The emergence of probability from randomness and games

Workshop: Modeling Market Dynamics and Equilibrium
- New Challenges, New Horizons
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- Kenshi Miyabe (University of Tokyo)
- Research area:
  - algorithmic randomness
  - computable analysis
  - game-theoretic probability

- algorithmic randomness
  - computable aspects of limit theorems

- Characterization of Kurtz randomness by a differentiation theorem, Theory of Computing Systems, 52(1):113-132, 2013.

- Van Lambalgen's Theorem for uniformly relative Schnorr and computable randomness (with J. Rute), Proceedings of the Twelfth Asian Logic Conference, 251-270, 2013.

- Truth-table Schnorr randomness and truth-table reducible randomness, Mathematical Logic Quarterly 57(3):323-338, 2011

- An extension of van Lambalgen's Theorem to infinitely many relative 1-random reals, Notre Dame Journal of Formal Logic, 51(3): 337-349, 2010.

- computable analysis
  - computability of real functions

- L1-computability, layerwise computability and Solovay reducibility, Computability, 2:15-29, 2013.

- An optimal superfarthingale and its convergence over a computable topological space, LNAI 7070, 273-284, 2013.

- Computable measure theory and algorithmic randomness, in preparation.

- game-theoretic probability
  - probability theory based on the idea of algorithmic randomness

- The law of the iterated logarithm in game-theoretic probability with quadratic and stronger hedges (with A. Takemura), Stochastic Processes and their Application, 123, 3132-3152, 2013.

- Convergence of random series and the rate of convergence of the strong law of large numbers in game-theoretic probability (with A. Takemura), Stochastic Processes and their Applications, 122:1-30, 2012.

- The aim of this talk is to give an introduction to game-theoretic probability, which is a new framework of probability theory and has many applications including finance.

- Prof. Akimichi Takemura (University of Tokyo)
- With Takeuchi and Kumon, he has been advocating game-theoretic probability in Japan. He has published more than 10 papers on game-theoretic probability.

Overview of the talk

- Motivation
  - A brief history on probability theory
- An introduction on game-theoretic probability
- Some results on Brownian motion

Motivation

Hilbert (1862-1943)

- Hilbert's 23 problmes on International Congress of Mathematicians at 1900
- 6th "Mathematical treatment of the axioms of physics"

- Many mathematical models of physical systems are deterministic.
- Deterministic models have limits.
(Consider the kinetic theory of gases.)

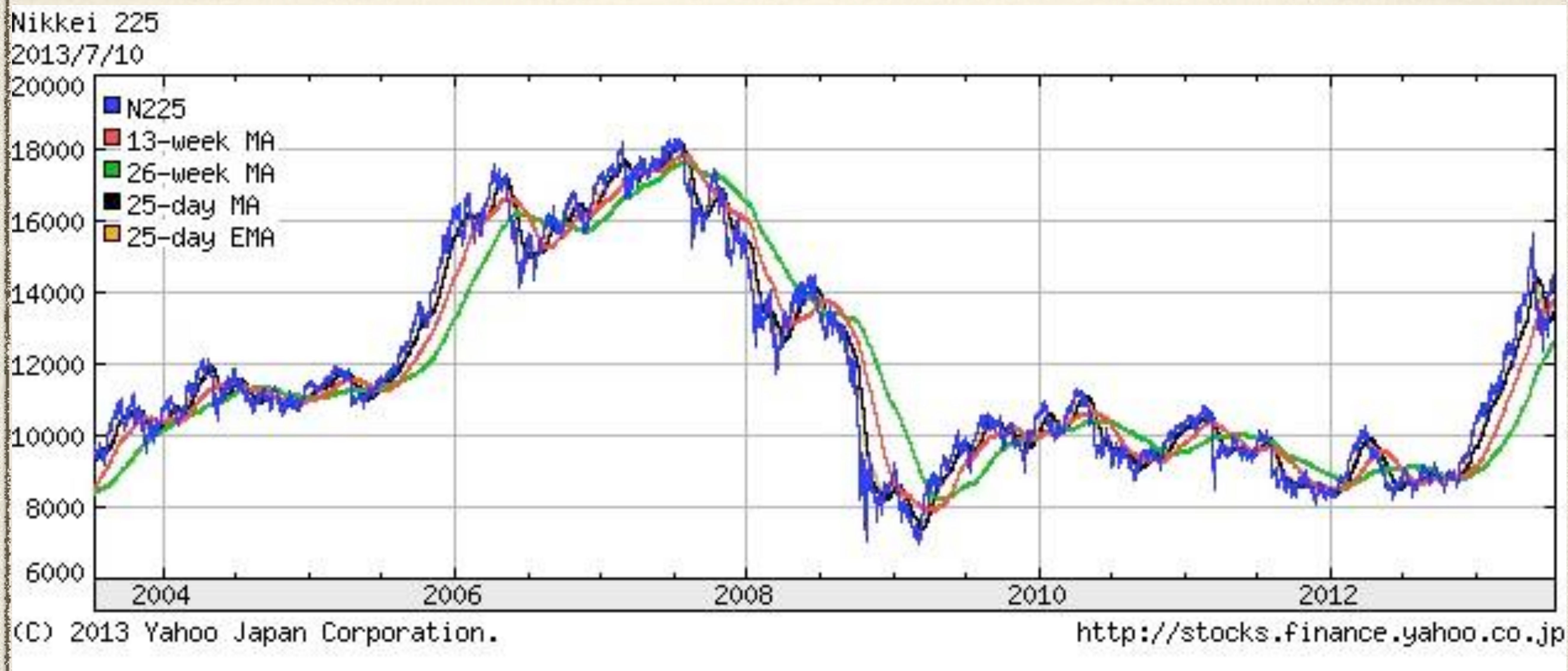
- Hilbert's idea can be summarized as follows.

- Find axioms of probability.
- Use it to make a model of physical systems.

Kolmogorov (1903-1987)

- Probability axioms (1933), which answers Hilbert's problem, are widely used.

Nikkei Stock Average



- Researchers usually start from the assumption that “stock-market prices follow Brownian motion”

  because one tries to make a probabilistic model.

- Mathematics start from axioms.
  Should a theory start from a (probabilistic) model?

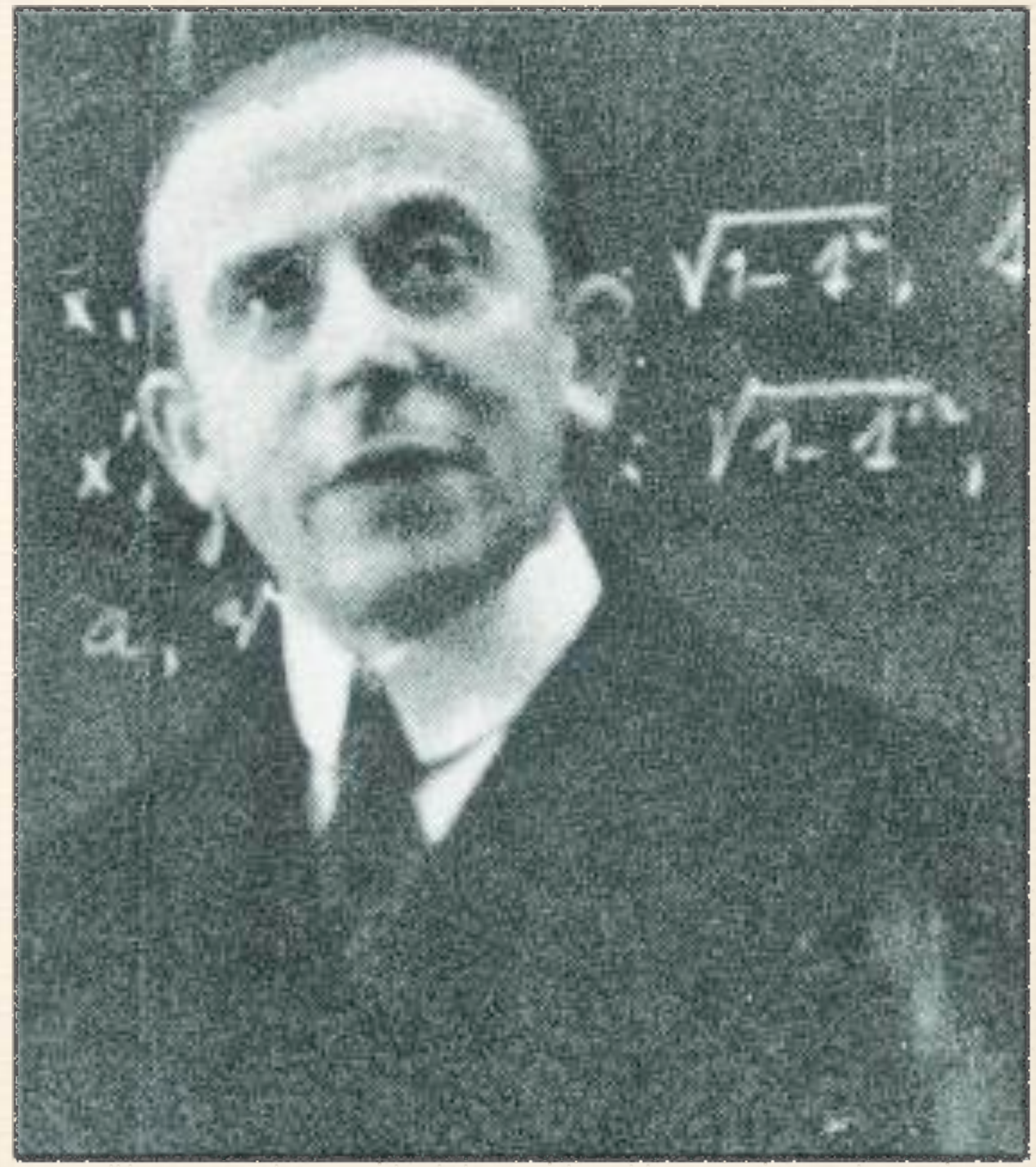
- How can we verify the assumption?
  We need another view to science.

Kolmogorov 1963

I have already expressed the point of view that the basis of the applicability of the mathematical theory of probability to random events of the real world is the frequency approach to probability in one form or another, which was so strongly advocated by von Mises.

von Mises (1883-1953)

- von Mises (1883-1953)
- Frequency approach (1919-)



Like all the other natural sciences, the theory of probability starts from observations, orders them, classifies them, derives from them certain basic concepts and laws and, finally, by means of the usual and universally applicable logic, draws conclusions which can be tested by comparison with experimental results. In other words, in our view the theory of probability is a normal science, distinguished by a special subject and not by a special method of reasoning.

"Probability, statistics and truth" by von Mises.

- What is given? - Data
- What is the goal? - Predict

- “First the Collective - then the Probability.”
- For von Mises, randomness yields probability. (The standard idea is that probability yields randomness.)
- However, he did not give a proper notion of randomness.

Algorithmic randomness

- Typical (Martin-Löf 1966)
  - there is no special property effectively found
- Unpredictable (Schnorr 1971; Ville)
  - there is no effective procedure to predict the next data
- Incompressible (Schnorr 1973, Levin 1973, Chaitin 1975; Kolmogorov)
  - there is no short program that produces the data

- Unpredictability yields probability.
  - Game-theoretic probability

- Incompressibility yields probability.
  - Algorithmic probability

An introduction to game-theoretic probability

- We will show the strong law of large numbers in game-theoretic probability.
- In other words, unpredictability yields SLLN.
- Note that we do not use measure (at least explicitly).

- “I can predict whether the stock-price will go up or down tomorrow”.
- Let’s test it.
- He can choose how many tickets he buys.
  You can buy one ticket for one euro.
  You will receive two euros for one ticket if the price goes up. So even odds.
- His initial capital is finite. He should keep his capital nonnegative. The number of tickets can be any real number such as 1/2 and -4/5.

- We do not care about how he predicts; he can use any model, the Internet and his friends.
- He need not buy tickets every time.
- He wins if he increases his capital to infinity.

unpredictable $\iff$ capital is bounded $\Rightarrow$ SLLN

We can sometimes believe the boundedness almost surely, sometimes not.

$\mathcal{K}_0 := 1.$

$n = 1, 2, \ldots$:

$\quad$ The player announces $M_n \in \mathbb{R}$.

$\quad$ $x_n \in \{-1, 1\}$ is announced.

$\quad$ $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

The player should keep $\mathcal{K}_n$ nonnegative.

The player wins if $\sup_n \mathcal{K}_n = \infty$.

The capital $\mathcal{K}_n$ depends on the move of $\{M_n\}$ and $\{x_n\}$.

$\mathcal{K}_0 := 1.$

$n = 1, 2, \ldots$:

Skeptic announces $M_n \in \mathbb{R}$.

Reality announces $x_n \in \{-1, 1\}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Skeptic must keep $\mathcal{K}_n$ nonnegative.

Reality must keep $\mathcal{K}_n$ from tending to infinity.

Recall that M_n should not depend on x_k for $k > n$.

- What we want to show is
“capital is bounded for any strategy => SLLN”,
in other words,
“not SLLN => capital is unbounded for some strategy”.

- Find such a strategy!

Definition

If there exists a strategy S of Skeptic such that E or $\lim_n \mathcal{K}_n = \infty$ happens, then we say that S forces E.

Theorem

In the protocol, Skeptic can force

$$\lim_n \frac{\sum_{k=1}^n x_k}{n} = 0.$$

Definition

If there exists a strategy S of Skeptic such that E or $\sup_n \mathcal{K}_n = \infty$ happens, then we say that S weakly forces E.

Lemma

For each $\epsilon > 0$, Skeptic can weakly forces

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i \leq \epsilon.$$

.

Proof

Let $\mathcal{K}_0 = 1$. At round n, Skeptic buys $\epsilon\mathcal{K}_{n-1}$-tickets. Then

$$\mathcal{K}_n = \prod_{i=1}^{n}(1+\epsilon x_i).$$

Since this is bounded,

$$D \geq \sum_{i=1}^{n} \ln(1+\epsilon x_i) \geq \epsilon \sum_{i=1}^{n} x_i - \epsilon^2 \sum_{i=1}^{n} x_i^2 \geq \epsilon \sum_{i=1}^{n} x_i - \epsilon^2 n,$$

$$\frac{\sum_{i=1}^{n} x_i}{n} \leq \frac{D}{\epsilon n} + \epsilon.$$

- One can show in game-theoretic probability
  - Strong Law of Large Numbers
  - Law of the Iterated Logarithm
  - Central Limit Theorem

  - probably almost all other limit theorems

- They logically imply the corresponding theorems in measure-theoretic probability.

- What’s probability in game-theoretic probability?

- It can be
  - tendency of statistical model
  - the degree of belief

$$\overline{\mathbb{P}}(E) := \inf\{\epsilon > 0 \;:\; (\exists S) \sup \mathcal{K}_n^S(w_0 \cdots w_n) \geq 1/\epsilon \text{ for all } w_1 w_2 \cdots \in E\},$$

$$\underline{\mathbb{P}}(E) := 1 - \overline{\mathbb{P}}(E^c).$$

S denotes a strategy of Skeptic.

$\mathcal{K}_n^S$ is the capital of Skeptic at round n with the strategy S.

Consider a ticket by which you can receive 1 euro when E happens.

$\overline{\mathbb{P}}(E)$ = Lowest price at which Skeptic can buy it.

$\underline{\mathbb{P}}(E)$ = Highest price at which Skeptic can sell it.

Some results on Brownian motion

- Unpredictability implies Wiener measure.
- Incompressibility implies Wiener measure.

Player: Forecaster, Skeptic, Reality

Protocol: $\mathcal{K}_0 := 1$.

$n = 1, 2, \ldots$:

Forecaster announces $m_n \in \mathbb{R}$, $v_n \geq 0$.

Skeptic announces $M_n \in \mathbb{R}$, $V_n \geq 0$.

Reality announces $x_n \in \mathbb{R}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - m_n) + V_n((x_n - m_n)^2 - v_n)$.

Skeptic must keep $\mathcal{K}_n$ nonnegative.

Reality must keep $\mathcal{K}_n$ from tending to infinity. .

Reality chooses a continuous function $\omega : [0, \infty) \to \mathbb{R}$

Skeptic chooses an increasing sequence of stopping times

$$\tau_1 \le \tau_2 \le \cdots$$

and M_n and V_n. The capital process is

$$\begin{aligned}\mathcal{K}_t^G(\omega) :=& c + \sum_{n=1}^{\infty} (M_n(\omega)(\omega(\rho_{n+1}) - \omega(\rho_n)) \\ &+ V_n(\omega)((\omega^2(\rho_{n+1}) - \rho_{n+1}) - (\omega^2(\rho_n) - \rho_n)))\end{aligned}$$

where $\rho_n = \tau_n \wedge t$, which is inspired from Lévy characterization of Wiener process.

Skeptic is allowed to use a strategy whose capital is represented by

$$\sum_i \mathcal{K}_t^{G_i}(\omega)$$

where $\sum_i c_i$ converges.

Theorem (Vovk 2008)

Let W be the Wiener measure on $(\Omega, \mathcal{F})$. Then,

$$\overline{\mathbb{P}}(A) = W(A)$$

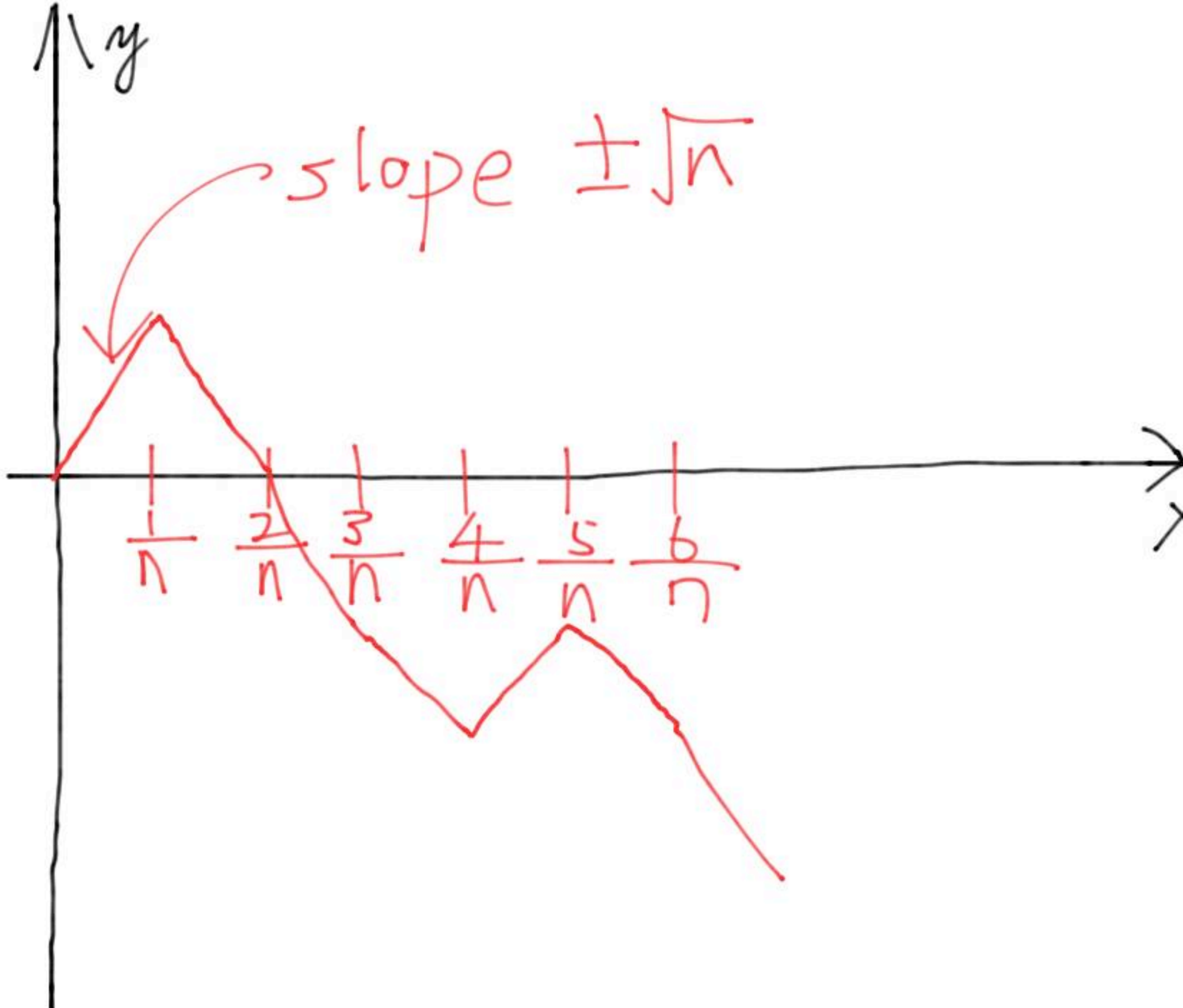
for each $A \in \mathcal{F}$.

For a finite binary string $\sigma \in 2^{<\omega}$, Kolmogorov complexity is the minimal length of a program that produces the string:

$$K(\sigma) = \min\{|\tau| \ : \ U(\tau) = \sigma\}$$

where U is a universal prefix-free Turing machine. One can think U as a programming language and τ as a program written in the language.

If $K(\sigma) > |\sigma| - c$ for $c \in \mathbb{N}$, then we say σ is c-compressible.



y
slope $\pm\sqrt{n}$
$\frac{1}{n}$ $\frac{2}{n}$ $\frac{3}{n}$ $\frac{4}{n}$ $\frac{5}{n}$ $\frac{6}{n}$
x

Let C_n be a set of such functions.

Each function $f \in C_n$ can be represented by a binary string of length n.

Definition (Asarin and Prokovskiy 1986)

Let $\{x_n\}$ be a sequence such that $x_n \in C_n$ for each n. The sequence $\{x_n\}$ is called complex if there is a constant $d > 0$ such that $K(x_n) > n - d$ for all n. A function $x \in C[0, 1]$ is a complex oscillation if there is a complex sequence $\{x_n\}$ such that $||x - x_n||$ converges effectively to 0 as $n \to \infty$.

Theorem (Asarin and Prokovskiy 1986)

The class of complex oscillations have Wiener measure 1.

Theorem (Fouché 2000)

A function is a complex oscillation if and only if it is Martin-Löf random in $C[0,1]$ with Wiener measure.

Summary - Three tools!

- Probabilistic model
  - the standard way and easy to use

  (Measure-theoretic probability does not care about null sets, in other words, does not care the case that the assumption was wrong.)
- Game-theoretic probability
  - direct modeling without probability
- Algorithmic randomness
  - direct analysis of information

If you want to know more ...

- Probability and Finance: It's Only a Game! by Glenn Shafer and Vladimir Vovk. New York: Wiley, 2001
- http://www.probabilityandfinance.com/