An integral test for Schnorr randomness and its applications

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The main result

Roughly speaking, the following are equivalent:

1. f is a difference between two integral tests for Schnorr randomness,
2. f is an effective L^1-computable function,
3. f is Schnorr layerwise computable and its L^1-norm is computable.

An integral test for Schnorr randomness

Randomness and differentiability

Theorem (Brattka, Miller and Nies)

A real is Martin-Löf random iff each computable function of bounded variation is differentiable at the real.

Let f be a computable function of bounded variation. Then

1. $f'(x)$ converges to a finite value if x is ML-random,
2. $f'(x)$ may not converge and may be ∞ if x is not ML-random.

The behavior f' is similar to an integral test!!

An integral test

Consider $[0,1]$ with the Lebesgue measure μ.

Definition

An integral test is a lower semicomputable function $t : [0,1] \to \mathbb{R}^+$ such that $\int t d\mu < \infty$.

Theorem

A real $x \in [0,1]$ is ML-random

iff $t(x) < \infty$ for all integral tests.

An integral test for Schnorr randomness

Definition(M.)

An integral test for Schnorr randomness is a lower semi-computable function $t : [0,1] \to \mathbb{R}^+$ such that $\int t d\mu$ is a computable real.

Theorem(M.)

A real $x \in [0,1]$ is Schnorr random

iff $t(x) < \infty$ for all integral tests for Schnorr randomness.

Effective L^1-computability

Lebesgue Differentiation Theorem

Theorem (Lebesgue Differentiation Theorem)
For each $f \in L^1([0,1])$,

$$f(x) = \lim_{r \to 0} \frac{\int_{(x-r,x+r)} f d\mu}{2r}$$

for almost all $x \in [0,1]$.

L^1-computability

Definition (Pour-El and Richards 1989)

A function $f \in L^1$ is L^1-computable if there exists a computable sequence $\{f_n\}$ of polynomials with rational coefficients such that

$$||f - f_n||_1 \leq 2^{-n}$$

for all n.

Effective L^1-computability

Definition (Pathak, Rojas and Simpson)

Given an L^1-computable function f, define

$$\hat{f}(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if } x \text{ is Schnorr random,} \\ 0 & \text{otherwise.} \end{cases}$$

where f_n is a computable sequence of approximation as in the definition of L^1-computability.

Effective LDT

Theorem (Effective Lebesgue Differentiation Theorem; Pathak-Rojas-Simpson, Rute)

For all $x \in [0,1]$, x is Schnorr random iff

$$\hat{f}(x) = \lim_{r \to 0} \frac{\int_{(x-r,x+r)} f \, d\mu}{2r}.$$

An integral test for Schnorr randomness should be related to the function $\hat{f}$ in some sense!

Coincidence

Definition (M.)

Two functions f, g are Schnorr equivalent if $f(x) = g(x)$ for all Schnorr random reals x.

Theorem (M.)

For an L^1-computable function $f :\subseteq [0,1] \to \mathbb{R}$,

$\hat{f}$ is Schnorr equivalent to a difference between two integral tests for Schnorr randomness,

and vice versa.

Schnorr layerwise computability

Layerwise computability

Proposition (Hoyrup-Rojas 2009) Let $f :\subseteq X \to \overline{\mathbb{R}}^+$ be a function such that

1. f is a layerwise lower semi-computable function,
2. $\int f d\mu$ is a computable real.

Then f is layerwise computable.

An integral test for Schnorr randomness should be related to layerwise computability in some sense!

Schnorr layerwise computability

Definition (Hoyrup-Rojas)

Let $\{U_n\}$ be a universal ML-test. A function $f :\subseteq [0,1] \to \mathbb{R}$ is layerwise computable if f is computable on $[0,1]\setminus U_n$ uniformlly in n.

Definition (M.)

A function $f :\subseteq [0,1] \to \mathbb{R}$ is Schnorr layerwise computable if there exists a Schnorr test $\{U_n\}$ such that f is computable on $[0,1]\setminus U_n$ uniformly in n.

Another coincidence

Theorem (M.)

Let f be a function such that

1. f is a Schnorr layerwise computable function,
2. $||f||_1$ is a computable real.

Then f is Schnorr equivalent to a difference between two integral tests for Schnorr randomness.

Conversely, any integral test for Schnorr randomness is Schnorr equivalent to a Schnorr layerwise computable function.

Summary

Roughly speaking, the following are equivalent.

1. A difference between two integral tests.
2. An effective L^1-computable function.
3. A Schnorr layerwise computable function whose L^1-norm is computable.