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L^1-computability, layerwise computability and Solovay reducibility

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Abstract. We propose a hierarchy of classes of functions that corresponds to the hierarchy of randomness notions. Each class of functions converges at the corresponding random points. We give various characterizations of the classes, that is, characterizations via integral tests, L^1-computability and layerwise computability. Furthermore, the relation among these classes is formulated using a version of Solovay reducibility for lower semicomputable functions.

Keywords: algorithmic randomness, computable anlaysis, L^1-computability, layerwise computability, Solovay reducibility

1. Introduction

We propose a hierarchy of classes of effective functions that corresponds to the hierarchy of algorithmic randomness notions. We give various characterizations of the classes using notions already defined in the literature. A similar attempt was made by Brattka, Miller and Nies [8], who characterized some randomness notions via differentiability.

A theorem of Lebesgue [18] states that every nondecreasing function $f : [0, 1] \to \mathbb{R}$ is differentiable almost everywhere. Thus, intuitively, at any "random" real, a function is differentiable.

The theory of algorithmic randomness defines which points in a measure space are "random". Roughly speaking, a point is random if it is not contained in any effectively presented set with measure zero. Many randomness notions have been proposed, and their hierarchy has been developed. Some commonly studied notions of randomness are weak 2-randomness, Martin-Löf randomness, computable randomness, Schnorr randomness and Kurtz randomness. (See [10, 22] for details.)

Brattka, Miller and Nies [8] proposed, for each randomness notion, finding a class of effective functions so that a real $z \in [0, 1]$ satisfies the randomness notion if and only if each function in the class is differentiable at z. For instance, Brattka et al. [8] showed that z is computably random if and only if each computable nondecreasing function is differentiable at z, and that z is Martin-Löf random if and only if each computable function of bounded variation is differentiable at z. (The second result was also proved by Demuth [9].) Brattka et al. [8] also gave a weak 2-randomness version and Freer, Kjos-Hanssen, Nies and Stephan [12] gave a Schnorr randomness version. These results have provided a hierarchy of classes of effective functions that corresponds to the hierarchy of randomness notions via differentiability.

The theorem of Lebesgue has a stronger form, namely the Lebesgue differentiation theorem [2, 18, 19]. We can thus consider a hierarchy of classes of effective functions that corresponds to the hierarchy of randomness notions via the Lebesgue differentiation theorem. Pathak, Rojas and Simpson [23] (and independently Jason Rute) showed that the class of an effective version of L^1-computable functions characterizes Schnorr randomness via the Lebesgue differentiation theorem. A Kurtz-randomness version was given by Miyabe [21], who showed that the class of the differences of two integral tests for Kurtz randomness characterizes Kurtz randomness via the Lebesgue differentiation theorem. According to this result and as we see in this paper, the classes of the differences of two integral tests are important. We will give a Schnorr-randomness version of this result and present a systematic study of the hierarchy of the classes.

This paper is organized as follows. The randomness notions studied in this paper are weak 2-randomness, Martin-Löf randomness, Schnorr randomness and Kurtz randomness. We characterize these randomness notions in various ways. Section 2 gives definitions and results used in subsequent sections. Section 3 characterizes weak 2-randomness and Schnorr randomness via integral tests. The characterizations of Martin-Löf randomness and Kurtz randomness via integral tests are known. Section 4 characterizes the class of the differences of two integral tests via L^1-computability. Section 5 characterizes an effective version of L^1-computability via a Schnorr version of layerwise computability. Section 6 introduces Solovay reducibility for lower semicomputable functions to explain the relation among the classes.

2. Background

2.1. Computable analysis

We recall some notions from computable analysis. See [5, 7, 27, 28] for details. We abbreviate "if and only if" as *"iff"*. Let Σ be a finite alphabet such that $0, 1 \in \Sigma$. By Σ^* we denote the set of finite words over Σ and by Σ^ω the set of infinite sequences over Σ. A *notation* of a set X is a surjective partial function $\nu :\subseteq \Sigma^* \to X$, and a *representation* is a surjective partial function $\delta :\subseteq \Sigma^\omega \to X$. A *naming system* is a notation or a representation. For $i \in \{1, 2\}$, let $Y_i \in \{\Sigma^*, \Sigma^\omega\}$ and $\gamma_i \subseteq Y_i \to X_i$ be naming systems. A point $x \in X_1$ is γ-*computable* if it has a computable γ-name. A function $h :\subseteq Y_1 \to Y_2$ *realizes* a partial function $f :\subseteq X_1 \to X_2$ if $\gamma_2 \circ h(y_1) = f \circ \gamma_1(y_1)$ whenever $y_1 \in \mathrm{dom}(\gamma_1)$ and $\gamma_1(y_1) \in \mathrm{dom}(f)$. The function f is called (γ_1, γ_2)-*computable* if it has a computable realization.

The canonical notations of the natural and the rational numbers are denoted by $\nu_{\mathbb{N}}$ and $\nu_{\mathbb{Q}}$, respectively. The representation $\rho_< :\subseteq \Sigma^\omega \to \mathbb{R}$ is defined by

$$\rho_<(p) = x \iff p \text{ enumerates all } q \in \mathbb{Q} \text{ with } q < x.$$

We use $\overline{\rho}_<$ for the representation of points in $\mathbb{R} \cup \{\infty\}$. The representation $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$ is defined by

$$\rho(p) = x \iff p \text{ encodes a sequence } \{q_n\} \text{ of rationals such that } |x - q_n| \le 2^{-n}.$$

Definition 2.1 (computable metric space). A computable metric space is a 3-tuple $\mathbf{X} = (X, d, \alpha)$ such that

(i) (X, d) is a metric space,
(ii) $\alpha :\subseteq \Sigma^* \to A$ is a notation of a dense subset A of X with a computable domain,
(iii) d restricted to $A \times A$ is (α, α, ρ)-computable.

We give some examples of computable metric spaces.

Example 2.1. (i) (unit interval) Let $\mathbf{I} = (\mathbb{I}, d, \alpha)$ be such that $\mathbb{I} = [0, 1]$, α is a canonical notation of $\mathbb{Q} \cap \mathbb{I}$ and $d(p, q) = |p - q|$.
(ii) (extended real line) Let $\overline{\mathbf{R}} = (\overline{\mathbb{R}}, d, \alpha)$ be such that $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, α is a canonical notation of $\mathbb{Q} \cup \{\pm\infty\}$ and $d(x, y) = |f(x) - f(y)|$ where $f(x) = \frac{x}{1+|x|}$, $f(\infty) = 1$ and $f(-\infty) = -1$.

Let $\mathbf{X} = (X, d, \alpha)$ be a computable metric space. A *fast Cauchy sequence* on a metric space is a sequence $\{x_n\}$ of points in the space such that $d(x_n, x_{n-1}) \le 2^{-n}$. The representation $\delta :\subseteq \Sigma^\omega \to X$ of points in $\mathbf{X}$ is defined by

$$\delta(p) = x \iff p \text{ encodes a fast Cauchy sequence } \{x_n\} \text{ in } A \text{ that converges to } x.$$

A *basic open ball* on $\mathbf{X}$ is denoted by $B(u, r) = \{x : d(u, x) < r\}$ and a *basic closed ball* is denoted by $\overline{B}(u, r) = \{x : d(u, x) \le r\}$ where $A \in \alpha$ and $r \in \mathbb{Q}$. By τ, we denote the class of open sets. The representation $\theta :\subseteq \Sigma^\omega \to \tau$ of open sets is defined by

$$\theta(p) = W \iff p \text{ encodes a sequence } \{B_i\} \text{ of basic open balls such that } W = \bigcup_i B_i.$$

For simplicity, we use the following terminology. A point on $\mathbf{X}$ is *computable* if it is δ-computable. A open set is *c.e.* if it is θ-computable. A closed set is *co-c.e.* if its complement is c.e. A total function $f : X_1 \to X_2$ is *computable* if it is (δ_1, δ_2)-computable. A total function $f : X \to \overline{\mathbb{R}}$ is *lower semi-computable* if it is $(\delta, \overline{\rho}_<)$-computable. A total function $f : X \to \overline{\mathbb{R}}$ is *extended computable* if it is $(\delta, \overline{\rho})$-computable. Here $\overline{\rho}$ is the representation δ of points in $\overline{\mathbf{R}}$.

2.2. Computable measures

For computability of measures on a computable metric space, see [3, 16, 25]. We only consider a Borel probabilistic computable measure. A measure on a computable metric space is *computable* if the measure of a finite union of basic open balls is uniformly lower semicomputable. Then the integral of a nonnegative lower semicomputable function over a c.e. open set with respect to a computable measure is uniformly lower semicomputable.

Fix a computable enumeration $\{\alpha_i\}$ of A.

Proposition 2.2. *Let μ be a computable measure on a computable metric space. Then there exists a computable sequence $\{r_j\}$ such that $\mu(\overline{B}(\alpha_i, r_j) \setminus B(\alpha_i, r_j)) = 0$ for all i and j.*

We call $B(\alpha_i, r_j)$ a *basic set* and $\overline{B}^c(\alpha_i, r_j)$ a *co-basic set* for each i and j. Let $\mathcal{I}$ be the set of all finite intersections of basic sets and co-basic sets. Note that $\mu(U)$ is computable uniformly in $U \in \mathcal{I}$. Let $B_{\langle i,j\rangle} = B(\alpha_i, r_j)$ and $\overline{B}^c_{\langle i,j\rangle} = \overline{B}^c(\alpha_i, r_j)$. As in [16], for $\sigma \in 2^{<\omega}$, the cell $\Gamma(\sigma)$ is defined by induction on $|\sigma|$:

$$\Gamma(\epsilon) = X,\ \Gamma(\sigma 0) = \Gamma(\sigma) \cap B_k,\ \Gamma(\sigma 1) = \Gamma(\sigma) \cap \overline{B}^c_k$$

where ϵ is the empty string and $k = |\sigma|$.

2.3. Algorithmic randomness

We refer the reader to two books [10, 22] for a survey on algorithmic randomness. A *Martin-Löf test* is a sequence $\{U_n\}$ of uniformly c.e. open sets with $\mu(U_n) \le 2^{-n}$. A point $x \in X$ is *Martin-Löf random* if $x \notin \bigcap_n U_n$ for each Martin-Löf test. A *Schnorr test* is a Martin-Löf test such that $\mu(U_n)$ is uniformly computable. A point $x \in X$ is *Schnorr random* if $x \notin \bigcap_n U_n$ for each Schnorr test. A point $x \in X$ is *Kurtz random* or *weakly random* if it is contained in each c.e. open set with measure 1. A *generalized Martin-Löf test* is a sequence $\{U_n\}$ of uniformly c.e. open sets with $\lim_n \mu(U_n) = 0$. A point $x \in X$ is *weakly 2-random* if $x \notin \bigcap_n U_n$ for each generalized Martin-Löf test.

Two functions $f, g :\subseteq X \to \mathbb{R}$ are Kurtz equivalent (denoted by $f =_{\mathrm{WR}} g$) if $f(x) = g(x)$ on each Kurtz random point. By $\mathbf{1}_U$, we denote the characteristic function of $U \subseteq X$; that is, $\mathbf{1}_U(x) = 1$ if $x \in U$ and $\mathbf{1}_U(x) = 0$ if $x \notin U$. Two subsets U, V are Kurtz equivalent if $\mathbf{1}_U$ and $\mathbf{1}_V$ are Kurtz equivalent.

A basic set or a co-basic set is Kurtz equivalent to a union of cells. A set $U \in \mathcal{I}$ is Kurtz equivalent to a union of cells. If $U = B_{i_1} \cap \cdots \cap B_{i_k} \cap \overline{B}^c_{j_1} \cap \cdots \cap \overline{B}^c_{j_l}$ for some $i_1, \ldots, i_k, j_1, \ldots, j_l$, then U is Kurtz equivalent to

$$\bigcup \{\Gamma(\sigma) \mid \sigma(i_1) = 0, \ldots, \sigma(i_k) = 0, \sigma(j_1) = 1, \ldots, \sigma(j_l) = 1, |\sigma| = m\}$$

for $m > \max\{i_1, \ldots, i_k, j_1, \ldots, j_l\}$. It is easy to see that a union of sets in $\mathcal{I}$ is Kurtz equivalent to a union of cells uniformly.

2.4. A rational step function

The notion of a rational step function has been used in the literature; e.g., [17, 23]. The following definition is Kurtz equivalent to the notions of [17, 23].

Definition 2.3. A rational step function is a finite sum $s = \sum_{k=1}^{n} q_k \mathbf{1}_{E_k}$ where $q_k \in \mathbb{Q}$ and $E_k \in \mathcal{I}$.

Note that there exists a canonical numbering of the collection of rational step functions. The following is immediate from Lemma 4.6 in [21].

Proposition 2.4. *For a nonnegative lower semicomputable function $f : X \to \overline{\mathbb{R}}$, there exists a computable increasing sequence $\{s_n\}$ of rational step functions such that $\lim_n s_n$ is Kurtz equivalent to f.*

We call the sequence $\{s_n\}$ an approximation of f by finite rational step functions.

Definition 2.5. A *rational cell function* is a finite sum

$$s = \sum_{k=1}^{n} q_k \mathbf{1}_{\Gamma(\sigma_k)}$$

where $q_k \in \mathbb{Q}$ and $\sigma_k \in 2^m$ for all k such that $1 \leq k \leq n$ and for some fixed $m \in \mathbb{N}$.

Proposition 2.6. *A rational cell function is a rational step function. A rational step function is Kurtz equivalent to a rational cell function.*

Proof. The former half is immediate. Let $s = \sum_{k=1}^{n} q_k \mathbf{1}_{E_k}$ be a finite rational step function where $q_k \in \mathbb{Q}$ and $E_k \in \mathcal{I}$. Replace E_k with a Kurtz equivalent union of cells with a fixed sufficiently large m to have a finite rational cell function. □

3. Integral tests

In this section, we give characterizations of several randomness notions via integral tests.

The following characterization of Martin-Löf randomness is a well-known result. Let $\overline{\mathbb{R}}^+$ be the set of nonnegative elements of $\overline{\mathbb{R}}$. An integral test is a nonnegative lower semicomputable function $t : X \to \overline{\mathbb{R}}^+$ such that $\mu(t) = \int t d\mu < \infty$.

Theorem 3.1 ([16, 20]). *A point z is Martin-Löf random iff $t(z) < \infty$ for each integral test t.*

The author has presented a version for Kurtz randomness.

Theorem 3.2 (Miyabe [21]). *A point $z \in X$ is Kurtz random iff $t(z) < \infty$ for each nonnegative extended computable function $t : X \to \overline{\mathbb{R}}$ such that $\mu(t)$ is computable.*

3.1. A weak 2-randomness version of integral tests

First we give a version for weak 2-randomness.

Proposition 3.3. *A point z is weakly 2-random iff $t(z) < \infty$ for each nonnegative lower semicomputable function $t : X \to \overline{\mathbb{R}}^+$ such that $t(x) < \infty$ almost everywhere.*

Proof. Suppose that z is not weakly 2-random. Then there exists a decreasing sequence $\{U_n\}$ of uniformly c.e. open sets such that $\mu(U_n) \to 0$ and $z \in \bigcap_n U_n$. Let $t(x) = \sup\{n \mid x \in U_n\}$. Then t is lower semicomputable, $t(x) < \infty$ almost everywhere and $t(z) = \infty$.

Suppose $t(z) = \infty$ for an nonnegative lower semicomputable function t such that $t(x) < \infty$ almost everywhere. Let $U_n = \{x \mid t(x) > n\}$. Then $\{U_n\}$ is a generalized ML-test and $z \in \bigcap_n U_n$. □

3.2. Integral tests for Schnorr randomness

Next we give a version for Schnorr randomness.

Definition 3.4. An integral test for Schnorr randomness is a nonnegative lower semicomputable function $t : X \to \overline{\mathbb{R}}^+$ such that $\mu(t) = \int t d\mu$ is computable.

Theorem 3.5. *A point z is Schnorr random iff $t(z) < \infty$ for each integral test t for Schnorr randomness.*

Lemma 3.6. *Let $\{x_n\}$ be a sequence of uniformly computable positive reals. If there exists a sequence $\{y_n\}$ of uniformly computable positive reals such that $x_n \le y_n$ for all n and $\sum_n y_n$ is computable, then $\sum_n x_n$ is also computable.*

This lemma was used in [21], where a proof of it can be found.

Proof of the "if" direction of Theorem 3.5. Suppose z is not Schnorr random. Then there exists a Schnorr test $\{U_n\}$ such that $z \in \bigcap_n U_n$. Let $t(x) = \#\{n \in \mathbb{N} : x \in U_n\}$ where $\#$ denotes the size of the set. Note that $t(z) = \infty$. Since U_n is uniformly c.e., t is lower semicomputable. Note that $\mu(t) = \sum_{n=0}^{\infty} \mu(U_n)$. Since $\mu(U_n) \le 2^{-n}$ and $\sum_n 2^{-n}$ is computable, $\mu(t)$ is computable by Lemma 3.6. Hence, t is an integral test for Schnorr randomness. □

The "only if" direction is more difficult to prove than the "if" direction. Intuitively, since the area $\mu(t)$ is computable, each area cut horizontally at two rationals p, q ($p < q$) is also computable. Then $\mu(\{x : t(x) > q\})$ and $\mu(\{x : t(x) < q\})$ can be approximated well unless $\mu(\{x : t(x) = q\}) > 0$. We then need a computable sequence $\{q_n\}$ such that $\mu(\{x : t(x) = q_n\}) = 0$. A similar sequence is also used in [16] to construct a base of uniformly almost decidable balls, where the computable Baire category theorem plays an important role.

Definition 3.7. A constructive G_δ-set is a set of the form $\bigcap_n U_n$ where $\{U_n\}$ is a sequence of uniformly θ-computable open sets.

Theorem 3.8 (Computable Baire theorem [4, 29]). *On a computable metric space, every dense constructive G_δ-set contains a dense sequence of uniformly computable points.*

We prove the "only if" direction of Theorem 3.5 via three lemmas.

Lemma 3.9. *Let $h_r(x) = \min\{r, t(x)\}$ where t is an integral test for Schnorr randomness and r is a computable real. Then $\int h_r d\mu$ is computable uniformly from r.*

Proof. We assume $r > 0$. Let $g_r(x) = \max\{r, t(x)\}$. Since h_r and g_r are lower semicomputable, so are $\int h_r d\mu$ and $\int g_r d\mu$. Since $h_r(x) + g_r(x) = t(x) + r$, we obtain $\mu(h_r) + \mu(g_r) = \mu(t) + r$. Since the right-hand side is computable, the left-hand side is also computable. Thus, $\mu(h_r)$ and $\mu(g_r)$ are computable. □

The measure $\mu(\{x : t(x) \ge r\})$ can then be approximated from above.

Lemma 3.10. *Let t be an integral test for Schnorr randomness and s be a computable real. Then $\mu(\{x : t(x) \ge s\})$ is upper semicomputable uniformly from s.*

Proof. For $0 < r < s$, let $I_r^s = \int (h_s - h_r) d\mu$. Then I_r^s is computable uniformly in r and s. Note that

$$h_s(x) - h_r(x) = \begin{cases} s - r & \text{if } t(x) \ge s \\ t(x) - r & \text{if } r \le t(x) < s \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\mu(\{x : t(x) \ge s\}) \le \frac{I_r^s}{s - r} \le \mu(\{x : t(x) \ge r\}).$$

Note that

$$\lim_{\epsilon\to+0}\mu(\{x : t(x)\geq s-\epsilon\})=\mu(\{x : t(x)\geq s\}).$$

Then

$$\mu(\{x : t(x)\geq s\})=\lim_{\epsilon\to+0}\frac{I^s_{s-\epsilon}}{\epsilon}=\lim_{n\to\infty}nI^s_{s-1/n}.$$

Since $nI^s_{s-1/n}$ is uniformly computable, it suffices to show that $nI^s_{s-1/n}$ decreasing in n. For $n\leq m$, we have

$$I^s_{s-1/n}=I^s_{s-1/m}+I^{s-1/m}_{s-1/n}$$

and

$$\frac{I^s_{s-1/m}}{s-(s-1/m)}\leq\mu(\{x\ :\ t(x)\geq s-1/m\})\leq\frac{I^{s-1/m}_{s-1/n}}{(s-1/m)-(s-1/n)}.$$

By combining them, we obtain

$$\frac{m-n}{nm}I^s_{s-1/m}\leq\frac{1}{m}(I^s_{s-1/n}-I^s_{s-1/m}),$$
$$mI^s_{s-1/m}\leq nI^s_{s-1/n}.$$

Hence, $\mu(\{x : t(x)\geq s\})$ has a computable approximation from above. □

Lemma 3.11. *There exists a sequence $\{r_n\}$ of uniformly computable reals such that $\mu(\{x : t(x)=r_n\})=0$ for all n.*

Proof. Define $U_k=\{r\in\mathbb{R}^+ : \mu(\{x : t(x)\geq r\})<\mu(\{x : t(x)>r\})+1/k\}$. By Lemma 3.10, $\mu(\{x : t(x)\geq r\})$ is upper semicomputable and $\mu(\{x : t(x)>r\})$ is lower semicomputable, thus U_k is a c.e. open set uniformly in k. Since μ is finite, the set of r for which $\mu(\{x : t(x)=r\})\geq 1/k$ is finite. Hence, U_k is dense. Note that $\mathbb{R}^+$ equipped with the standard metric is a computable metric space. Then, by the computable Baire Theorem, the dense constructive G_δ-set $\bigcap_k U_k$ contains a sequence r_n of uniformly computable reals that is dense in $\mathbb{R}^+$. By construction, $\mu(\{x : t(x)=r_n\})=0$ for all n. □

Proof of the "only if" direction of Theorem 3.5. Let t be an integral test for Schnorr randomness. Let $\{r_n\}$ be a sequence of uniformly computable reals such that $\mu(\{x\ :\ t(x)=r_n\})=0$ for all n. Then $\mu(\{x : t(x)\geq r_n\})=\mu(\{x : t(x)>r_n\})$. It follows that $\mu(\{x : t(x)>r_n\})$ is computable uniformly in n.

Select an increasing computable subsequence $\{s_n\}\subseteq\{r_n\}$ such that $s_n\geq 2^n\mu(t)$. Let $V_n=\{x\ :\ t(x)>s_n\}$. Then $\{V_n\}$ is uniformly c.e. open and the measure $\mu(V_n)$ is computable. Since $s_n\mu(V_n)\leq\mu(t)$, we have $\mu(V_n)\leq 2^{-n}$. Hence, $\{V_n\}$ is a Schnorr test. If $t(y)=\infty$, then y is not Schnorr random. □

4. L^1-computability

4.1. Effective L^1-computability

The notion of L^1-computability was defined by Pour-El and Richard [24] and has been widely used in the literature. Pathak, Rojas and Simpson [23] considered an effective version of L^1-computability on a finite dimensional cube with the Lebesgue measure. We give the definition in a slightly different formulation.

The L^1-norm of a function f is denoted by $||f||_1=\int_X|f|d\mu$. The L^1-computability is being a computable point under the L^1-norm.

Definition 4.1. A function $f :\subseteq X \to \mathbb{R}$ is an *effective L^1-computable function* if there exists a computable sequence $\{s_n\}$ of finite rational step functions such that $f(x) = \lim_n s_n(x)$ and $||s_n - s_{n-1}||_1 \leq 2^{-n}$ for all $n \geq 1$.

Pathak et al. [23] showed that an effective L^1-computable function characterizes Schnorr randomness via the Lebesgue differentiation theorem. In contrast, Miyabe [21] showed that the class of differences between two integral tests for Kurtz randomness characterizes Kurtz randomness via the Lebesgue differentiation theorem. It is natural to ask how L^1-computability is related to being a difference between two integral tests for Schnorr randomness. We answer this question by showing that they are essentially the same.

Let $f : X \to \overline{\mathbb{R}}^+$ be functions. We use the same symbol f to mean the partial function $f :\subseteq X \to \mathbb{R}$. For instance, for $f, g : X \to \overline{\mathbb{R}}^+$, $f - g$ is the partial function to $\mathbb{R}$ and $\mathrm{dom}(f - g) = \{x \ : \ f(x) < \infty, \ g(x) < \infty\}$.

Definition 4.2. Two functions $f, g :\subseteq X \to \mathbb{R}$ are *Schnorr equivalent* (denoted by $f =_{\mathrm{SR}} g$) if $f(x)$ and $g(x)$ are defined and equal for each Schnorr random point $x \in X$.

Theorem 4.3. *Every difference between two integral tests for Schnorr randomness is Schnorr equivalent to an effective L^1-computable function. Conversely, every effective L^1-computable function is Schnorr equivalent to a difference between two integral tests for Schnorr randomness.*

Proof. First, we prove the former half. Let f be the difference between two integral tests t, u for Schnorr randomness. Then there exist approximations $\{t_n\}$ and $\{u_n\}$ by rational step functions such that $||t - t_n|| \leq 2^{-n-2}$ and $||u - u_n|| \leq 2^{-n-2}$. This is possible because the integrals of t and u are computable. Let $s_n = t_n - u_n$ for all n. Then

$$\lim_n s_n = \lim_n (t_n - u_n) =_{\mathrm{SR}} t - u = f$$

and

$$\begin{aligned} ||s_n - s_{n-1}||_1 =& ||t_n - t_{n-1} - u_n + u_{n-1}||_1 \\ \leq& ||t - t_n||_1 + ||t - t_{n-1}||_1 + ||u - u_n||_1 + ||u - u_{n-1}||_1 < 2^{-n}. \end{aligned}$$

Hence, f is Schnorr equivalent to an effective L^1-computable function.

Next, we prove the latter half. Let g be an effective L^1-computable function. Then there exists a computable sequence $\{s_n\}$ of rational step functions such that $g(x) = \lim_n s_n(x)$ and $||s_{n+1} - s_n||_1 \leq 2^{-n}$ for all n. Since $s_{n+1} - s_n$ is a rational step function, it is Kurtz equivalent to a finite cell function $c_n = \sum_k q_k \mathbf{1}_{\Gamma(\sigma_k)}$ by Proposition 2.6. Let

$$c_n^+ = \sum_{q_k \geq 0} q_k \mathbf{1}_{\Gamma(\sigma_k)} \text{ and } c_n^- = -\sum_{q_k < 0} q_k \mathbf{1}_{\Gamma(\sigma_k)}.$$

Then $||c_n||_1 = ||s_{n+1} - s_n||_1 \leq 2^{-n}$ and

$$||c_n||_1 = \sum_k |q_k| \mu(\Gamma(\sigma_k)) = ||c_n^+||_1 + ||c_n^-||_1.$$

It follows that $||c_n^+||_1 \leq 2^{-n}$ and $||c_n^-||_1 \leq 2^{-n}$. Hence, $\sum_n c_n^+$ and $\sum_n c_n^-$ are integral tests for Schnorr randomness. Here g is Schnorr equivalent to $\sum_n c_n^+ - \sum_n c_n^-$. □

Pathak et al. [23] showed some properties of effective L^1-computable functions on a finite dimensional cube with the Lebesgue measure. We obtain the following generalizations to a computable metric space with a computable measure.

Corollary 4.4. *Let f be an effective L^1-computable function. Then $f(x)$ is defined for all Schnorr random points x.*

Proposition 4.5. *Let f, g be effective L^1-computable functions. Then f, g are Schnorr equivalent iff $||f - g||_1 = 0$.*

Proof. If f and g are Schnorr equivalent, then $\mu(\{x \mid f(x) \not\simeq g(x)\}) = 0$. Thus, $||f - g||_1 = 0$.

Suppose that $||f - g||_1 = 0$. There are integral tests t, u for Schnorr randomness such that $f - g =_{\mathrm{SR}} t - u$. Let $\{t_n\}$ and $\{u_n\}$ be their approximations by rational step functions according to Proposition 2.4. Since $\mu(t)$ and $\mu(u)$ are computable, we can assume that $||t - t_n||_1 \leq 2^{-n-1}$ and $||u - u_n||_1 \leq 2^{-n-1}$. Then

$$||t_n - u_n||_1 \leq ||t_n - t||_1 + ||t - u||_1 + ||u - u_n||_1 \leq 2^{-n}.$$

Since t_k and u_k are uniformly rational step functions, there exists a computable sequence $\{h_n\}$ of rational step functions such that h_n is Kurtz equivalent to $|t_k - u_k|$. Then $h = \lim_n \sum_{k \leq n} h_k$ is lower semicomputable. Since $||t_k - u_k||_1 \leq 2^{-k}$, h is an integral test for Schnorr randomness by Lemma 3.6. Then $h(x) < \infty$ for each Schnorr random point x. It follows that $|t_n(x) - u_n(x)| \to 0$ as $n \to \infty$. Hence, $(f - g)(x) = t(x) - u(x) = \lim_n (t_n(x) - u_n(x)) = 0$.
□

4.2. Weak L^1-computability

Here we give a Martin-Löf randomness version of Theorem 4.3. Since the class of integral tests is larger than the class of integral tests for Schnorr randomness, one needs a weaker notion than L^1-computability. Recall that an L^1-computable function is a computable point in the L^1-space equipped with the metric $|| \cdot ||_1$. Ambos-Spies, Weihrauch and Zheng [1] defined a *weakly computable* real as a real $r = \lim_n q_n$ for a computable sequence $\{q_n\}$ of rationals such that $\sum_n |q_{n+1} - q_n| < \infty$, and proved that a real is weakly computable iff it is the difference of two left-c.e. reals. We define a weakly computable point in the L^1-space as follows.

Definition 4.6. A function $f :\subseteq X \to \mathbb{R}$ is *weakly L^1-computable* if there exists a computable sequence $\{s_n\}$ of finite rational step functions such that $f(x) = \lim_n s_n(x)$ and $\sum_n ||s_{n+1} - s_n||_1 < \infty$.

Definition 4.7 (ML-equivalence). Two functions $f, g :\subseteq X \to \mathbb{R}$ are ML-equivalent (denoted by $f =_{\mathrm{MLR}} g$) if $f(x), g(x)$ are defined and equal for each Martin-Löf random point $x \in X$.

Proposition 4.8. *Every difference between two integral tests is ML-equivalent to a weakly L^1-computable function. Conversely, every weakly L^1-computable function f is ML-equivalent to a difference between two integral tests.*

Proof. First we prove the former half. Let $f = t - u$ where t, u are integral tests and $\{t_n\}, \{u_n\}$ be computable increasing approximations of t, u respectively. Let $s_n = t_n - u_n$. Then $\{s_n\}$ is a computable sequence of finite rational step functions. Furthermore,

$$\begin{aligned}\sum_n ||s_{n+1} - s_n||_1 =& \sum_n ||t_{n+1} - u_{n+1} - t_n + u_n||_1 \\ \leq& \sum_n ||t_{n+1} - t_n||_1 + \sum_n ||u_{n+1} - u_n||_1 \\ =& ||t||_1 + ||u||_1 < \infty.\end{aligned}$$

Finally, note that

$$f(x) = t(x) - u(x) = \lim_n t_n(x) - \lim_n u_n(x) = \lim_n (t_n(x) - u_n(x)) = \lim_n s_n(x)$$

for each ML-random point x.

Next we prove the latter half. Let $\{s_n\}$ be a computable sequence of finite rational step functions such that $g = \sum_n s_n$ and $\sum_n ||s_n||_1 < \infty$. Let $\{t_n\}$ be a computable sequence of finite rational cell functions such that s_n and

t_n are Kurtz equivalent for each n. Let t_n^+, t_n^- be the positive part and the negative part of t_n respectively. Since t_n is a rational cell function, so is t_n^+, thus $\sum_n t_n^+$ is a lower semicomputable function. Note that

$$\mu(\sum_n t_n^+) = \sum_n \mu(t_n^+) \leq \sum_n ||t_n||_1 = \sum_n ||s_n||_1 < \infty.$$

Hence, $\sum_n t_n^+$ is an integral test and so is $\sum_n t_n^-$. Then

$$g = \sum_n s_n =_{\mathrm{WR}} \sum_n t_n = \sum_n (t_n^+ - t_n^-) =_{\mathrm{MLR}} \sum_n t_n^+ - \sum_n t_n^-.$$

Hence, g is ML-equivalent to a difference between two integral tests. □

Corollary 4.9. *Let f be a weakly L^1-computable function. Then $f(x)$ is defined for all ML-random points x.*

Proposition 4.10. *Let f, g be the differences of two integral tests. Then f, g are ML-equivalent iff $||f - g||_1 = 0$.*

Proof. The "only if" direction is immediate.

Suppose that $||f - g||_1 = 0$. There exist two integral tests t, u such that $f - g =_{\mathrm{MLR}} t - u$. Let $\{t_n\}$ and $\{u_n\}$ be their computable approximations by rational step functions. Note that

$$||t_n - u_n||_1 \leq ||t_n - t||_1 + ||t - u||_1 + ||u - u_n||_1 \to 0$$

as $n \to \infty$ and $||t_n - u_n||_1$ is uniformly computable. Then there exists a computable sequence $\{n_i\}$ such that

$$||t_{n_i} - u_{n_i}||_1 \leq 2^{-i}.$$

Let h_i be a rational step function Kurtz equivalent to $|t_{n_i} - u_{n_i}|$ uniformly in i. Let $h = \sum_i h_i$. Then h is lower semicomputable. Note that $\mu(h) = \sum_i ||t_{n_i} - u_{n_i}||_1 < \infty$. Hence, h is an integral test. (In fact, h is an integral test for Schnorr randomness.)

Let $x \in X$ be Martin-Löf random. Then $f(x), g(x), t(x)$ and $u(x)$ are defined. Since $h(x) < \infty$, we have $|t_{n_i} - u_{n_i}|(x) \to 0$ as $i \to \infty$. Thus,

$$f(x) - g(x) = t(x) - u(x) = \lim_i t_{n_i}(x) - \lim_i u_{n_i}(x) = \lim_i (t_{n_i}(x) - u_{n_i}(x)) = 0.$$

This completes the proof. □

4.3. Computably approximable functions

Next we give a version of Proposition 4.8 and 4.10 for weak 2-randomness. A real is called *computable approximable* (or *c.a.*) if it is the limit of a computable sequence of rationals. We define a c.a. point in the L^1-space as follows.

Definition 4.11. A function $f :\subseteq X \to \mathbb{R}$ is *computably L^1-approximable* (or *c.a.*) if there exists a computable sequence $\{s_n\}$ of rational step functions such that

$$f(x) = \lim_n s_n(x)$$

and $f(x)$ is defined almost everywhere.

Note that a c.a. function may not be in L^1. Consider the function $f :\subseteq [0, 1] \to \mathbb{R}$ such that $f(x) = 1/x$. It is not difficult to see that a function is computably L^1-approximable iff it is limit computable [6] and defined almost everywhere.

Proposition 4.12. *Let $f :\subseteq X \to \mathbb{R}$ be a function whose domain is the set of weakly 2-random points. Then f is c.a. iff f is the difference between two integral tests for weak 2-randomness.*

The proofs are straightforward modifications of the proof of Theorem 4.8.

Definition 4.13. Two functions $f, g :\subseteq X \to \mathbb{R}$ are weakly 2-equivalent if $f(x), g(x)$ are defined and equal for each weakly 2-random point x.

Theorem 4.14. *Let f, g be the differences between two integral tests for weak 2-randomness. Then f, g are weakly 2-equivalent iff $||f - g||_1 = 0$.*

The proof is a straightforward modification of that of Theorem 4.10.

4.4. An application

The effectivization of L^1-computability makes some results more precise. We give an example.

Definition 4.15 (Hoyrup et al. [17]). A finite measure μ is *computably normable relative to* some other finite measure λ if the norm of the operator L_μ is computable from μ and λ.

We do not give the definition of the operator L_μ, since we do not use it explicitly in this paper.

Theorem 4.16 (Hoyrup et al. [17]). *Let μ, λ be such that $\mu \ll \lambda$ and μ is computably normable relative to λ. Then the Radon–Nikodym derivative $\frac{d\mu}{d\lambda}$ can be computed as an element of $L^1(\lambda)$ from μ and λ.*

In the proof, Hoyrup et al. [17] constructed a finite rational step function v_n such that $||h - v_n||_\lambda < 2^{-n}$ where h is a Radon–Nikodym derivative. By letting $g = \lim_n v_n$, g is a Radon–Nikodym derivative and an effective L^1-computable function. The following is immediate.

Theorem 4.17. *Let μ, λ be computable measures such that $\mu \ll \lambda$ and μ is computably normable relative to λ. Then there exists an effective $L^1(\lambda)$-computable Radon–Nikodym derivative f of μ with respect to λ. Furthermore, if g is another effective $L^1(\lambda)$-computable Radon–Nikodym derivative of μ with respect to λ, then f and g are Schnorr equivalent.*

This result is an effective version of the classical theorem that states that two Radon–Nikodym derivatives (for the same measures) are equal almost everywhere.

5. Schnorr layerwise computability

Layerwise computability [14, 15] has desirable properties for the study of effective probability theory. Hoyrup and Rojas [14] showed that, if a function is layerwise lower semicomputable and has a computable integral, then it is layerwise computable. In the following, we show that, if a function is lower semicomputable and has a computable integral, then it is Schnorr layerwise computable. Furthermore, the converse also holds in the sense of Theorem 5.2

Definition 5.1. A function $f :\subseteq X \to \mathbb{R}$ is *Schnorr layerwise computable* if there exists a Schnorr test $\{U_n\}$ such that the restriction $f|_{X \setminus U_n}$ is uniformly computable.

Theorem 5.2. *A function is Schnorr equivalent to a Schnorr layerwise computable function whose L^1-norm is computable iff the function is Schnorr equivalent to a difference between two integral tests for Schnorr randomness.*

Proof. (if direction) It suffices to show that an integral test f for Schnorr randomness is Schnorr layerwise computable. Let $\{s_n\}$ be a computable sequence of nonnegative finite rational step functions such that $||s_n||_1 \le 2^{-2n}$ and $f =_{\mathrm{WR}} \sum_n s_n$.

Let $U_n = \{x : s_n(x) > 2^{-n}\}$. Then U_n is uniformly c.e. Since $2^{-n}\mu(U_n) \le ||s_n||_1 \le 2^{-2n}$, we have $\mu(U_n) \le 2^{-n}$. Note that the real $\mu(U_n)$ is uniformly computable.

Let $V_k = \bigcup\{U_n : n > k\}$. Then $\mu(V_k) \le \sum_{n>k} \mu(U_n) \le \sum_{n>k} 2^{-n} = 2^{-k}$. The real $\mu(V_k)$ is uniformly computable by Lemma 3.6. Hence, $\{V_k\}$ is a Schnorr test.

Suppose $x \in X \setminus V_k$ and x is Schnorr random. Then $s_n(x) \le 2^{-n}$ for each $n > k$. Hence,

$$f(x) - \sum_{m=1}^{n} s_m(x) = \sum_{m=n+1}^{\infty} s_m(x) \le \sum_{m=n+1}^{\infty} 2^{-m} = 2^{-n}$$

for each $n > k$. Hence, $f(x)$ is computable from x and k.

Finally notice that, for integral tests f, g for Schnorr randomness, $||f - g||_1$ is computable by Theorem 4.3

(only if direction) Let f be a Schnorr layerwise computable function whose L^1-norm is computable. Then there exists a Schnorr test $\{U_n\}$ such that $f_n = f|_{X\setminus U_n}$ is uniformly computable. Let f_n' be total and uniformly lower semicomputable functions such that $f_n'|_{X\setminus U_n} = f_n$. Let $f_n'' = \min\{f_n', n\}$.

Let $t_n(x) = \sum\{k \mid x \in U_k,\ k \le n\}$. Then $\int \sup_n t_n d\mu = \sum_n n \cdot \mu(U_n)$ is computable. Let $t = \sup_n t_n$.

Let

$$g_n(x) = \begin{cases} t_n(x) & \text{if } x \in U_n \\ t_{n-1}(x) + f_n''(x) & \text{otherwise.} \end{cases}$$

Note that g_n is lower semicomputable. Let $g(x) = \sup_n g_n(x)$. Then g is lower semicomputable and

$$g(x) = \begin{cases} \infty & \text{if } x \in \bigcap_n U_n \\ t(x) + f(x) & \text{if } x \notin \bigcap_n U_n. \end{cases}$$

Note that $\mu(g) = \mu(t) + \mu(f)$ is computable. If x is Schnorr random, then $f(x) = g(x) - t(x)$. □

6. Solovay reducibility for lower semicomputable functions

6.1. Motivation

We begin with the following observation. Let f be a layerwise computable function. Then $f(x)$ is computable from x for each ML-random point x. In contrast, let g be a function such that $g(x) = \Omega$ for each $x \in X$ where Ω is Chaitin's omega. Then g is an integral test. However, $g(x)$ is not computable from x for each computable point x.

This means that, in this context, layerwise computability is not a Martin-Löf randomness version of Schnorr layerwise computability. To calculate the value $g(x)$, we need more precise information; the information of $x \notin U_k$ is insufficient because the information is finite. We need to know how close to $g(x)$ an approximation is. With this motivation, we introduce Solovay reducibility for lower semicomputable functions.

For simplicity, we generalize the following characterization of Solovay reducibility for left-c.e. reals.

Theorem 6.1 (Downey, Hirschfeldt and Nies [11]). *Let α, β be left-c.e. reals. Then α is Solovay reducible to β ($\alpha \le_S \beta$) iff there are a computable real d and a left-c.e. real γ such that $d\beta = \alpha + \gamma$.*

6.2. Some characterizations

Definition 6.2. Let f, g be nonnegative lower semicomputable functions. We say that f is Solovay reducible to g (denoted by $f \le_S g$) if there exists a computable real d and a nonnegative lower semicomputable function h such that

$$d \cdot g =_{\mathrm{WR}} f + h.$$

The following is immediate.

Proposition 6.3. *Let f, g be nonnegative lower semicomputable functions such that $f \leq_S g$.*

(i) *If g is a.e. computable, then so is f.*
(ii) *If g has a computable integral, then so does f.*
(iii) *If g is integrable, then so is f.*
(iv) *If $g(x) < \infty$ almost everywhere, then f also has the property.*

We give characterizations of classes of such functions.

Proposition 6.4. *Let f be a nonnegative lower semicomputable function that is bounded by a natural number M. Then f is a.e. computable iff $f \leq_S M$.*

Note that M is taken to be a constant function.

Proof. Suppose f is an a.e. computable function. Then there exists a nonnegative lower semicomputable function h Kurtz equivalent to $M - f$. It follows that $M = f + (M - f) =_{\mathrm{WR}} f + h$, which implies $f \leq_S M$.

Let $f \leq_S M$. Since M is a.e. computable in particular, so is f by the theorem above. □

Proposition 6.5. *There exists an integral test t such that a lower semicomputable function f is integrable iff $f \leq_S t$.*

Proof. Let f_n be a computable enumeration of nonnegative lower semicomputable functions such that $\mu(f_n) \leq 1$. Let $t = \sum_n 2^{-n} \cdot f_n$. Then $\mu(t) = \sum_n 2^{-n}\mu(f_n) < \infty$. Hence, t is integrable.

If $f \leq_S t$, then f is integrable. If f is integrable, then there exist natural numbers e, m such that $f_e = f/m$. Let $h = \sum_{n \neq e} 2^{-n} f_n$. Then

$$t = h + 2^{-e} f_e = h + 2^{-e} f/m \text{ and } m \cdot 2^e \cdot t = m \cdot 2^e \cdot h + f.$$

Hence, $f \leq_S t$. □

For a characterization of a nonnegative lower semicomputable function with a computable integral, we use the notion of a Solovay test. A Solovay test on the Cantor space was introduced by Solovay [26] and it characterizes Martin-Löf randomness. Schnorr randomness is characterized by a finite total Solovay test, which was introduced by Rupprecht; see [10]. The generalizations to a computable metric space were given by Gács, Hoyrup and Rojas [13], who called them a Borel–Cantelli test (BC-test) and a strong BC-test. However, here we call the latter a Solovay test for Schnorr randomness.

Definition 6.6. A *Solovay test for Schnorr randomness* is a sequence $\{U_n\}$ of uniformly c.e. open sets such that $\sum_n \mu(U_n)$ is computable.

Proposition 6.7 (Gács, Hoyrup and Rojas [13]). *A point is Schnorr random iff $x \in U_n$ for at most finitely many n for each Solovay test $\{U_n\}$ for Schnorr randomness.*

Theorem 6.8. *A nonnegative lower semicomputable function f has a computable integral iff there exist a computable sequence $\{a_n\}$ of natural numbers and a Solovay test $\{U_n\}$ for Schnorr randomness such that*

$$f \leq_S \sum_n a_n \cdot \mathbf{1}_{U_n}$$

and $\sum_n a_n \mu(U_n)$ is computable.

We have seen in Theorem 5.2 that, if a nonnegative lower semicomputable function f has a computable integral, then f is Schnorr layerwise computable. Note that Theorem 6.8 also implies this fact.

Proof. The "if" direction is immediate.

Suppose that $\mu(f)$ is computable. Let $\{s_n\}$ be a computable sequence of finite rational cell functions such that $f =_{\mathrm{WR}} \sum_n s_n$ and $||s_n||_1 \leq 2^{-2n}$ for each $n \geq 1$. The finite rational cell functions can be written in the form

$$s_n = \sum_{k=1}^{l_n} q_{n,k} \mathbf{1}_{\Gamma(\sigma_{n,k})}$$

where $q_{n,k} \in \mathbb{Q}$ and $\sigma_{n,k} \in 2^{<\omega}$. We separate each s_n into parts by the size of $q_{n,k}$. Let $s_{n,0}$ be the sum of the terms where $q_{n,k} \leq 2^{-n}$, $s_{n,1}$ be the sum of the terms where $2^{-n} < q_{n,k} \leq 2$, and $s_{n,i}$ be the sum of the terms where $2^{i-1} < q_{n,k} \leq 2^i$ for $i \geq 2$.

Let $g = s_0 + \sum_{n \geq 1} s_{n,0}$. Then g is bounded, is an a.e. computable function and has a computable integral. Then there exist a natural number M and a nonnegative lower semicomputable function h such that $M =_{\mathrm{WR}} g + h$. Note that $\mu(h)$ is computable.

We define $\{a_{n,k}\}$, $\{p_{n,k}\}$ and $\{U_{n,k}\}$ as follows.

$$\begin{aligned}
&a_{0,0} = M,\ p_{0,0} = 0,\ U_{0,0} = X,\\
&a_{0,k} = 0,\ p_{0,0} = 0,\ U_{0,k} = \emptyset \text{ for } k \geq 1,\\
&a_{n,k} = 0,\ p_{n,k} = 0,\ U_{n,k} = \emptyset \text{ if } q_{n,k} \leq 2^{-n},\\
&a_{n,k} = 2^i \text{ where } i \text{ is the least natural number such that } 2^{-n} < q_{n,k} \leq 2^i,\\
&p_{n,k} = 2^i - q_{n,k},\ U_{n,k} = \Gamma(\sigma_{n,k}) \text{ for } n \geq 1.
\end{aligned}$$

Then $\{a_n\}$ is a computable sequence of natural numbers.

We show that $\{U_{n,k}\}$ is a Solovay test for Schnorr randomness. Note that $\{U_{n,k}\}$ is a cell and has a uniformly computable measure. Since $||s_n||_1 \leq 2^{-2n}$ for each $n \geq 1$, we have

$$\sum_k \mu(U_{n,k}) = \mu(\bigcup_k U_{n,k}) = \mu(\bigcup_{k: q_{n,k} > 2^{-n}} U_{n,k}) \leq 2^{-n}$$

for $n \geq 1$. It follows that $\sum_{n,k} \mu(U_{n,k})$ is finite and computable.

We show that $f \leq_S \sum_{n,k} a_{n,k} \cdot \mathbf{1}_{U_{n,k}}$. This is because

$$\begin{aligned}
\sum_{n,k} a_{n,k} \cdot \mathbf{1}_{U_{n,k}} =& g + h + \sum_{n,k: n \geq 1 \& q_{n,k} > 2^{-n}} (q_{n,k} + p_{n,k}) \cdot \mathbf{1}_{\Gamma_{n,k}}\\
=& f + h + \sum_{n,k: n \geq 1 \& q_{n,k} > 2^{-n}} p_{n,k} \cdot \mathbf{1}_{\Gamma_{n,k}}.
\end{aligned}$$

Finally, we show that $\sum_{n,k} a_{n,k} \mu(U_{n,k})$ is computable. This is because

$$\begin{aligned}
\sum_k p_{n,k} \mu(U_{n,k}) \leq& \sum_{k: q_{n,k} \leq 2} 2\mu(U_{n,k}) + \sum_{k: q_{n,k} > 2} 2 q_{n,k} \mu(U_{n,k})\\
\leq& 2 \cdot 2^{-n} + 2 \cdot 2^{-2n} < 2^{-n+2}
\end{aligned}$$

for each $n \geq 1$, and thus, $\sum_{n,k} p_{n,k} \mu(U_{n,k})$ is computable. □

6.3. Computability for each point

Consider again the computability for each point.

Proposition 6.9. *Let f, g be nonnegative lower semicomputable functions. If $f \leq_S g$ and $g(x) < \infty$, then $f(x)$ is computable from x and $g(x)$.*

Proof. Since $f \leq_S g$, there exist a computable real d and a nonnegative lower semicomputable function h such that $d \cdot g =_{\mathrm{WR}} f + h$. By the fact that $g(x) < \infty$, we have $d \cdot g(x) = f(x) + h(x)$. Since $f(x)$ and $h(x)$ are left-c.e. from x, $f(x)$ can be computed from x and $g(x)$. □

Corollary 6.10. *Let f be a nonnegative lower semicomputable function. If f has a computable integral and x is Schnorr random, then $f(x)$ is computable from x. If f is integrable and x is ML-random, then $f(x)$ is computable from x and $t(x)$ where t is as in Theorem 6.5.*

Discussion

We have discussed four classes of functions which are closely related to weak 2-randomness, Martin-Löf randomness, Schnorr randomness and Kurtz randomness. These functions include the difference between the associated integral tests, and nice points in the L^1-space. The Solovay reducibility for c.e. reals was generalized to the function space and studied here.

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