# Weak $L^1$-computability and Limit $L^1$-computability

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Brattka, Miller and Nies [1] showed that some randomness notions are characterized by differentiability of some classes of functions. They also proposed to study which class corresponds to which randomness notion. Pathak, Rojas and Simpson [3] and independently Rute [4] showed that a real in the unit interval is Schnorr random if and only if the Lebesgue differentiation theorem for the point holds for all effective version of $L^1$-computable functions. Then its other randomness versions are of our interest. The author [2] gave several characterizations of the class of the effective version of $L^1$-computable functions. Then we also study its other randomness versions.

Let $(X, d, \alpha)$ be a computable metric space and $\mu$ be a computable measure on it. The following definition and result are by [2]. A *integral test for Schnorr randomness* is a nonnegative lower semicomputable function $f :\subseteq X \to \overline{\mathbb{R}}$ whose integral is computable. A function $f$ is *$L^1$-computable with an effective code* if there exists a computable sequence $\{s_n\}$ of finite rational step functions such that $f = \lim_n s_n$ and $||s_{n+1} - s_n||_1 \leq 2^{-n}$ for all $n$.

**Definition 1** ([2]). *Let $f :\subseteq X \to \mathbb{R}$ be a function whose domain is the set of Schnorr random points. Then $f$ is $L^1$-computable with an effective code iff $f$ is the difference between two integral tests for Schnorr randomness.*

The following is the Martin-löf randomness verions of this result. Recall that an *integral test* is a nonnegative lower semicomputable function $t : X \to \overline{\mathbb{R}}$ with $\int t d\mu < \infty$.

**Definition 2.** *A function $f :\subseteq X \to \mathbb{R}$ is weakly $L^1$-computable if there exists a computable sequence $\{s_n\}$ of finite rational step functions such that $f(x) = \lim_n s_n(x)$ and $\sum_n ||s_{n+1} - s_n||_1 < \infty$.*

**Theorem 3.** *Let $f :\subseteq X \to \mathbb{R}$ be a function whose domain is the set of Martin-Löf random points. Then $f$ is weakly $L^1$-computable iff $f$ is the difference between two integral tests.*

Similarly we can give the weak 2-randomness version.

The author gave another characterization of the effective $L^1$-computability via Schnorr layerwise computability. We say a function $f :\subseteq X \to \mathbb{R}$ is *Schnorr*

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*layerwise computable* if there exists a Schnorr test $\{U_n\}$ such that the restriction $f|_{X\setminus U_n}$ is uniformly computable.

**Theorem 4** ([2])**.** *Let $f :\subseteq X \to \mathbb{R}$ be a function whose domain is the set of Schnorr random points. Then $f$ is Schnorr layerwise computable and its $L^1$-norm is computable iff $f$ is the difference between two integral tests for Schnorr randomness.*

To study the other randomness versions of this result, we introduce Solovay reducibility for nonnegative lower semicomputable functions. Recall the following characterization of Solovay reducibility. For left-c.e. reals $\alpha$ and $\beta$, $\alpha \leq_S \beta$ iff there are a constant $d$ and a left-c.e. real $\gamma$ such that $d\beta = \alpha + \gamma$.

**Definition 5.** *Let $f, g$ be nonnegative lower semicomputable functions. We say that $f$ is Solovay reducible to $g$ (denoted by $f \leq_S g$) if there exists a computable real $d$ and a nonnegative lower semicomputable function $h$ such that*

$$d \cdot g =_{\mathrm{WR}} f + h.$$

Recall that a *Solovay test for Schnorr randomness* is a sequence $\{U_n\}$ of uniformly c.e. open sets such that $\sum_n \mu(U_n)$ is computable.

**Theorem 6.** *A nonnegative lower semicomputable function $f$ has a computable integral iff there exist a computable sequence $\{a_n\}$ of natural numbers and a Solovay test $\{U_n\}$ for Schnorr randomness such that $f \leq_S \sum_n a_n \cdot \mathbf{1}_{U_n}$ and $\sum_n a_n \mu(U_n)$ is computable.*

This theorem implies one implication of Theorem 4. Hence Solovay reducibility for nonnegative lower semicomputable functions will be a useful tool to study the relation between randomness notions and computability (like Schnorr layerwise computability).

# References


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