous randomness notion would be Martin-Löf randomness and there are some weaker or stronger randomness notions.

Some recent researches show the connection between algorithmic randomness and computable analysis. Computable analysis studies some effective versions of classical theorems in analysis. Then some classical theorems with "almost everywhere" can be converted to effective versions with "for each random point". Furthermore we sometimes have the converse. An example is that a real is Martin-Löf random iff each computable function of bounded variation is differentiable at the real [5, 4]. Recall that a classical theorem says that a function of bounded variation is differentiable almost everywhere.

The author proposed in [13] that an integral test is a useful tool to study the relation between algorithmic randomness and computable analysis. In this paper we give a version of Schnorr randomness.

In Section 3 we give a characterization of Schnorr randomness by integral tests. This is the main result of this paper and we will demonstrate the value of this result in later sections. In Section 4 we prove some basic results of step functions. These results are used in the next section.

In Section 5 we prove that the following three notions are essentially equivalent.

1. A function is the difference between two integral tests for Schnorr randomness.
2. A function is L^1-computable with an effective code.

3. A function is Schnorr layerwise computable and has a computable integration.

Then this is an important class of functions.

In Section 6 we give a simple application of the results. A classical result says that two Radon-Nikodym derivatives of a measure with respect to a measure are equal almost everywhere. We prove that, with an appropriate condition two Radon-Nikodym derivatives with effective codes are equal at Schnorr random points.

2 Preliminary

2.1 Computable analysis

We recall some results from computable analysis [17, 3, 18]. A *representation* is a surjective partial function $\delta :\subseteq \Sigma^\omega \to X$. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We use the canonical representation ρ for $\mathbb{R}$ and the lower representation $\overline{\rho}_<$ and the upper representation $\overline{\rho}_>$ for $\overline{\mathbb{R}}$ as is in [17]. Let δ_1, δ_2 be representations of points on X_1, X_2 respectively. We say a function $f :\subseteq X_1 \to X_2$ is (δ_1, δ_2)-*computable* if there is a computable mapping from a δ_1-representation of a point $x \in X_1$ to a δ_2-representation of $f(x) \in X_2$.

A *computable metric space* is a triple (X, d, α) such that (i) $d : X \times X \to \mathbb{R}$ is a metric on X, (ii) $\alpha : \mathbb{N} \to X$ is a sequence such that the set $\{\alpha_i \mid i \in \mathbb{N}\}$ is dense in X, (iii) $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable sequence in $\mathbb{R}$. A point α_i is called an *ideal point.* A *basic open ball* is the set $B(\alpha_i, q_j) = \{x \mid d(\alpha_i, x) < q_j\}$, and a *basic closed ball* is $\overline{B}(\alpha_i, q_j) = \{x \mid d(\alpha_i, x) \leq q_j\}$ where α_i is an ideal point and q_i is a nonnegative rational.

We say a sequence $\{x_i\}$ of points *converges rapidly* to x if $d(x, x_i) < 2^{-i}$ for all i. The *Cauchy representation* is the representation $\delta_X :\subseteq \Sigma^\omega \to X$ such that $\delta_X(p) = x$ when p encodes a sequence of ideal points which converges rapidly to x.

We say a function $f :\subseteq X \to \mathbb{R}$ is computable if it is (δ_X, ρ)-computable. A function $f :\subseteq X \to \overline{\mathbb{R}}$ is lower semicomputable if it is $(\delta_X, \overline{\rho}_<)$-computable, and upper semicomputable if $(\delta_X, \overline{\rho}_>)$-computable.

2.2 Computable measure

We recall some results on computability of a measure on a computable metric space. For more detail, see [1, 11]. A probabilistic measure on a computable metric space is *computable* if the measures of a finite union of basic open balls are uniformly lower semicomputable. Then the integral of a nonnegative lower semicomputable function over a c.e. open set with respect to a computable measure is lower semicomputable uniformly.

In the following we fix a computable metric space (X, d, α) and a computable probabilistic measure μ on it.

Proposition 1. *There exists a dense computable sequence* $\{r_j\}$ *such that*

$$\mu(\overline{B}(\alpha_i, r_j) \backslash B(\alpha_i, r_j)) = 0$$

for all i *and* j.

In the following we call $B(\alpha_i, r_j)$ a *base set* and $\overline{B}^c(\alpha_i, r_j)$ a *quasi-base set* for each i and j. Let $\mathcal{I}$ be the set of all finite intersections of base sets and quasi-base sets. Note that $\mu(U)$ is computable uniformly in $U \in \mathcal{I}$. Let $B_{\langle i,j\rangle} = B(\alpha_i, r_j)$ and $\overline{B}^c_{\langle i,j\rangle} = \overline{B}^c(\alpha_i, r_j)$. As is in [11], for $\sigma \in 2^*$, the cell $\Gamma(\sigma)$ is defined by induction on $|\sigma|$:

$$\Gamma(\epsilon) = X,\ \Gamma(\sigma 0) = \Gamma(\sigma) \cap B_k,\ \Gamma(\sigma 1) = \Gamma(\sigma) \cap \overline{B}^c_k$$

where ϵ is the empty string and $k = |\sigma|$.

2.3 Algorithmic randomness

We refer the reader to two recent books [6, 14] for a survey on algorithmic randomness on Cantor space. In this paper we consider randomness notion on the computable metric space with the computable probabilistic measure.

A *Martin-Löf test* is a sequence $\{U_n\}$ of uniformly c.e. open sets with $\mu(U_n) \leq 2^{-n}$. A point $x \in X$ is *Martin-Löf random* if $x \notin \bigcap_n U_n$ for each Martin-Löf test. A *Schnorr test* is a Martin-Löf test such that $\mu(U_n)$ is uniformly computable. A point $x \in X$ is *Schnorr random* if $x \notin \bigcap_n U_n$ for each Schnorr test. A point $x \in X$ is *Kurtz random* if it contains in each c.e. open set with measure 1. Two functions $f, g :\subseteq X \to \mathbb{R}$ are *Kurtz equivalent* (denoted by $f =_{\mathrm{WR}} g$) if $f(x) = g(x)$ on each Kurtz random point x.

Let $\mathbf{1}_U$ be the characteristic function of $U \subseteq X$, that is, $\mathbf{1}_U(x) = 1$ if $x \in U$ and $\mathbf{1}_U(x) = 0$ if $x \notin U$. Two subsets U, V are Kurtz equivalent if $\mathbf{1}_U$ and $\mathbf{1}_V$ are Kurtz equivalent.

A base set or a quasi-base set is Kurtz equivalent to a union of cells. A set $U \in \mathcal{I}$ is Kurtz equivalent to a union of cells. If $U = B_{i_1} \cap \cdots \cap B_{i_k} \cap \overline{B}^c_{j_1} \cap \cdots \cap \overline{B}^c_{j_l}$ for some $i_1, \ldots, i_k, j_1, \ldots, j_l$, then U is Kurtz equivalent to

$$\bigcup \{\Gamma(\sigma) \mid \sigma(i_1) = 0, \ldots, \sigma(i_k) = 0, \sigma(j_1) = 1, \ldots, \sigma(j_l) = 1, |\sigma| = m\}$$

for $m > \max\{i_1, \ldots, i_k, j_1, \ldots, j_l\}$. It is easy to see that a union of sets in $\mathcal{I}$ is Kurtz equivalent to a union of cells uniformly.

3 An integral test for Schnorr randomness

It is well known that Martin-Löf randomness has a characterization by integral tests [12, 11]. Let $\overline{\mathbb{R}}^+$ be the set of nonnegative elements in $\overline{\mathbb{R}}$. An integral test is a nonnegative lower semicomputable function $t : X \to \overline{\mathbb{R}}^+$ such that $\mu(t) = \int t d\mu < \infty$. A point x is Martin-Löf random iff $t(x) < \infty$ for each integral test

t. The author [13] showed that a point x is Kurtz random iff $t(x) < \infty$ for each nonnegative computable function $f : X \to \overline{\mathbb{R}}$ such that $\mu(f)$ is computable. In this section we give a version of Schnorr randomness.

Definition 1. *An integral test for Schnorr randomness is a nonnegative lower semicomputable function $t : X \to \overline{\mathbb{R}}^+$ such that $\mu(t) = \int t d\mu$ is computable.*

Theorem 1. *A point z is Schnorr random iff $t(z) < \infty$ for each integral test t for Schnorr randomness.*

Lemma 1. *Let x_n be uniformly computable positive reals. If there exists a uniformly computable sequence $\{y_n\}$ such that $x_n \leq y_n$ for all n and $\sum_n y_n$ is computable, then $\sum_n x_n$ is also computable.*

The proof is not difficult. The author used this lemma in [13] too with a proof.

Proof (of "if" direction of Theorem 1). Suppose z is not Schnorr random. Then there exists a Schnorr test $\{U_n\}$ such that $z \in \bigcap_n U_n$. Let $t(x) = \#\{n \in \mathbb{N} : x \in U_n\}$ where $\#$ denotes the size of the set. Note that $t(z) = \infty$. Since U_n is uniformly c.e., t is lower semicomputable. Note that $\mu(t) = \sum_{n=0}^{\infty} \mu(U_n)$. Since $\mu(U_n) \leq 2^{-n}$ and $\sum_n 2^{-n}$ is computable, $\mu(t)$ is computable by Lemma 1. Hence t is an integral test for Schnorr randomness.

The "only if direction" is a little more difficult than the other. Intuitively, since the area $\mu(f)$ is computable, each area cut horizontally at two rationals p, q ($p < q$) is also computable. Then $\mu(\{x : t(x) > q\})$ and $\mu(\{x : t(x) < q\})$ are approximated well unless $\mu(\{x : t(x) = q\}) > 0$. Then we need a computable sequence $\{q_n\}$ such that $\mu(\{x : t(x) = q_n\}) = 0$. A similar sequence is also used in [11] to construct a base of uniformly almost decidable balls. Before giving a proof, we cite computable Baire Category theorem (proved in [19, 2]) which they use in [11].

Definition 2. *A constructive G_δ-set is a set of the form $\bigcap_n U_n$ where $\{U_n\}$ is a sequence of uniformly θ-computable open sets.*

Theorem 2 (Computable Baire theorem). *On a computable metric space, every dense constructive G_δ-set contains a dense sequence of uniformly computable points.*

We prove "only if direction" of Theorem 1 by giving three lemmas.

Lemma 2. *Let $h_r(x) = \min\{r, t(x)\}$ where t is an integral test for Schnorr randomness and r is a computable real. Then $\int h_r d\mu$ is computable from r.*

Proof. We assume $r > 0$. Let $g_r(x) = \max\{r, t(x)\}$. Since h_r and g_r are lower semicomputable, so are $\int h_r d\mu$ and $\int g_r d\mu$. Since $h_r(x) + g_r(x) = t(x) + r$, we obtain $\mu(h_r) + \mu(g_r) = \mu(t) + r$. Since the right-hand side is computable, the left-hand side is also computable. It follows that $\mu(h_r)$ and $\mu(g_r)$ are computable.

Then $\mu(\{x : t(x) \geq r\})$ can be approximated from above.

Lemma 3. *Let t be an integral test for Schnorr randomness and s be a computable real. Then $\mu(\{x : t(x) \geq s\})$ is upper semicomputable uniformly in s.*

Proof. For $0 < r < s$ let $I_r^s = \int(h_s - h_r)d\mu$. Then I_r^s is computable uniformly in r and s. Note that

$$h_r(x) - h_s(x) = \begin{cases} r - s & \text{if } t(x) \geq s \\ t(x) - s & \text{if } r \leq t(x) < s \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\mu(\{x : f(x) \geq s\}) \leq I_r^s/(r-s) \leq \mu(\{x : f(x) \geq r\})$. Since $\lim_{\epsilon \to +0} \mu(\{x : f(x) \geq s - \epsilon\}) = \mu(\{x : f(x) \geq s\})$, we have $\lim_{\epsilon \to +0} I_{s-\epsilon}^s/\epsilon = \mu(\{x : f(x) \geq s\})$. Hence $\mu(\{x : f(x) \geq s\})$ has a computable approximation from above.

Lemma 4. *There exists a sequence $\{r_n\}$ of uniformly computable reals such that $\mu(\{x : f(x) = r_n\}) = 0$ for all n.*

Proof. Define $U_k = \{r \in \mathbb{R}^+ : \mu(\{x : f(x) \geq r\}) < \mu(\{x : f(x) > r\}) + 1/k\}$. By computability of μ, U_k is a c.e. open set uniformly in k. Since μ is finite, the set of r for which $\mu(\{x : f(x) = r\}) \geq 1/k$ is finite. Hence U_k is dense. Note that $\mathbb{R}^+$ equipped with a structure is a computable metric space. Then by the computable Baire Theorem, dense constructive G_δ-set $\bigcap_k U_k$ contains a sequence r_n of uniformly computable reals which is dense in $\mathbb{R}^+$. By construction $\mu(\{x : f(x) = r_n\}) = 0$ for all n.

Proof (of "only if" direction of Theorem 1). Let t be an integral test for Schnorr randomness. Let $\{r_n\}$ be a sequence of uniformly computable reals such that $\mu(\{x : t(x) = r_n\}) = 0$ for all n. Then $\mu(\{x : f(x) \geq r_n\}) = \mu(\{x : f(x) > r_n\})$. It follows that $\mu(\{x : f(x) > r_n\})$ is computable uniformly in n.

Pick up an increasing computable subsequence $\{s_n\} \subseteq \{r_n\}$ such that $s_n \geq 2^n \mu(t)$. Let $V_n = \{x : t(x) > s_n\}$. Then $\{V_n\}$ is uniformly c.e. open and the measure $\mu(V_n)$ is computable. Since $s_n \mu(V_n) \leq \mu(t)$, $\mu(V_n) \leq 2^{-n}$. Hence $\{V_n\}$ is a Schnorr test. If $t(y) = \infty$ then y is not Schnorr random.

4 Lower semicomputability

Here we prove some basic results of step functions.

4.1 A finite rational step function

The notion of a finite rational step function has been used in some literatures such as [7, 15]. The following definition is Kurtz equivalent to theirs.

Definition 3. *A finite rational step function is a finite sum $s = \sum_{k=1}^{n} q_k \mathbf{1}_{E_k}$ where $q_k \in \mathbb{Q}$ and $E_k \in \mathcal{I}$.*

Note that there exists a canonical numbering of the collection of rational step functions. The following is immediate from a result in [13].

Proposition 2. *For a nonnegative lower semicomputable function* $f : X \to \overline{\mathbb{R}}^+$*, there exists a sequence* $\{s_n\}$ *of finite rational step functions such that* $\lim_n s_n$ *is Kurtz equivalent to* f*.*

We call the sequence $\{s_n\}$ an approximation of f by finite rational step functions.

4.2 A rational cell function

Definition 4. *A* finite rational cell function *is a finite sum*

$$s = \sum_{k=1}^{n} q_k \mathbf{1}_{\Gamma(\sigma_k)}$$

where $q_k \in \mathbb{Q}$ *and* $\sigma_k \in 2^m$ *for all* k *such that* $1 \leq k \leq n$ *and for some fixed* $m \in \mathbb{N}$*.*

Proposition 3. *A finite rational cell function is a finite rational step function. A finite rational step function is Kurtz equivalent to a finite rational cell function.*

Proof. The former half is immediate. Let $s = \sum_{k=1}^{n} q_k \mathbf{1}_{E_k}$ be a finite rational step function where $q_k \in \mathbb{Q}$ and $E_k \in \mathcal{I}$. Replace E_k with a Kurtz equivalent union of cells with a fixed sufficiently large m to have a finite rational cell function.

5 A function defined on Schnorr random points

The class of the differences between integral tests was an important class in [13]. Here we study the class for the integral tests for Schnorr randomness.

5.1 The difference between two integral tests

Let S be the set of integral tests for Schnorr randomness. Let $D = \{f - g : f, g \in S\}$ where $\mathrm{dom}(f - g) = \{x : f(x) < \infty,\ g(x) < \infty\}$.

Theorem 3. *A point* x *is Schnorr random iff* $f(x)$ *is defined for each* $f \in D$*.*

Proof. Suppose that x is not Schnorr random. Then there exists an integral test t for Schnorr randomness such that $t(x) = \infty$. Let $f = t - 0$ where 0 is the constant function. Then $f \in D$ and $f(x)$ is not defined.

Suppose that x is Schnorr random. For each $h \in D$, there exist $f, g \in S$ such that $h = f - g$. Since $f(x) < \infty$ and $g(x) < \infty$, $h(x)$ is defined.

Definition 5. *Two functions* $f, g :\subseteq X \to \mathbb{R}$ *are* Schnorr equivalent *if* $f(x) = g(x)$ *for each Schnorr random point* $x \in X$.

Let $f :\subseteq X \to \mathbb{R}$ be the function defined almost everywhere. The L^1-norm of a function f is $||f||_1 = \int_X |f| d\mu$.

Theorem 4. *Two functions* $f, g \in D$ *are Schnorr equivalent iff* $||f - g||_1 = 0$.

Proof. If f and g are Schnorr equivalent, then $\mu(\{x \mid f(x) \not\simeq g(x)\}) = 0$. It follows that $||f - g||_1 = 0$.

Suppose that $||f - g||_1 = 0$. Since $f - g \in D$, $f(x)$ and $g(x)$ are defined. Let $f - g = t - u$ where $t, u \in S$. Let $\{t_n\}$ and $\{u_n\}$ be their approximations by finite rational step functions by Proposition 2. Since $\mu(t)$ and $\mu(u)$ are computable, we can assume that $||t - t_n||_1 \leq 2^{-n-1}$ and $||u - u_n||_1 \leq 2^{-n-1}$. Then $||t_n - u_n||_1 \leq ||t_n - t||_1 + ||t - u||_1 + ||u - u_n||_1 \leq 2^{-n}$.

Let $h_n = \sum_{k=1}^{n} |t_k - u_k|$. Then $h = \lim_n h_n$ is lower semicomputable. Since $||t_k - u_k||_1 \leq 2^{-k}$, $h \in S$ by Lemma 1. Then $h(x) < \infty$ for each Schnorr random point x. It follows that $|t_n(x) - u_n(x)| \to 0$ as $n \to \infty$. Hence $(f - g)(x) = t(x) - u(x) = \lim_n (t_n(x) - u_n(x)) = 0$.

5.2 L^1-computability

The notion of L^1-computability is defined by Pour-El and Richard [16] and is used in many literatures. Here we show that L^1-computability is essentially the same as the difference between two integral tests for Schnorr randomness.

Definition 6. *A function* f *is* L^1*-computable with an effective code if there exists a computable sequence* $\{s_n\}$ *of finite rational step functions such that* $f = \lim_n s_n$ *and* $||s_{n+1} - s_n||_1 \leq 2^{-n}$ *for all* n.

Let L be the set of L^1-computable functions with effective codes.

Theorem 5. *A function* $f \in D$ *is Kurtz equivalent to a function* $g \in L$.

Proof. Let $f = t - u$ where $t, u \in S$. Let $\{t_n\}$ and $\{u_n\}$ be their approximations by finite rational step functions such that $||t - t_n|| \leq 2^{-n-2}$ and $||u - u_n|| \leq 2^{-n-2}$. Let $s_n = t_n - u_n$. Then $\lim_n s_n = \lim_n (t_n - u_n) =_{\mathrm{WR}} t - u = f$ and

$$\begin{aligned} ||s_{n+1} - s_n||_1 =& ||t_{n+1} - t_n - u_{n+1} + u_n||_1 \\ \leq& ||t - t_{n+1}||_1 + ||t - t_n||_1 + ||u - u_{n+1}||_1 + ||u - u_n||_1 < 2^{-n}. \end{aligned}$$

Theorem 6. *A function* $g \in L$ *is Kurtz equivalent to a function* $f \in D$.

Proof. For $g \in L$, let $\{s_n\}$ be a computable sequence of finite rational step functions such that $g = \lim_n s_n$ and $||s_{n+1} - s_n||_1 \leq 2^{-n}$ for all n. Since $s_{n+1} - s_n$ is a finite rational step function, it is Kurtz equivalent to a finite cell function $c_n = \sum_k q_k \mathbf{1}_{\Gamma(\sigma_k)}$ by Proposition 3. Let

$$c_n^+ = \sum_{q_k \geq 0} q_k \mathbf{1}_{\Gamma(\sigma_k)} \text{ and } c_n^- = -\sum_{q_k < 0} q_k \mathbf{1}_{\Gamma(\sigma_k)}.$$

Then $||c_n||_1 = ||s_{n+1} - s_n||_1 \leq 2^{-n}$ and

$$||c_n||_1 = \sum_k |q_k|\mu(\Gamma(\sigma_k)) = ||c_n^+||_1 + ||c_n^-||_1.$$

It follows that $||c_n^+||_1 \leq 2^{-n}$ and $||c_n^-||_1 \leq 2^{-n}$. Hence $\sum_n c_n^+$ and $\sum_n c_n^-$ are integral tests for Schnorr randomness. Here g is Kurtz equivalent to $\sum_n c_n^+ - \sum_n c_n^-$.

Theorem 7. *Two functions $f, g \in L$ are Schnorr equivalent iff $||f - g||_1 = 0$.*

Proof. The "only if" direction is immediate. Suppose that $||f - g||_1 = 0$. Since $f - g \in L$, there exists a function $h \in D$ which is Kurtz equivalent to $f - g$. By Theorem 4, $h(x) = 0$ for each Schnorr random point x. It follows that $f(x) - g(x) = h(x) = 0$.

5.3 Schnorr layerwise computability

A layerwise computability [9, 10] has some desirable properties to study effective probability theory. In [9] Hoyrup and Rojas showed that, if the integral of a function is computable then layerwise lower semicomputability implies layerwise computability. In the following we show that lower semicomputability implies a stronger notion, that is, Schnorr layerwise computability. The converse also holds in the sense of Theorem 8.

Definition 7. *A function $f :\subseteq X \to \mathbb{R}$ is* Schnorr layerwise computable *if there exists a Schnorr test U_n such that the restriction $f|_{X \setminus U_n}$ is uniformly computable.*

Theorem 8. *A function is Schnorr equivalent to a Schnorr layerwise computable function whose L^1-norm is computable iff the function is Schnorr equivalent to a function in D.*

Proof. (if direction) It suffices to show that a function $f \in S$ is Schnorr layerwise computable. Let $\{s_n\}$ be a computable sequence of nonnegative finite rational step functions such that $||s_n||_1 \leq 2^{-2n}$ and $f =_{\mathrm{WR}} \sum_n s_n$.

Let $U_n = \{x \ : \ s_n(x) > 2^{-n}\}$. Then U_n is uniformly c.e. Since $2^{-n}\mu(U_n) \leq ||s_n||_1 \leq 2^{-2n}$, we have $\mu(U_n) \leq 2^{-n}$. Note that the real $\mu(U_n)$ is uniformly computable.

Let $V_k = \bigcup\{U_n \ : \ n > k\}$. Then $\mu(V_k) \leq \sum_{n>k} U_n \leq \sum_{n>k} 2^{-n} = 2^{-k}$. The real $\mu(V_k)$ is uniformly computable by Lemma 1. Hence $\{V_k\}$ is a Schnorr test.

Suppose $x \in X \setminus V_k$. Then $x \notin V_k$. It follows that $s_n(x) \leq 2^{-n}$ for each $n > k$. Hence

$$f(x) - \sum_{m=1}^{n} s_m(x) = \sum_{m=n+1}^{\infty} s_m(x) \leq \sum_{m=n+1}^{\infty} 2^{-m} = 2^{-n}$$

for each $n > k$. Hence $f(x)$ is computable from x and k.
(only if direction) Let f be a Schnorr layerwise computable function whose L^1-norm is computable. Then there exists a Schnorr test $\{U_n\}$ such that $f_n =$

$f|_{X\setminus U_n}$ is uniformly computable. Let f'_n be total uniformly lower semicomputable functions such that $f'_n|_{X\setminus U_n} = f_n$. Let $f''_n = \min\{f'_n, n\}$.

Let $t_n(x) = \sum\{k \mid x \in U_k,\ k \le n\}$. Then $\int t_n d\mu = \sum_{k=1}^n k \cdot \mu(U_k)$. Let $t = \sup_n t_n$. Since $\mu(U_k)$ is computable and $\le 2^{-n}$, $\int t d\mu = \sum_n n \cdot \mu(U_n)$ is computable by Lemma 1.

Let

$$g_n(x) = \begin{cases} t_n(x) & \text{if } x \in U_n \\ t_{n-1}(x) + f''_n(x) & \text{otherwise.} \end{cases}$$

Note that g_n is lower semicomputable. Let $g(x) = \sup_n g_n(x)$. Then g is lower semicomputable and

$$g(x) = \begin{cases} \infty & \text{if } x \in \bigcap_n U_n \\ t(x) + f(x) & \text{if } x \notin \bigcap_n U_n. \end{cases}$$

Note that $\mu(g) = \mu(t) + \mu(f)$ is computable. If x is Schnorr random, then $f(x) = g(x) - d(x)$.

Theorem 9. *Let f, g be Schnorr layerwise computable functions whose L^1-norms are computable. Then f, g are Schnorr equivalent iff $||f - g||_1 = 0$.*

The proof is exactly the same as that of Theorem 7.

6 An application

Definition 8 ([8]). *A finite measure μ is* computably normable relative to *some other finite measure λ, if the norm of the operator L_μ is computable from μ and λ.*

Theorem 10 ([8]). *Let μ, λ be such that $\mu \ll \lambda$ and μ is computably normable relative to λ. Then the Radon-Nikodym derivative $\frac{d\mu}{d\lambda}$ can be computed as an element of $L^1(\lambda)$ from μ and λ.*

In the proof they constructed a finite rational step function v_n such that $||h - v_n||_\lambda < 2^{-n}$ where h is a Radon-Nikodym derivative. By letting $h' = \lim_n v_n$, h' is a Radon-Nikodym derivative and L^1-computable with an effective code. Hence one can compute a Radon-Nikodym derivative which is L^1-computable with an effective code. Furthermore if another L^1-computable function h'' with an effective code is a Radon-Nikodym derivative, then h' and h'' are Schnorr equivalent. This result is an effective version of the classical theorem which says that two Radon-Nikodym derivatives are equal almost everywhere.

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References

1. Bossehoff, V.: Notions of probabilistic computability on represented spaces. Journal of Universal Computer Science 14(6), 956–995 (2008)
2. Brattka, V.: Computable versions of baires category theorem. Mathematical Foundations of Computer Science 2001 pp. 224–235 (2001)
3. Brattka, V., Hertling, P., Weihrauch, K.: A tutorial on computable analysis. New Computational Paradigms pp. 425–491 (2008)
4. Brattka, V., Miller, J.S., Nies, A.: Randomness and differentiability, submitted
5. Demuth, O.: The differentiability of constructive functions of weakly bounded variation on pseudo numbers. Comment. Math. Univ. Carolin. 16(3), 583–599 (1975)
6. Downey, R., Hirschfeldt, D.R.: Algorithmic Randomness and Complexity. Springer, Berlin (2010)
7. Hoyrup, M., Rojas, C., Weihrauch, K.: Computability of the radon-nikodym derivative. In: Löwe, B., Normann, D., Soskov, I., Soskova, A. (eds.) Models of Computation in Context. Lecture Notes in Computer Science, vol. 6735, pp. 132–141. Springer (2011)
8. Hoyrup, M., Rojas, C., Weihrauch, K.: Computability of the radon-nikodym derivative. Models of Computation in Context pp. 132–141 (2011)
9. Hoyrup, M., Rojas, C.: An Application of Martin-Löf Randomness to Effective Probability Theory. In: CiE. pp. 260–269 (2009)
10. Hoyrup, M., Rojas, C.: Applications of Effective Probability Theory to Martin-Löf Randomness. In: ICALP (1). pp. 549–561 (2009)
11. Hoyrup, M., Rojas, C.: Computability of probability measures and Martin-Löf randomness over metric spaces. Information and Computation 207(7), 830–847 (2009)
12. Li, M., Vitányi, P.: An introduction to Kolmogorov complexity and its applications. Graduate Texts in Computer Science, Springer-Verlag, New York, third edition edn. (2009)
13. Miyabe, K.: Characterization of kurtz randomness by a differentiation theorem, submitted
14. Nies, A.: Computability and Randomness. Oxford University Press, USA (2009)
15. Pathak, N., Rojas, C., Simpson, S.G.: Schnorr randomness and the lebesgue differentiation theorem, to appear
16. Pour-El, M., Richards, J., El, M.: Computability in analysis and physics. Springer (1989)
17. Weihrauch, K.: Computable Analysis: an introduction. Springer, Berlin (2000)
18. Weihrauch, K., Grubba, T.: Elementary Computable Topology. Journal of Universal Computer Science 15(6), 1381–1422 (2009)
19. Yasugi, M., Mori, T., Tsujii, Y.: Effective properties of sets and functions in metric spaces with computability structure. Theoretical Computer Science 219(1–2), 467–486 (1999)