An integral test for Schnorr randomness and its applications

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Some recent researches show that some classical theorems with "almost everywhere" can be converted to effective version with "for each random point". One example is that a real is Martin-Löf random iff each computable function of bounded variation is differentiable at the real. The author [1] has given a Kurtz randomness version where we find that an integral test is a useful tool to study the relation between algorithmic randomness and computable analysis. Here we give a version of Schnorr randomness. Consider a computable metric space (X, d, α) and a computable measure μ on it.

Definition 1. *An integral test for Schnorr randomness is a nonnegative lower semicomputable function $t : X \to \overline{\mathbb{R}}^+$ such that $\int t d\mu$ is a computable real.*

Theorem 1. *A point z is Schnorr random iff $t(z) < \infty$ for each integral test t for Schnorr randomness.*

As an application we have another effectivized version of a classical theorem.

Theorem 2. *Let $f, g :\subseteq X \to \mathbb{R}$ be the differences between two integral tests for Schnorr randomness. Then $f(x) = g(x)$ for each Schnorr random point iff $||f - g||_1 = 0$.*

The class of the differences between two integral tests for Schnorr randomness has some characterizations and is an important class.

Definition 2. *A function $f :\subseteq X \to \mathbb{R}$ is L^1-computable with an effective code if there exists a computable sequence $\{s_n\}$ of finite rational step functions such that $f(x) = \lim_n s_n(x)$ and $||s_{n+1} - s_n||_1 \leq 2^{-n}$ for all n.*

Definition 3. *A function $f :\subseteq X \to \mathbb{R}$ is* Schnorr layerwise computable *if there exists a Schnorr test U_n such that the restriction $f|_{X \setminus U_n}$ is uniformly computable.*

Theorem 3. *Let* SR *be the set of Schnorr random points. Then*

$$\begin{aligned}&\{f|_{\mathrm{SR}} \mid f \text{ is the difference between two integral tests for Schnorr randomness}\}\\ =&\{f|_{\mathrm{SR}} \mid f \text{ is an } L^1\text{-computable function with an effective code}\}\\ =&\{f|_{\mathrm{SR}} \mid f \text{ is Schnorr layerwise computable and its } L^1\text{-norm is computable}\}\end{aligned}$$

References

1. Miyabe, K.: Characterization of Kurtz randomness by a differentiation theorem, submitted