Weak L^1-computability and Limit L^1-computability

Ninth International Conference on
Computability and Complexity in Analysis
24 - 27 June, 2012, University of Cambridge

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Motivation

Randomness and differentiability

Theorem (Demuth, Brattka-Miller-Nies)

A real is Martin-Löf random iff each computable function of bounded variation is differentiable at the real.

Other randomness versions have been obtained such as

1. weak 2-randomness,
2. computable randomness,
3. Schnorr randomness,
4. Kurtz randomness.

Effective LDT

Theorem (Effective Lebesgue Differentiation Theorem; Pathak-Rojas-Simpson, Rute)

For all $x \in [0,1]$, x is Schnorr random iff

$$\hat{f}(x) = \lim_{r \to 0} \frac{\int_{x-r}^{x+r} f d\mu}{2r}$$

for all effective L^1-computable functions.

What are other randomness versions of effective L^1-computable functions?

Coincidence

Miyabe showed that, roughly speaking, the following are equivalent.

1. A difference between two integral tests for Schnorr randomness.
2. An effective L^1-computable function.
3. A Schnorr layerwise computable function whose L^1-norm is computable.

Do we have other randomness versions of this coincidence?

Weak and limit L^1-computability

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Finite rational step function

Definition

A finite rational step function is a finite sum

$$s = \sum_{k=1}^{n} q_k \mathbf{1}_{(q_k, r_k)}$$

where $q_k \in \mathbb{Q}$ and $q_k, r_k \in \mathbb{Q} \cap [0, 1]$. .

Effective L^1-computability

Definition

A function $f :\subseteq [0,1] \to \mathbb{R}$ is effective L^1-computable if there exists a computable sequence $\{s_n\}$ of finite rational step functions such that

$$f(x) = \lim_n s_n(x) \text{ and } ||s_{n+1} - s_n||_1 \leq 2^{-n} \text{ for all } n.$$

An integral test for Schnorr randomness

Definition (M.)

An integral test for Schnorr randomness is a nonnegative lower semicomputable function $f : [0,1] \to \overline{\mathbb{R}}^+$ such that $\int f d\mu$ is a computable real.

Theorem (M.)

A real x is Schnorr random iff $f(x) < \infty$ for each integral test for Schnorr randomness.

Coincidence

Definition (M.)

Two functions f, g are Schnorr equivalent if $f(x) = g(x)$ for all Schnorr random reals x.

Theorem (M.)

For an effective L^1-computable function $f :\subseteq [0,1] \to \mathbb{R}$, f is Schnorr equivalent to a difference between two integral tests for Schnorr randomness, and vice versa.

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Weak L^1-computability

A weakly computable real $r = \lim_n q_n$ such that

$$\sum_n |q_{n+1} - q_n| < \infty$$

by Ambos-Spies et al. (2000).

Definition (M.)

A function $f :\subseteq [0,1] \to \mathbb{R}$ is weakly L^1-computable if

$$f(x) = \lim_n s_n(x) \text{ and } \sum_n \|s_{n+1} - s_n\|_1 < \infty.$$

Coincidence

Definition (M.)

Two functions f, g are ML-equivalent if $f(x) = g(x)$ for all ML-random reals x.

Theorem (M.)

For a weakly L^1-computable function $f :\subseteq [0,1] \to \mathbb{R}$, f is ML-equivalent to a difference between two integral tests, and vice versa.

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Limit L^1-computability

A real $r \in \Delta^0_2$ iff it is a limit computable point.

Definition(M.)

A function $f :\subseteq [0,1] \to \mathbb{R}$ is limit L^1-computable if f is defined a.e. and

$$f(x) = \lim_n s_n(x).$$

An (weakly or limit) L^1-computable function is, roughly speaking, a (weakly or limit) computable point in the L^1-space.

An integral test for weak 2-randomness

Definition (M.)

An integral test for weak 2-randomness is a nonnegative lower semicomputable function $f : [0,1] \to \overline{\mathbb{R}}^+$ such that $f(x) < \infty$ almost everywhere.

Theorem (M.)

A real x is weakly 2-random iff $f(x) < \infty$ for each integral test for weak 2-randomness.

Coincidence

Definition (M.)

Two functions f, g are weakly 2-equivalent if $f(x) = g(x)$ for all weakly 2-random reals x.

Theorem (M.)

For a limit L^1-computable function $f :\subseteq [0, 1] \to \mathbb{R}$,

f is weakly 2-equivalent to a difference between two integral tests for weak 2-randomness,

and vice versa.

Solovay reducibility for functions

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Schnorr layerwise computability

Definition (Hoyrup-Rojas)

Let $\{U_n\}$ be a universal ML-test. A function $f :\subseteq [0,1] \to \mathbb{R}$ is layerwise computable if f is computable on $[0,1]\setminus U_n$ uniformlly in n.

Definition (M.)

A function $f :\subseteq [0,1] \to \mathbb{R}$ is Schnorr layerwise computable if there exists a Schnorr test $\{U_n\}$ such that f is computable on $[0,1]\setminus U_n$ uniformly in n.

Another coincidence

Theorem (M.)

Let f be a function such that

1. f is a Schnorr layerwise computable function,
2. $||f||_1$ is a computable real.

Then f is Schnorr equivalent to a difference between two integral tests for Schnorr randomness.

Conversely, any integral test for Schnorr randomness is Schnorr equivalent to a Schnorr layerwise computable function.

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Observation

- For a layerwise computable function f,
  f(x) is computable from x if x is ML-random.
- However, an integral test does not have such a property,
  e.g., f(x) is Chaitin's omega for all x.
- To calculate f(x), we need to know how close a rational step function is. This reminds me of Solovay reducibility.

Solovay reducibility

Theorem (Downey, Hirschfeldt and Nies)

Let α, β be left-c.e. reals.

$$\alpha \leq_S \beta \iff \exists d \exists \gamma \text{ s.t. } d\beta = \alpha + \gamma.$$

Definition (M.)

Let f, g be nonnegative lower semicomputable functions.

$$f \leq_S g \iff \exists d \exists h \text{ s.t. } dg =_{\mathrm{WR}} f + h$$

where $=_{\mathrm{WR}}$ denotes Kurtz equivalence.

Basic properties

Proposition(M.)

Let $f \leq_S g$.

1. If g is a.e. computable, then so is f.
2. If g has a computable integral, then so is f.
3. If g is integrable, then so is f.
4. If $g(x) < \infty$ almost everywhere, then so is f.

.

Characterizations

Proposition (M.)

Let f be bounded by $M \in \mathbb{N}$. Then f is a.e. computable iff $f \leq_S M$.

Proposition (M.)

There exsits t such that f is integrable iff $f \leq_S t$.

Solovay test

Definition

A Solovay test for Schnorr randomness is a sequence $\{U_n\}$ of uniformly c.e. open sets such that $\sum_n \mu(U_n)$ is computable.

Proposition

A real is Schnorr random iff $x \in U_n$ for at most finitely many n for each Solovay test for Schnorr randomness.

Schnorr version

Theorem

Let f be a nonnegative lower semicomputable function.

Then f has a computable integral

iff there exists

1. a computable sequence $\{a_n\}$ of natural numbers and
2. a Solovay test $\{U_n\}$ for Schnorr randomness such that

$$f \leq_S \sum_n a_n \cdot \mathbf{1}_{U_n}$$

and $\sum_n a_n \mu(U_n)$ is computable.

Summary

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Summary

- We study the relation between integral tests for randomness notions and variants of L^1-computability.
- Solovay reducibility for functions can be seen as a generalization of Schnorr layerwise computability.