Uniform Kurtz randomness

van Lambalgen's theorem

- holds for (uniform) ML-randomness
- holds for uniform Schnorr randomness
- holds for uniform computable randomness
- holds for uniform Demuth randomness

Kurtz randomness

Theorem(Franklin-Stephan '11)

- If A is Kurtz random and B is A-Kurtz random, then $A \oplus B$ is Kurtz random.
- There exists a pair A, B such that $A \oplus B$ is Kurtz random and neither A nor B is Kurtz random relative to the other.

The "difficult direction" holds but the "easy direction" does not hold.

Definition

A uniform Kurtz test is a total computable function $f : 2^\omega \to \tau$ such that $\mu(f(Z)) = 1$ for all $Z \in 2^\omega$.

A set B is called Kurtz random uniformly relative to A if $B \in f(A)$ for each uniform Kurtz test f.

Definition

A uniform Kurtz null test is a computable function $f : 2^\omega \times \omega \to (2^{<\omega})^{<\omega}$ such that, for each $Z \in 2^\omega$ and $n \in \omega$, $\mu([\![f(Z,n)]\!]) \leq 2^{-n}$. For a fixed set X, we say that $\{[\![f(X,n)]\!]\}$ is a Kurtz null test uniformly relative to X.

Proposition

B is Kurtz random uniformly relative to A off B passes all Kurtz null tests uniformly relative to A.

Proposition

The following are equivalent for sets A and B:

1. B is not Kurtz random uniformly relative to A.
2. There are a computable function $d : 2^\omega \times 2^{<\omega} \to \mathbb{R}^+$ and a computable order h such that $d(Z, -)$ is a martingale for each $Z \in 2^\omega$ and $d(A, B \restriction n) > h(n)$ for all $n \in \omega$.
3. There are an oracle prefix-free machine M and a computable function h such that $Z \mapsto \mu(\mathrm{dom}(M^Z))$ is a computable function and $K_{M^A}(B \restriction h(n)) < h(n) - n$ for all $n \in \omega$.

easy direction

Theorem (M.-Kihara)

If $A \oplus B$ is Kurtz random,

then B is Kurtz random uniformly relative to A.

Proof

Supppose B is not Kurtz random uniformly relative to A. Then there is a total computable function $f : 2^\omega \to \tau$ such that $\mu(f(Z)) = 1$ for all $Z \in 2^\omega$ and $B \notin f(A)$. We define a c.e. set U by

$$U = \{X \oplus Y \ : \ Y \in f(X)\}.$$

Then $\mu(U) = 1$ and $A \oplus B \notin U$. Hence $A \oplus B$ is not Kurtz random.

Corollary

There is a pair $A, B \in 2^\omega$ such that B is Kurtz random uniformly relative to A and not Kurtz random relative to A.

difficult direction

Theorem (M.-Kihara)
There is a pair A, B such that A and B are mutually uniformly Kurtz random and $A \oplus B$ is not Kurtz random.

So, the "easy direction" does hold but the "difficult direction" does not hold!!

Lemma

If $A(n) = 0$ or $B(n) = 0$ for all n, then $A \oplus B$ is not Kurtz random.

Proof

Let $\{f_i\}$ be an enumeration of all uniform Kurtz tests. At stage s, we define $\alpha_s \prec A$ and $\beta_s \prec B$ such that $|\alpha_s| = |\beta_s|$.

At stage $s = 2i$, search $\beta \succeq \beta_s$ and m such that

$$[\![\beta]\!] \subseteq f_i(\alpha_s 0^m).$$

Such β and m always exist. We assume $|\alpha_s 0^m| \geq |\beta|$. Define

$$\alpha_{s+1} = \alpha_s 0^m,\ \beta_{s+1} = \beta 0^{|\alpha_s|+m-|\beta|}.$$

At stage $s = 2i+1$, define α_{s+1} and β_{s+1} similarly by replacing α and β.

Weaker form

Mekle's criterion

Theorem (due to Merkle)

The following are equivalent for a se $X \in 2^\omega$.

1. X is not ML-random.
2. $X = x_0 x_1 x_2 \cdots$ for a sequence $\{x_i\}$ of strings such that $K(x_i) \leq |x_i| - 1$ for all i.
3. There is a prefix-free machine M such that $X = x_0 x_1 x_2 \cdots$ for a sequence $\{x_i\}$ of strings such that $K_M(x_i) \leq |x_i| - 1$ for all i.

Schnorr version

Theorem

The following are equivalent for a se $X \in 2^\omega$.

1. X is not Schnorr random.
2. There is a c.m.m. M such that $X = x_0 x_1 x_2 \cdots$ for a sequence $\{x_i\}$ of strings such that $K_M(x_i) \leq |x_i| - 1$ for all i.

$$X \upharpoonright [m,n) = X(m)X(m+1)\cdots X(n-1) \in 2^{n-m}.$$

Theorem (M.-Kihara)

The following are equivalent for a set $X \in 2^\omega$.

1. X is not Kurtz random.
2. There exists a computable order l and a c.m.m. M such that

$$K_M(X \upharpoonright [l(n), l(n+1)) \leq l(n+1) - l(n) - 1$$

for all n.

"c.m.m." can be replaced by "prefix-free decidable machine".

Suppose that A is not Kurtz random. Then there is a computable function $f : \omega \to (2^{<\omega})^{<\omega}$ and a computable order u such that

1. $f(n) \subseteq 2^{u(n)}$,
2. $|f(n)| = 2^{u(n)-n}$,
3. $X \upharpoonright u(n) \in f(n)$.

We assume $u(0) = 0$ and u is strictly increasing.

Let

$$k(0) = 0,\ k(n+1) = u(k(n)) + n + 2,\ l(n) = u(k(n)).$$

Construct a KC set

$$\langle l(n+1) - (n) - 1, \sigma \restriction [l(n), l(n+1)) \rangle_{n \in \omega, \sigma \in f(k(n+1))}.$$

For $\sigma \in f(k(n+1))$, we have $|\sigma| = u(k(n+1)) = l(n+1)$. The weight is

$$\sum_{n} \sum_{\sigma \in f(k(n+1))} 2^{-(l(n+1)-l(n)-1)} = 1.$$

The constructed machine M has a computable measure.

Note that $\sigma_n = X \upharpoonright u(n) \in f(n)$. Then there is $\tau \in 2^{l(n+1)-l(n)-1}$ such that

$$M(\tau_n) = X \upharpoonright [l(n), l(n+1)).$$

Thus, $K_M(X \upharpoonright [l(n), l(n+1))) \leq l(n+1) - l(n) - 1$ for each n.

Suppose that there is a computable order u and a c.m.m. M such that

$$K_M(A \upharpoonright [l(n), l(n+1))) \leq l(n+1) - l(n) - 1$$

for all n. We construct a Kurtz null test as follows. Let $S_0 = \{\lambda\}$ and

$$S_{n+1} = \{M(\sigma) \,:\, \sigma \in 2^{<l(n+1)-l(n)},\ M(\sigma) = l(n+1)-l(n)\}.$$

Since M is prefix-free, we have $\mu([\![S_{n+1}]\!]) \leq 2^{-1}$.

We define $f : \omega \to (2^{<\omega})^{<\omega}$ by

$$f(n) = \{x_1 \cdots x_n \ : \ x_i \in S_i \text{ for } i = 1, \cdots, n\}.$$

Then

$$\mu([\![f(n)]\!]) = \prod_{i=1}^{n} \mu([\![S_i]\!]) \leq 2^{-n}.$$

Thus $\{[\![f(n)]\!]\}_n$ is a Kurtz null test. Since $A \in [\![f(n)]\!]$ for all n, A is not Kurtz random.

- This intuitively says that a set is not Kurtz random iff there is a computable separation each of which has regularity.

- Thus, even if one can find a such computable separation of A+B, one may not find such separation in neither of A nor B.

computable union

Definition (M.)

Let $h, g : \omega \to \omega$ be strictly increasing computable functions such that

$$\omega = \{h(n) \ : \ n \in \omega\} \cup \{g(n) \ : \ n \in \omega\}.$$

We write $A \oplus_h B$ to mean the set X such that

$$X(h(n)) = A(n), \ X(g(n)) = B(n).$$

Theorem (M.-Kihara)

The following are equivalent for a set $X \in 2^\omega$.

1. X is Kurtz random.
2. For each computable union $\oplus_h$, letting $X = A \oplus_h B$, the sets A, B are mutually uniform Kurtz random.
3. For each computable union $\oplus_h$, letting $X = A \oplus_h B$, at least one of A and B is Kurtz random.