The preordering related to uniform Schnorr randomness

RPTC2013
14 May 2013

Kenshi Miyabe
JSPS Research Fellow at The University of Tokyo

One line summary

In this talk, we give a uniform-Schnorr-randomness version of the equivalence between

$$A \leq_{LR} B \iff A \leq_{LK} B.$$

Lowness and triviality

The following are equivalent for a set $A \in 2^\omega$:

(i) A is K-trivial, $K(A \upharpoonright n) \leq K(n) + O(1)$.

(ii) A is low for Martin-Löf randomness, that is, each Martin-Löf random set is Martin-Löf random relative to A.

(iii) A is low for K, that is, $K(\sigma) \leq K^A(\sigma) + O(1)$.

(iv) A is a base for Martin-Löf randomness, that is, A is Turing reducible to a Martin-Löf random set relative to A.

(v) And many others.

Many by Nies.

$$A \leq_{LR} B \iff \mathrm{MLR}(B) \subseteq \mathrm{MLR}(A),$$

$$A \leq_{LK} B \iff K^B(\sigma) \leq K^A(\sigma) + O(1).$$

Kjos-Hanssen, Miller and Solomon (2012) proved that

$$A \leq_{LK} B \iff A \leq_{LR} B.$$

Open cover

Proposition

$A \leq_{LR} B \iff$ each $\Sigma^0_1(A)$ class G such that $\mu(G) < 1$ is contained in a $\Sigma^0_1(B)$ class S such that $\mu(S) < 1$.

Theorem (Bienvenu and Miller 2012)

The following are equivalent for a set $A \in 2^\omega$:

(i) $A \in \mathrm{Low}(\mathrm{MLR})$

(ii) For every A-left-c.e. summable function $f : \omega \to \mathbb{R}^+$, there exists a left-c.e. summable function g such that $f \leq g$.

f is *summable* if $\sum_n f(n) < \infty$.

The following are equivalent for a set $A \in 2^\omega$:

(i) A is Schnorr trivial, that is, $K_N(A \upharpoonright n) \leq K_M(n) + O(1)$.

(ii) A is low for uniform Schnorr randomness, that is, each Schnorr random set is Schnorr random uniformly relative to A. (Franklin and Stephan 2010)

(iii) A is low for u.c.m.m., that is, $K(\sigma) \leq K_{M^A}(\sigma) + O(1)$. (M. 2011)

(iv) A is a base for uniform Schnorr tests. (M. presented in the CCR2012)

Uniform Schnorr randomness

Definition

A uniform Schnorr test is a computable function $f : 2^\omega \times \omega \to \tau$ such that

- the function $\langle X, n\rangle \mapsto \mu(f(X,n))$ is computable,
- $\mu(f(X,n)) \leq 2^{-n}$ for all $X \in 2^\omega$ and $n \in \omega$.

The sequence $\{f(A,n)\}$ of open sets is called a Schnorr test uniformly relative to A. A set B is called Schnorr random uniformly relative to A if $B \notin \bigcap_n U_n^A$ for each Schnorr test $\{U_n^A\}$ uniformly relative to A.

Definition

An oracle prefix-free machine M is called a uniformly computable measure machine if $X \mapsto \mu(\llbracket \mathrm{dom}(M^X) \rrbracket)$ is computable.

Theorem

A set B is Schnorr random uniformly relative to A iff $K_{M^A}(B \upharpoonright n) > n - O(1)$ for all u.c.m.m. M.

Theorem

The following are equivalent for $A, B \in 2^\omega$:

(i) Every Schnorr random set uniformly relative to B is Schnorr random uniformly relative to A.

(ii) Every Schnorr test uniformly relative to A is covered by a Schnorr test uniformly relative to B.

(iii) For each u.c.m.m. M, there exists a u.c.m.m. N such that

$$K_{N^B}(\sigma) \leq K_{M^A}(\sigma) + O(1).$$

(iv) For each strictly bounded and uniformly Schnorr function $g : 2^\omega \to \tau$, there is a strictly bounded and uniformly Schnorr function $h : 2^\omega \to \tau$ such that $g(A) \subseteq h(B)$.

A computable function $g : 2^\omega \to \tau$ is called a uniformly Schnorr function if $X \mapsto \mu(g(X))$ is computable. A computable function $g : 2^\omega \to \tau$ is called strictly bounded if $\sup_{X \in 2^\omega} \mu(g(X)) < 1$.

(v) For every computable function $f : 2^\omega \times \omega \to \mathbb{R}^+$ such that the function $X \mapsto \sum_n f(X,n)$ is computable, there is a computable function $g : 2^\omega \times \omega \to \mathbb{R}^+$ such that the function $X \mapsto \sum_n g(X,n)$ is computable and $f(A,n) \leq g(B,n)$ for all n.

(iii)$\Rightarrow$(ii)$\Rightarrow$(i) $\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(iii)

(iii)$\Rightarrow$(ii)$\Rightarrow$(i) and (v)$\Rightarrow$(iii) are not difficult.

(i)$\Rightarrow$(iv) can be shown by the usual way.

(iv)$\Rightarrow$(v) is the hardest direction.

We identify

$$2^\omega = 2^{\omega\times\omega} = (2^\omega)^\omega = [0,1]^\omega.$$

Let

$$\mathcal{B}_{n,\alpha} = \{X \in [0,1]^\omega \ : \ X_n \in [0,\alpha)\}.$$

Let $f : 2^\omega \times \omega \to \mathbb{R}^+$ be a computable function such that $X \mapsto \sum_n f(X,n) \leq 1$ is computable. Consider $h : 2^\omega \to \tau$ defined by

$$h(X) = \bigcup_n \mathcal{B}_{n,f(X,n)}.$$

$$\mu(h(X)) = 1 - \prod_n (1 - \mu(\mathcal{B}_{n,f(X,n)})) = 1 - \prod_n (1 - f(X,n)).$$

$$\log(1 - \mu(h(X))) = \sum_n \log(1 - f(X, n)).$$

Then h is a strictly bounded and uniformly Schnorr function. By (iv), there is a strictly bounded and uniformly Schnorr function $k : 2^\omega \to \tau$ such that $h(A) \subseteq k(B)$.

One may want to define

$$g(X,n) = \sup\{\alpha \in [0,1] \ : \ \mathcal{B}_{n,\alpha} \subseteq k(X)\}.$$

Unfortunately, this does not work. Instead, we define

$$g(X,n) = \max\{\alpha \in [0,1] \ : \ \mu(\mathcal{B}_{n,\alpha} \setminus k(X,n)) \leq 2^{-n-c}\}$$

where $k(X,n)$ is the approximation of $k(X)$ within 2^{-n-c}. Then g is computable and $g(B,n) \geq f(A,n)$ because

$$\mathcal{B}_{n,f(A,n)} \subseteq h(A) \subseteq k(B),$$

$$\mu(\mathcal{B}_{n,f(A,n)} \setminus k(B,n)) \leq \mu(k(B) \setminus k(B,n)) \leq 2^{-n-c}.$$

$$\mu\Big(\bigcup_{n>N} \mathcal{B}_{n,g(X,n)} \setminus k(X,m)\Big) = (1-\mu(k(X,m)))\mu\Big(\bigcup_{n>N} \mathcal{B}_{n,g(X,n)}\Big).$$

$$1 - \prod_{n>N} (1 - g(X,n)) = \mu\Big(\bigcup_{n<N} \mathcal{B}_{n,g(X,n)}\Big) \leq \frac{2^{-N-c} + 2^{-m-c}}{\delta}$$

where

$$\sup_{X \in 2^\omega} \mu(k(X)) < 1 - \delta.$$

(vi) For every uniform Schnorr integral test $f : 2^\omega \times 2^\omega \to \mathbb{R}^+$, there is another uniform Schnorr integral test g such that $f(A, X) \leq g(B, X)$ for all X.

f is a uniform Schnorr integral test if f is lower semi computable function and $X \mapsto \int_{2^\omega} f(X, Z) d\mu(Z)$ is computable.

We can show (v)$\Rightarrow$(vi)$\Rightarrow$(i) by identifying

$$\omega = (2^{<\omega})^\omega.$$