Almost uniform relativization

Theorem (M.-Kihara)

The following are equivalent for a set $X \in 2^\omega$.

1. X is Kurtz random.
2. For each computable union $\oplus_h$, letting $X = A \oplus_h B$, the sets A, B are mutually uniform Kurtz random.
3. For each computable union $\oplus_h$, letting $X = A \oplus_h B$, at least one of A and B is Kurtz random.

Question

- Is this theorem really natural?

- The usual relativization is too strong for the easy direction to hold.
- The uniform relativization may be too weak for the difficult direction to hold

Theorem (Frankline and Stephan '11)

If A is Kurtz random and B is A-Kurtz random, then $A \oplus B$ is Kurtz random.

Proof

Let A be a Kurtz-random set and U be an arbitrary c.e. open set U with measure 1. For each rational $r < 1$, let

$$U_r = \{P \ : \ \mu(\{Q \ : \ P \oplus Q \in U\}) > r\}.$$

Then U_r is a c.e. open set.

For each r, we have $\mu(U_r) = 1$.

Since A is Kurtz random, $A \in U_r$ for each r. Let

$$T = \{Q \ : \ A \oplus Q \in U\}.$$

Then T is a A-c.e. open set with measure 1. Since B is A-Kurtz random, we have $B \in T$. Hence $A \oplus B \in U$. Since U is arbitrary, $A \oplus B$ is Kurtz random.

Definition

A almost uniform (a.u.) Kurtz test is a computable function $f : 2^\omega \to \tau$ such that $\mu(f(Z)) = 1$ for almost every $Z \in 2^\omega$. A set B is Kurtz random a.u. relative to A if $B \in f(A)$ for each a.u. Kurtz test f such that $\mu(f(A)) = 1$.

random $\Rightarrow$ a.u. random $\Rightarrow$ uniformly random

Theorem (M.)

$A \oplus B$ is Kurtz random iff A is Kurtz random and B is Kurtz random a.u. relative to A.

Proposition (M.)

If $A \oplus B$ is Kurtz random, then B is Kurtz random a.u. relative to A.

Proof

Suppose B is not Kurtz random a.u. relative to A. Then there is a.u. Kurtz test f such that $B \notin f(A)$ and $\mu(f(A)) = 1$. Let

$$U = \{X \oplus Y \ : \ Y \in f(X)\}.$$

Then U is a c.e. open set with measure 1. Since $A \oplus B \notin U$, $A \oplus B$ is not Kurtz random.

Proposition (M.)

If A is Kurtz random and B is Kurtz random a.u. relative to A, then $A \oplus B$ is Kurtz random.

Proof

Let U be a c.e. open set with measure 1. Consider

$$g(X) = \{Y \ : \ X \oplus Y \in U\}.$$

Then g is computable. Since U is measure 1, $g(X)$ has measure 1 for almost every X. Hence g is an a.u. Kurtz test.

Note that $g(A)$ has measure 1. For each rational $r < 1$, let

$$V_r = \{X \ : \ \mu(g(X)) > r\}.$$

Then V_r is a c.e. open set with measure 1. Then $A \in V_r$ for each r.

Since B is Kurtz random a.u. relative to A, we have $B \in g(A)$, which implies $A \oplus B \in U$. Hence $A \oplus B$ is Kurtz random.

Definition

An a.u. weak n-test is a computable function $f : 2^\omega \to \Sigma^0_n$ such that $\mu(f(Z)) = 1$ for almost every $Z \in 2^\omega$. A set B is weakly n-random a.u. relative to A if $B \in f(A)$ for each a.u. weak n-test f such that $\mu(f(A)) = 1$.

Definition (Brattka 2005)

Let (X, d, α) be a separable metric space. We define representations $\delta_{\Sigma^0_k(X)}$ of $\Sigma^0_k(X)$, $\delta_{\Pi^0_k(X)}$ of $\Pi^0_k(X)$ for $k \geq 1$ as follows:

- $\delta_{\Sigma^0_1(X)}(p) := \bigcup_{(i,j)\ll(p)} B(\alpha(i), \overline{j})$,
- $\delta_{\Pi^0_k(X)}(p) := X \setminus \delta_{\Sigma^0_k(X)}(p)$,
- $\delta_{\Sigma^0_{k+1}(X)}\langle p_0, p_1, p_2, \cdots\rangle := \bigcup_{i=0}^{\infty} \delta_{\Pi^0_k(X)}(p_i)$,

for all $p, p_i \in \omega^\omega$.

Theorem (M.)

$A \oplus B$ is weak n-random iff A is weak n-random and B is weak n-random a.u. relative to A.

Question

- Is almost uniform relativization a natural notion?

		a.u.	uniform
Demuth	Fail	?	Hold
weak 2	Fail	Hold	?
ML	Hold	Hold	Hold
computable	Fail	?	Hold in a weak sense
Schnorr	Fail	Hold	Hold
Kurtz	Fail	Hold	Fail

Lowness

Question

- Is lowness for a.u. Kurtz randomness equivalent to lowness for uniform Kurtz randomness?

$$\mathrm{W2R} \subsetneq \mathrm{auW2R} \subseteq ?\mathrm{uW2R}$$

$$\mathrm{MLR} = \mathrm{auMLR} = \mathrm{uMLR}$$

$$\mathrm{SR} \subsetneq \mathrm{auSR} = \mathrm{uSR}$$

$$\mathrm{WR} \subsetneq \mathrm{auWR} \subsetneq \mathrm{uWR}$$

$\text{Low}(R, S)$ is the set of oracles A such that $R \subseteq S^A$.

$$\text{Low}(\text{W2R}) = \text{Low}(\text{W2R}, \text{MLR}) = \text{Low}(\text{MLR}).$$

$$S \subseteq S' \Rightarrow \text{Low}(R, S) \subseteq \text{Low}(R, S'),$$

$$R \subseteq R' \Rightarrow \text{Low}(R, S) \supseteq \text{Low}(R', S),$$

$$\begin{aligned}\mathrm{Low}(\mathrm{auW2R}) &= \mathrm{Low}(\mathrm{auW2R}, \mathrm{auW2R}) \\ &= \mathrm{Low}(\mathrm{W2R}, \mathrm{auW2R}) \\ &\subseteq \mathrm{Low}(\mathrm{W2R}, \mathrm{auMLR}) \\ &= \mathrm{Low}(\mathrm{W2R}, \mathrm{MLR}) \\ &= \mathrm{Low}(\mathrm{MLR}).\end{aligned}$$

$$\begin{aligned}\mathrm{Low}(\mathrm{auW2R}) &= \mathrm{Low}(\mathrm{W2R}, \mathrm{auW2R}) \\ &\supseteq \mathrm{Low}(\mathrm{W2R}, \mathrm{W2R}) \\ &= \mathrm{Low}(\mathrm{MLR}).\end{aligned}$$

		a.u.	uniform
Demuth	studied	?	studied
weak 2	K-trivial	K-trivial	K-trivial
ML	K-trivial	K-trivial	K-trivial
computable	computable	?	?
Schnorr	Low(SR)	?	Schnorr trivial
Kurtz	studied	?	studied

Question

- Is there any relation with "uniformly almost everywhere dominating"?
- u.a.e.d iff High(W2R,2MLR))
- Study High(W2R,W3R)

Question

- MLR - T
- SR - tt
- WR, W2R - au?

Definition

$A \leq_{\mathrm{au}} B$ if there is a reduction Φ such that $A = \Phi^B$ and

$$\Phi^Z(n)\downarrow \text{ for all } n \text{ almost every } Z \in 2^\omega.$$

$$\leq_{\mathrm{tt}} \Rightarrow \leq_{\mathrm{au}} \Rightarrow \leq_{\mathrm{T}} .$$