The law of the iterated logarithm in game-theoretic probability

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This is a joint work with A. Takemura.

Overview

- Brief history of the LIL
- The main result and its related results
- The proof

Brief history of the LIL

The law of the iterated logarithm

Theorem (Khintchine 1924)

Let $\{X_n\}$ be the sequence of i.i.d. random variables such that $P(X_1 = 0) = P(X_1 = 1) = \frac{1}{2}$. Then

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} X_k - \frac{n}{2}}{\sqrt{2n \ln \ln n}} = \frac{1}{2}$$

almost surely.

Actually, this equation holds for each Schnorr random set.

The Kolmogorov LIL

Theorem (Kolmogorov 1929)

Let $\{X_n\}$ be the sequence of independent random variables with $\mathbf{E}X_n = 0$ and $\mathbf{E}X_n^2 < \infty$. Put $A_n = \sum_{k=1}^n \mathbf{E}X_k^2$ and $S_n = \sum_{k=1}^n X_k$. If $A_n \to \infty$ and there exists a sequence $\{c_n\}$ such that

$$|X_n| \le c_n = o\left(\sqrt{\frac{A_n}{\ln\ln A_n}}\right) \text{ a.s.},$$

then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2A_n \ln\ln A_n}} = 1 \text{ a.s.}$$

The Hartman-Wintner LIL

Theorem (Hartman-Wintner 1941)

Let $\{X_n\}$ be the sequence of i.i.d. random variables with $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$. Then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2A_n \ln\ln A_n}} = 1 \text{ a.s.}$$

Strassen (1966) showed that the converse also holds.

Predictably Unbounded Forecasting

Players: Forecaster, Skeptic, Reality

Protocol:

$\mathcal{K}_0 := 1$.

FOR $n = 1, 2, \ldots$:

Forecaster announces $m_n \in \mathbb{R}$, $c_n \geq 0$, and $v_n \geq 0$.

Skeptic announces $M_n \in \mathbb{R}$ and $V_n \in \mathbb{R}$.

Reality announces $x_n \in \mathbb{R}$ such that $|x_n - m_n| \leq c_n$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n) + V_n((x_n - m_n)^2 - v_n)$.

Collateral Duties: Skeptic must keep $\mathcal{K}_n$ non-negative. Reality must keep $\mathcal{K}_n$ from tending to infinity.

The GTP counterpart

Theorem (Shafer and Vovk 2001)

In the predictably unbounded forecasting protocol, Skeptic can force

$$\left(A_n \to \infty \ \& \ c_n = o\left(\sqrt{\frac{A_n}{\ln \ln A_n}} \right) \right)$$

$$\Longrightarrow \limsup_{n \to \infty} \frac{\sum_{i=1}^{n} (x_i - m_i)}{\sqrt{2 A_n \ln \ln A_n}} = 1.$$

Question

- Q. How to express the HW LIL in GTP?
- The importance:
  (1) How to express i.i.d. in GTP?
  (2) How much can we weaken the condition of the LIL?

The main result

THE UNBOUNDED FORECASTING GAME WITH QUADRATIC AND STRONGER HEDGES (UFQSH)

Players: Forecaster, Skeptic, Reality

Protocol:

$\mathcal{K}_0 := 1.$

FOR $n = 1, 2, \ldots$:

Forecaster announces $m_n \in \mathbb{R}$, $v_n \geq$ and $w_n \geq 0$.

Skeptic announces $M_n \in \mathbb{R}$, $V_n \in \mathbb{R}$ and $W_n \geq 0$.

Reality announces $x_n \in \mathbb{R}$.

$$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n) + V_n((x_n - m_n)^2 - v_n)$$
$$+W_n(h(x_n - m_n) - w_n).$$

With the usual collateral duties.

Assumption

1. h is an even function.
2. $h \in C^2$ and $h(0) = h'(0) = h''(0) = 0$.
3. $h''(x)$ is strictly increasing, unbounded and concave (upward convex) for $x \geq 0$.

Example

The function $h(x) = |x|^\alpha$ for $2 < \alpha \leq 3$ satisfies the assumption above.

The main theorem

Theorem (Miyabe and Takemura)

In UFQSH with h satisfying the assumption, Skeptic can force

$$\left(A_n \to \infty \text{ and } \sum_n \frac{w_n}{h(b_n)} < \infty \right)$$

$$\Rightarrow \limsup_{n\to\infty} \frac{S_n - \sum_{i=1}^n m_i}{\sqrt{2A_n \ln\ln A_n}} = 1$$

where $b_n = \sqrt{\frac{A_n}{\ln\ln A_n}}$.

Corollary

Let h be an extra hedge satisfying the assumption and

$$\sum_n \frac{1}{h(\sqrt{n/\ln\ln n})} < \infty.$$

In UFQSH with this h and $m_n \equiv m$, $v_n \equiv v$ and $w_n \equiv w$, the following are equivalent for $m' \in \mathbb{R}$ and $v' \geq 0$.

1. $m' = m$ and $v' = v$.
2. Skeptic can force
$$\limsup_{n\to\infty} \frac{S_n - m'n}{\sqrt{2n\ln\ln n}} = \sqrt{v'}. \qquad (1)$$
3. Reality can comply with (1).

- Marcinkiewicz and Zygmund (1937) constructed a sequence of independent random variables for which $A_n \to \infty$ and $|X_n| = O(\sqrt{A_n/\ln\ln A_n})$ and which does not obey the LIL.
- There are many sufficient conditions for the LIL.
- Egorov's sufficient condition (1984) is

$$\frac{\sum_{k=1}^{n} X_k^2}{A_n} \to 1 \text{ a.s. and}$$

$$\sum_{k=1}^{n} \mathbf{E} X_k^2 I(|X_k| > \frac{\epsilon A_n}{\ln\ln A_n}) = o(A_n)$$

for any $\epsilon > 0$.

The related results

- Takazawa (2012,201?) showed a weaker upper bound with double hedges.
- Miyabe and Takemura (2012) showed a necessary and sufficient condition for the SLLN.

The proof

Proposition

In UFQSH with h satisfying the assumption, Skeptic can force

$$\left(A_n \to \infty \text{ and } \sum_n \frac{w_n}{h(b_n)} < \infty \right) \Longrightarrow \limsup_{n\to\infty} \frac{S_n}{\sqrt{2A_n \ln\ln A_n}} \leq 1.$$

Lemma

In UFQSH with h satisfying the assumption, Skeptic can force

$$A_n \to \infty \text{ and } \sum_n \frac{w_n}{h(b_n)} < \infty \Longrightarrow |x_n| = o(b_n)$$

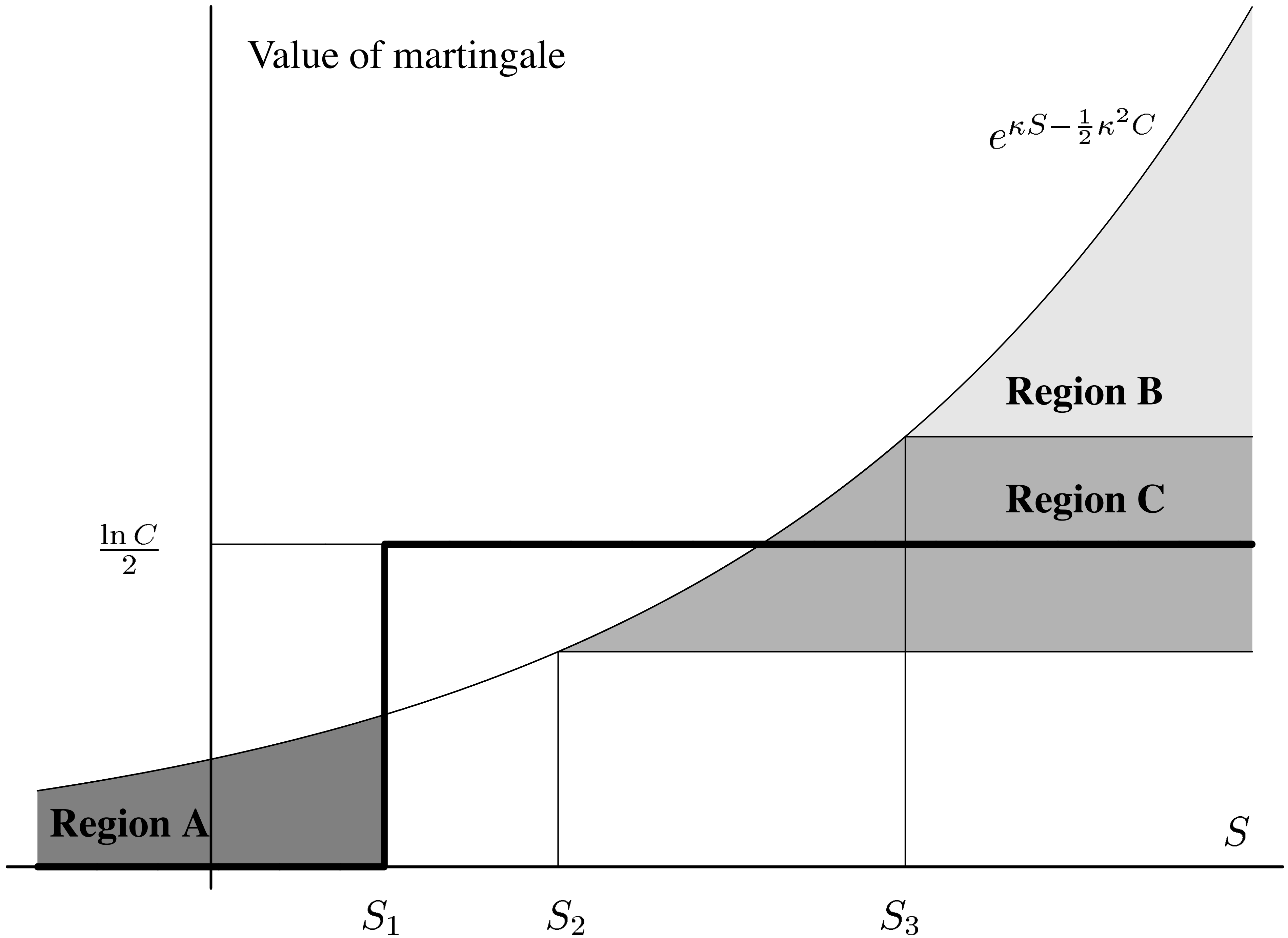

Fig. 5.1 Proving the large-deviation inequality by reducing an initial payoff. Here $S_1 = (1-\epsilon)\sqrt{2C \ln \ln C}$, $S_2 = (1+\epsilon^*)^2(1-\epsilon)\sqrt{2C \ln \ln C}$, and $S_3 = 2\sqrt{2C \ln \ln C}$.

taken from Shafer and Vovk 2001

In Shafer and Vovk (2001), they showed that, as long as Reality is required to ensure that

$$|x_n| \le \delta\sqrt{\frac{C}{\ln\ln C}},$$

for all n,

$$\frac{\mathcal{L}(s)}{\exp(\kappa\mathcal{S}(s) - \frac{\kappa^2}{2}C)} \le (\ln C)^{8\delta}$$

for every situation $s \in \tau$ where $\tau := \min\{n \mid A_n \ge C\}$ by considering the strategy

$$\mathcal{L}_{i+1} = \mathcal{L}_i \frac{1 + \kappa x_i + (1-\delta)\kappa^2 x_i^2/2}{1 + (1-\delta)\kappa^2 v_i/2}.$$

Define stopping time τ_1, τ_2, τ_3 by

$$\tau_1 = \min\{n \mid \cdots\},$$
$$\tau_2 = \min\{n \mid A_n \geq C\},$$
$$\tau_3 = \min\left\{n \mid |x_n| > \delta\sqrt{\frac{C}{\ln\ln C}}\right\}.$$

Lemma

In UFQSH with h satisfying the assumption, there exists a martingale $\mathcal{L}$ such that $\mathcal{L}(\square) = 1$

$$\frac{\mathcal{L}_n}{\exp(\kappa S_n - \kappa^2 C/2)} \leq (\ln C)^{4\delta}$$

for n such that $n = \tau_2 < \tau_1, \tau_3$.

A martingale that satisfies the property is defined by

$$\mathcal{L}_i = \mathcal{L}_{i-1} \frac{1 + \kappa x_i + \frac{\kappa^2 x_i^2}{2} - \frac{h(x_i)}{h(\kappa^{-1})}}{1 + \frac{\kappa^2 v_i}{2} - \frac{w_i}{h(\kappa^{-1})}}$$

for all i. .

Summary

The question again

- How do we express i.i.d. in GTP?
- An idea is to add stronger hedges.
- This idea seems to work well to some extent but not completely.