Uniform relativization and almost uniform relativization

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The theme of this talk

Theorem (van Lambalgen 1987)

$A \oplus B$ is Martin-Löf random

$\iff$ A is Martin-Löf random

and B is Martin-Löf random relative to A.

$\Rightarrow$: easy direction

$\Leftarrow$: difficult direction

Overview

- uniform relativization
- van Lambalgen's theorem for uniform Kurtz randomness
- almost uniform relativization

Uniform relativization

Failure of vL-theorem for Schnorr

- “The analog of Theorem 6.9.1 fails, however. That is, there are sets A and B such that A+B is Schnorr random but A and B are not relatively Schnorr random,”
- in “7.1.2 Limitations” of Downey’s book
- Easy direction does not hold!!

Suppose B is not ML-random relative to A. Then there is a ML-test $\{U_n^A\}$ relative to A such that $B \in \bigcap_n U_n^A$. Let

$$V_n = \{X \oplus Y \ : \ Y \in \tilde{U}_n^X\}$$

where $\tilde{U}_n^X$ is U_n^X enumerated as long as $\mu(U_n^X) \leq 2^{-n}$. Then $\{V_n\}$ is a ML-test and $A \oplus B \in \bigcap_n V_n$.

Definition

A uniform Schnorr test is a computable function $f : 2^\omega \times \omega \to \tau$ such that $X \times n \mapsto \mu(f(X, n)) \leq 2^{-n}$ is computable.

We call $\{f(A, n)\}$ a Schnorr test uniformly relative to A.

B is Schnorr random uniformly relative to A if B passes all Schnorr tests uniformly relative to A.

Theorem (M. 2011 and M.-Rute 2013)

$A \oplus B$ is Schnorr random

$\iff$ A is Schnorr random

and B is Schnorr random uniformly relative to A.

Why uniform not tt?

- It can take reals not only rationals.
- tt is very sensitive

Proposition

The following are equivalent:

(i) X is not Schnorr random.

(ii) There are a computable martingale d and a computable function such that $d(X \upharpoonright f(n)) \geq n$ for infinitely many n.

(iii) There are a computable martingale d and a computable order f such that $d(X \upharpoonright n) \geq f(n)$ for infinitely many n.

Proposition

The following are equivalent:

(i) B is not Schnorr random uniformly relative to A

(ii) There are a computable function $d : 2^\omega \times 2^{<\omega} \to \mathbb{R}^+$ and a computable function $f : 2^\omega \times \omega \to \omega$ such that $d^X = d(X, -)$ is a martingale for each $X \in 2^\omega$ and $d^A(B \upharpoonright f^A(n)) \geq n$ for infinitely many n.

(iii) There are a martingale $d \leq_{tt} A$ and a computable function $f : \omega \to \omega$ such that $d(B \upharpoonright f(n)) \geq n$ for infinitely many n.

Consider the following statement:

There are a martingale $d \leq_{tt} A$ and an order function $f : \omega \to \omega$ truth-table reducible to A such that $d(B \upharpoonright n) \geq f(n)$ for infinitely many n.

Essentially by Stephan and Franklin (2010), there exist A and B such that B is Schnorr random uniformly relative to A but the above statement holds.

Proof

Definition (M.)

A Schnorr integral test is a nonnegative lower semicomputable function $t : X \to \overline{\mathbb{R}}^+$ with a computable integral $\int t \, d\mu$.

Theorem (M.)

A point z is Schnorr random iff $t(z) < \infty$ for each Schnorr integral test t.

Definition

A uniform Schnorr integral test is a nonnegative lower semi-computable function $t : 2^\omega \times 2^\omega \to \overline{\mathbb{R}}^+$ such that $X \mapsto \int t(X, Y)\ \mu(Y)$ is a computable function.

Proposition

A point B is Schnorr random uniformly relative to A off $t(A, B) < \infty$ for each uniform Schnorr integral test t.

Theorem

If $A \oplus B$ is Schnorr random, then B is Schnorr random uniformly relative to A.

Proof

Let $t : 2^\omega \times 2^\omega \to \overline{\mathbb{R}}^+$ be a uniformly Schnorr integral test such that $t(A, B) = \infty$. Let

$$h(X) = \int t(X, Y)\ \mu(Y).$$

Then $h(X)$ is a computable function. We can assume that $h(X) \leq 1$ for all $X \in 2^\omega$. Let $s(X, Y) = 1 - h(X)$. Then $t + s$ is a Schnorr integral test on $2^\omega \times 2^\omega$. Thus $A \oplus B$ is not Schnorr random.

Lemma

Let t be a nonnegative lower semicomputable function with a computable integral $\int t \; d\mu$. There is a uniformly computable sequence $\{h_n\}$ of total computable functions $h_n : 2^\omega \to [0, \infty)$ such that $h_n \leq t$ everywhere and if A is Schnorr random, there is some n such that $h_n(A) = t(A)$.

Definition (M.)

A function $f :\subseteq X \to \mathbb{R}$ is Schnorr layerwise computable if there exists a Schnorr test $\{U_n\}$ such that the restriction $f|_{X \setminus U_n}$ is uniformly computable.

Theorem (M.)

A function is Schnorr equivalent to a Schnorr layerwise computable function whose L^1-norm is computable iff the function is Schnorr equivalent to a difference between two Schnorr integral tests.

Theorem

If A is Schnorr random and B is Schnorr random uniformly relative to A, then $A \oplus B$ is Schnorr random.

Proof

Suppose that (A, B) is not Schnorr random and A is Schnorr random. Then there is a Schnorr integral test t such that $t(A, B) = \infty$. Let

$$u(X) = \int t(X \oplus Y)\ \mu(Y).$$

Then u is lower semicomputable with a computable integral. By the lemma, there is a total computable function h such that $h \leq u$ everywhere and $h(A) = u(A)$.

Let $\tilde{t}$ be t enumerated as long as $\int \tilde{t}(X,Y)\ \mu(Y) \leq h(X)$. Since $h \leq u$, the equality holds. Thus $\tilde{t}$ is a uniform Schnorr integral test.

Furthermore, $\tilde{t}(A,Y) = t(A,Y)$ for all Y. Then $\tilde{t}(A,B) = t(A,B) = \infty$. Hence B is not Schnorr random uniformly relative to A.

Other randomness notions

- vL-theorem holds for
- - MLR for usual relativization
- - SR for uniform relativization

Let $\mathcal{R}$ be a relativizable randomness notion.

Then van Lambalgen's theorem holds for $\mathcal{R}$ if

$$A \oplus B \in \mathcal{R} \iff A \in R \wedge B \in \mathcal{R}^A.$$

Thus such notion is essentially unique.

I previously claimed that "such relativization is the natural one", which is called Miyabe's thesis in Diamondstone-Greenberg-Turetsky's paper.

Definition

A uniformly computable martingale is $d : 2^\omega \times 2^{<\omega} \to \mathbb{R}^+$ such that $d(X, -)$ is a martingale for each X. B is computably random uniformly relative to A if $\sup_n d(A, B \upharpoonright n) < \infty$ for each uniformly computable martingale.

Theorem (M.-Rute)

$A \oplus B$ is computably random

$\iff$ A and B are computably random

uniformly relative to each other.

Van Lambalgen's theorem holds for Kolmogorov-Loveland randomness in a similar form.

The natural relativization should hold at least

$$A \oplus B \in \mathcal{R} \iff A \in \mathcal{R}^B \wedge B \in \mathcal{R}^A.$$

Definition

A Demuth test is a sequence of c.e. open sets $\{V_n\}$ such that $\mu(V_n) \leq 2^{-n}$ for all n, and there is an ω-c.e. function f such that $V_n = [\![W_{f(n)}]\!]$.

A $\text{Demuth}_{\text{BLR}}$ test is a Demuth test relative to A where f is ω-c.e. by A, that is, the approximation is A-computable but the bound on the number of changes is computable.

Theorem (Diamondstone-Greenberg-Turetsky)

Van Lambalgen's theorem holds for $\text{Demuth}_{\text{BLR}}$ randomness.

- Demuth_BLR is equivalent to uniform Demuth randomness.

- MLR is equivalent to uniform MLR.

- full relativization vs partial relativization
- uniform relativization vs partial relativization
- total relativization?

Summary

- Uniform relativization may be the natural relativization for all randomness notions.