L^1-computability and the computability of conditional probability

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The topic

- differentiability
- van Lambalgen's theorem for non-uniform measures

Overview

- integral test
- L^1-computability
- convergence of martingales
- differentiability
- layerwise computability
- conditional probability

Integral tests and L^1-computability

integral test

Definition

An integral test is a lower semicomputable function $f : [0,1] \to \overline{\mathbb{R}}^+$ such that $\int f \, d\mu < \infty$.

Theorem (see the book by Li & Vitányi)

A real $x \in [0,1]$ is ML-random if and only if $f(x) < \infty$ for each integral test f.

Theorem (M. 20xx)

A real $x \in [0,1]$ is weakly 2-random if and only if $f(x) < \infty$ for each loser semicomputable function f such that f is finite almost everywhere.

Theorem (M. 20xx)

A real $x \in [0,1]$ is Schnorr random if and only if $f(x) < \infty$ for each lower semicomputable function $f : [0,1] \to \overline{\mathbb{R}}^+$ such that $\int f \, d\mu$ is a computable real.

Theorem (M. 2013)

A real $x \in [0,1]$ is Kurtz random if and only if $f(x) < \infty$ for each extended computable function $f : [0,1] \to \overline{\mathbb{R}}^+$ such that $\int f \, d\mu$ is a computable real.

Definition (Pour-El& Richards, Pathak et al, M. 20xx)

A function $f \;:\subseteq\; [0,1] \to \mathbb{R}$ is called effectively L^1-computable if there is a computable sequence $\{f_n\}$ of rational step functions such that $||f_n - f_{n-1}||_1 \leq 2^{-n-1}$ for each n and $f(x) = \lim_n f_n(x)$.

Theorem (M. 20xx)

The following are equivalent up to Schnorr null:

1. the difference between two Schnorr integral tests
2. an effectively L^1-computable function

Let $f :\subseteq 2^\omega \to \mathbb{R}$ be an effectively L^1-computable function. Then,

$$M(\sigma) = \frac{\int_{[\sigma]} f \ d\mu}{\mu([\sigma])}$$

is a computable martingale.

1. Such a martingale M converges along each Schnorr random real.
2. Each Schnorr random real is a Lebesgue point for each L^1-computable function.
3. A function that is a computable point in the variation norm is differentiable at each Schnorr random point.

Roughly speaking,

the difference between two Kurtz integral tests

= a.e. computable functions.

Theorem (M. 2013)

The following are equivalent for z:

1. z is Kurtz random.
2. z is a Lebesgu point for each a.e. computable function.

finite for integral tests	integral test	difference btw two integral tests	induced martingale	integral of integral tests	Lebesgue point
W2R	lower semicomp. + finite a.e.	?	?	?	?
MLR	lower semicomp. + integrable	?	?	?	?
CR	-	-	comp. mar	non-dec. comp. func	CR
SR	lower semicomp. + comp. integral	L^1-comp	-	comp. in variation norm	SR
WR	extended computable + integrable	a.e. comp	-	differentiab le a.e.	WR

A real $r \in [0,1]$ is computable if there is a computable sequence $\{q_n\}$ of rationals such that $|q_n - q_{n-1}| \leq 2^{-n-1}$ for each n and $r = \lim_n q_n$.

r is called weakly computable if replace $\sum_n |q_n - q_{n-1}| < \infty$ in the above. It is equivalent to the difference between two left-c.e. reals.

r is called computably approximable (c.a.) if $r = \lim_n q_n$ for a computable sequence $\{q_n\}$ of rationals. It is equivalent to Δ^0_2 and $\leq_T \emptyset'$.

Definition (M. 20xx)

A function $f :\subseteq [0,1] \to \mathbb{R}$ is called weakly L^1-computable if there is a computable sequence $\{f_n\}$ of rational step functions such that $\sum_n ||f_n - f_{n-1}||_1 < \infty$ and $f(x) = \lim_n f_n(x)$.

Definition (M. 20xx)

A function $f :\subseteq [0,1] \to \mathbb{R}$ is called L^1-c.a. if there is a computable sequence $\{f_n\}$ of rational step functions such that $f(x)$ is defined almost everywhere and $f(x) = \lim_n f_n(x)$.

Theorem (M. 20xx)

The following are equivalent up to ML-null:

1. the difference between two integral tests
2. a weakly L^1-computable function

Theorem (M. 20xx)

The following are equivalent up to W2-null:

1. the difference between two weak 2-integral tests
2. an L^1-c.a. function

finite for integral tests	integral test	difference btw two integral tests	induced martingale	integral of integral tests	Lebesgue point
W2R	lower semicomp. + finite a.e.	L^1-c.a.	?	?	?
			left-c.e.	interval c.e.	density-one
MLR	lower semicomp. + integrable	weak L^1-comp	-	-	density-one
MLR	lower semicomp. + integrable	-	-	comp. with bounded variation	MLR

Layerwise computability and conditional probability

Theorem (Hoyrup & Rojas 2009)

Let $f : X \to \overline{\mathbb{R}}^+$ be a layerwise lower semi-computable function. If $\int f \, d\mu$ is computable then f is layerwise computable.

Definition (M. 20xx)

A function $f :\subseteq [0,1] \to \mathbb{R}$ is called Schnorr layerwise computable if there is a Schnorr test $\{U_n\}$ such that $f|_{[0,1]\setminus U_n}$ is uniformly computable.

Theorem (M. 20xx)

The following are equivalent up to Schnorr null:

1. an effectively L^1-computable function
2. a Schnorr layerwise computable function with a computable integral

- Does van Lambalgen’s theorem hold for non-uniform measures?

Let μ be a measure on $2^\omega \times 2^\omega$. The first marginal μ_1 is defined by

$$\mu([\sigma]) = \mu([\sigma] \times 2^\omega).$$

The conditional measure μ_2^x with respect to x is defined by

$$\mu_2^x([\tau]) = \lim_n \frac{\mu([x \upharpoonright n] \times [\tau])}{\mu([x \upharpoonright n] \times 2^\omega)}.$$

Theorem (Bienvenu-Hoyrup-Shen)

Let μ be a computable measure on $2^\omega \times 2^\omega$. Consider the following statement:

(x, y) is μ-random if and only if x is μ_1-random and y is μ_2^x-random relative to x.

1. This holds if $x \mapsto \mu_2^x$ is layerwise computable.
2. This does not hold in general.
3. This holds for 2-randomness.

Observation

Let d be a computable martingale. Then, $\lim_n d(x \upharpoonright n)$ is not computable from x in general. Probably, there is a computable martingale d such that, for all x,

$$\lim_n d(x \upharpoonright n) \equiv_T x'.$$

Let d be a martingale with L^1-computable density. Then, the function

$$x \mapsto \lim_n d(x \upharpoonright n)$$

is effectively L^1-computable.

$$\mu \longrightarrow x \mapsto \mu_2^x$$

$$f \longrightarrow x \mapsto f_2^x$$

Show the following:

1. If f is effectively $L^1(\lambda \times \lambda)$-computable, then $x \mapsto f_2^x$ is effectively $L^1(\mu_1)$-computable.
2. If g is effectively L^1-computable, it is Schnorr layer-wise computable.
3. $h \mapsto h\lambda$ is computable.

Definition (Brattka 2001)

A represented space (x, δ) is called a computable vector space if $(X, +, \cdot, 0)$ is a vector space such that (i)$+$ is computable, (ii)$\cdot$ is computable, (iii)0 is a computable point.

A tuple $(X, ||\cdot||, e)$ is called a computable normed space if (i)$||\cdot||$ is a norm, (ii)the linear span of the range of e is dense, (iii)(X, d, α_e) is a computable metric space, (iv)(x, δ_X) is a computable vector space.

If (x, d) is a complete metric space, then we call it a computable Banach space.

Proposition

Let X be a computable metric space and μ be a computable measure on it. Then, the L^1-space equipped with a structure is a computable Banach space.

Proposition

1. If f is effectively $L^1(\lambda \times \lambda)$-computable, then $x \mapsto f_2^x$ is effectively $L^1(\mu_1)$-computable.
2. If g is effectively L^1-computable, it is Schnorr layerwise computable.
3. $h \mapsto h\lambda$ is computable.

Theorem

Let μ be a measure on $2^\omega \times 2^\omega$ with an L^1-computable density f w.r.t. λ. Then, μ_2^x has density f_2^x w.r.t. λ up to μ_1-Schnorr null.

To prove this we use an effective Levy 0-1 law.

Theorem

If μ has an L^1-computable density, then $x \mapsto \mu_2^x$ is μ_1-Schnorr layerwise computable.

Theorem (M. with a result in BHS)
Let μ be a measure with an L^1-computable density. Then van Lambalgen's theorem holds for Martin-Löf randomness and μ.

Theorem (M. with a result by Rute)
Let μ be a measure with an L^1-computable density. Then van Lambalgen's theorem holds for uniform Schnorr randomness and μ.

$$\mu \longrightarrow x \mapsto \mu_2^x$$

$$f \longrightarrow x \mapsto f_2^x$$

Theorem

Let f be a Schnorr layerwise computable function with a computable integral to a computable Banach space. Then f is effectively L^1-computable up to Schnorr null.

Theorem

Let μ be a measure with an $L^1(\lambda)$-computable density f. Then x is μ-Schnorr random if and only if x is λ-Schnorr random and $f(x) > 0$.

Summary

- An effective L^1-computable function works well almost everywhere!
- Assuming the existence of L^1-computable density makes it easier to study the measure.
- How strong is the assumption?

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