L^1-computability and Schnorr randomness

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L^1-computability

- Lebesgue differentiation theorem
- Radon-Nikodym derivative
- van Lambalgen's theorem

Theorem (Lebesgue 1904)

Every nondecreasing function $f : [0,1] \to \mathbb{R}$ is differentiable almost everywhere.

Theorem (Brattka-Miller-Nies 20xx)

A real $x \in [0,1]$ is computably random if and only if every nondecreasing computable function $f : [0,1] \to \mathbb{R}$ is differentiable at x.

randomness notions	class of functions
weak 2-randomness	differentiable a.e.
Martin-Löf randomness	bounded variation
computable randomness	Lipschitz or nondecreasing

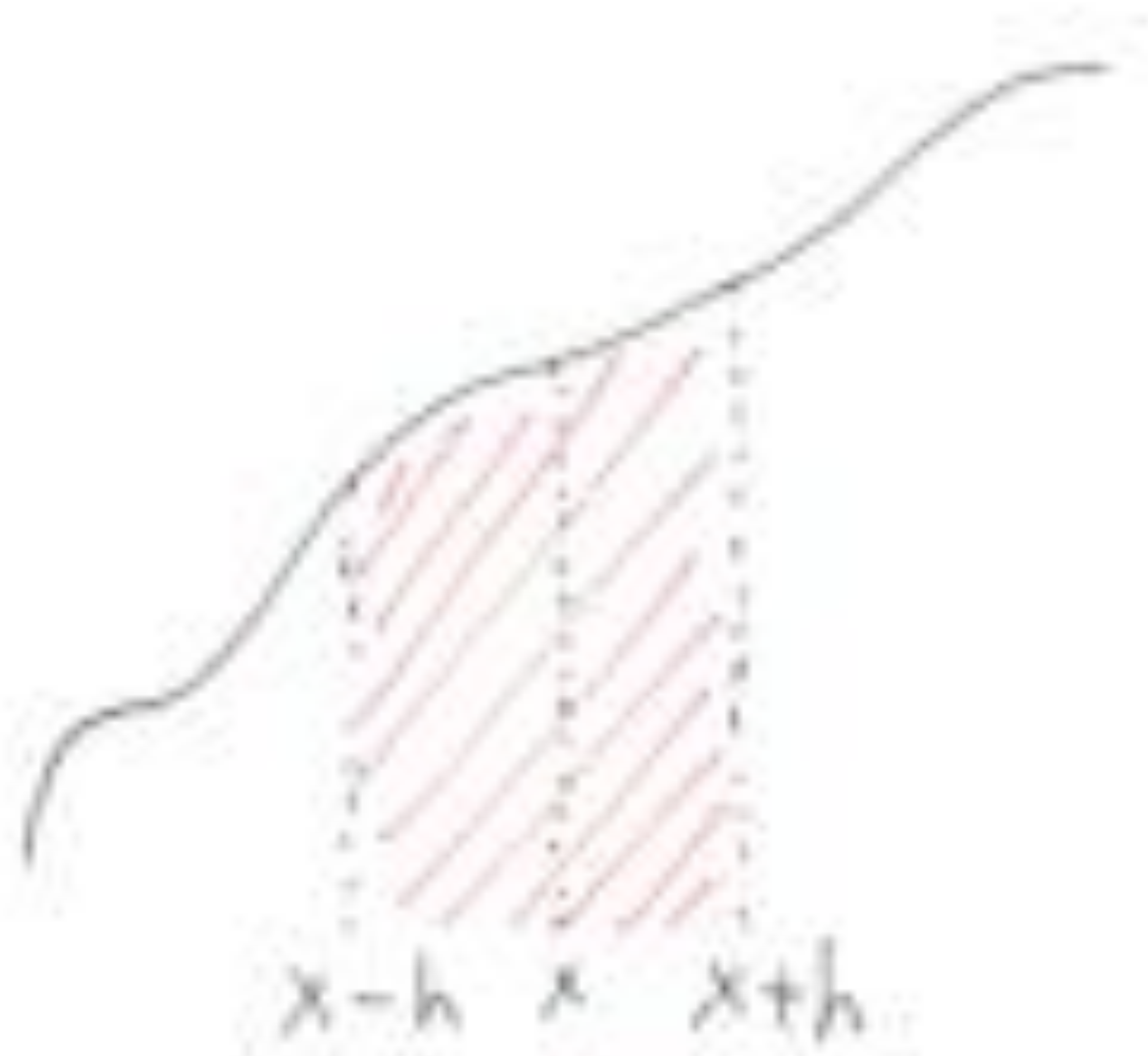
Theorem (Lebesgue 1910)

Let $f : [0,1] \to \mathbb{R}$ be an integrable function. Then,

$$\lim_{\epsilon \to 0} \frac{1}{\lambda(B(x,\epsilon))} \int_{B(x,\epsilon)} f \, d\lambda = f(x)$$

almost everywhere. Here, λ is the Lebesgue measure.

Such a point x is called a Lebesgue point for f. We will show some effectivizations of this result.


x-h
x
x+h

Theorem (Pathak-Rojas-Simpson)

A real $x \in [0,1]$ is Schnorr random if and only if x is a Lebesgue point for each effective L^1-computable function.

L^1-computability and uniqueness

Let $f : [0,1] \to \mathbb{R}$ be a function. The L^1-norm $||f||_1$ is defined by

$$||f||_1 = \int |f| \, d\mu$$

where μ is the Lebesgue measure.

A function $f : [0,1] \to \mathbb{R}$ is L^1-computable if it can be approximated by simple functions in the L^1-norm. (The precise definition is two pages later.)

Definition

A rational step function is a finite sum

$$f = \sum_{k=1}^{n} r_k \mathbf{1}_{(p_k, q_k)}$$

where $r_k \in \mathbb{Q}$ and $p_k, q_k \in \mathbb{Q} \cap [0, 1]$.

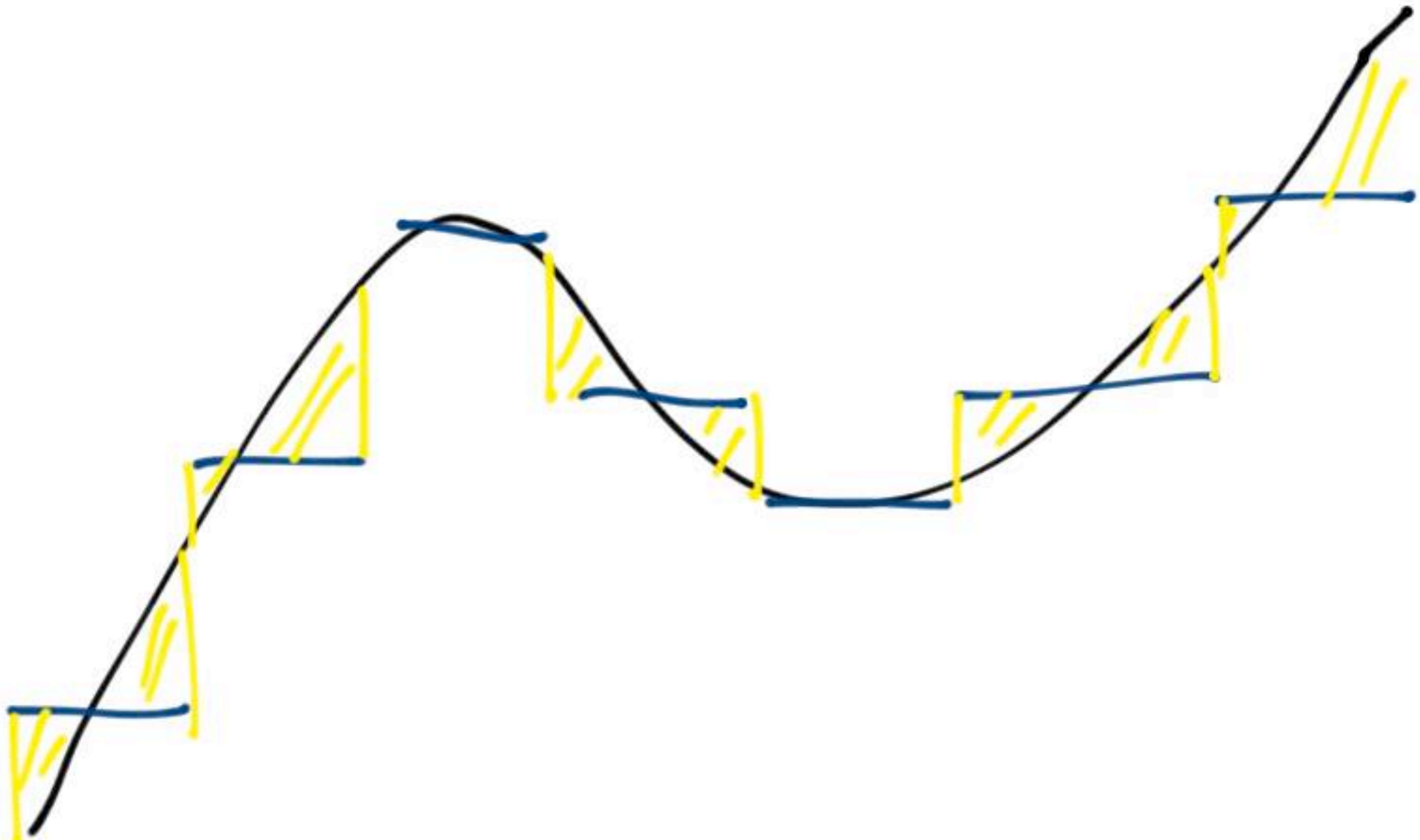
We do not require the intervals pairwise disjoint. It does not matter whether it is an open interval or closed interval.

Definition (Pour-El & Richard)

A function $f : [0,1] \to \mathbb{R}$ is L^1-computable if there exists a computable sequence $\{s_n\}$ of rational step functions such that

$$||f - s_n|| < 2^{-n}$$

for all n.



- An L^1-computable function is a computable point in the L^1-space.
- If f is L^1-computable and g is equivalent to f up to null set, then g is also L^1-computable.
- Being a Lebesgue point does not make sense!

Definition (Pathak-Rojas-Simpson, M.)

A function $f :\subseteq [0,1] \to \mathbb{R}$ is effectively L^1-computable if there exists a computable sequence $\{s_n\}$ of rational step functions such that $||f - s_n|| < 2^{-n}$ for all n and

$$f(x) = \lim_n s_n(x)$$

for all $x \in [0,1]$.

Note that f can be partial!

Theorem (M.; Pathak-Rojas-Simpson)

Let f, g be effective L^1-computable functions.

(i) f is defined up to Schnorr null.

(ii) $f = g$ a.e. if and only if f and g are equal up to Schnorr null. Thus, it does not depend on the representation up to Schnorr null.

A basic open set on $[0,1]$ is

$$(p,q),[0,q),(p,1]$$

for $p,q \in \mathbb{Q}$. There is a computable enumeration of basic open sets. An open set $U \subseteq [0,1]$ is c.e. if $U = \bigcup_i B_i$ for a computable sequence $\{B_i\}$ of basic open sets.

A Schnorr test is a sequence $\{U_n\}$ of uniformly c.e. open sets such that $\mu(U_n) \leq 2^{-n}$ and $\mu(U_n)$ is uniformly computable.

A class N is Schnorr null if it is covered by a Schnorr test $\{U_n\}$, that is, $N \subseteq \bigcap_n U_n$.

A real $x \in [0, 1]$ is Schnorr random if it avoids each Schnorr null class, that is, $x \notin \bigcap_n U_n$.

Definition

A Schnorr Solovay test is a computable sequence $\{B_n\}$ of basic open sets such that $\sum_n \mu(B_n)$ is a computable real.

Theorem

A set A is Schnorr random if and only if $A \in B_n$ for at most finitely many n for each Schnorr test $\{B_n\}$.

Definition (M.)

A Schnorr integral test is a lower semicomputable function $f : [0,1] \to \overline{\mathbb{R}}^+$ such that $\int f \, d\mu$ is a computable real.

Theorem (M.)

A point $x \in [0,1]$ is Schnorr random if and only if $f(x) < \infty$ for each Schnorr integral test.

Theorem

Each difference between two Schnorr integral tests is equivalent to an effective L^1-computable function up to Schnorr null. Each effective L^1-computable function is equivalent to the difference between two Schnorr integral tests up to Schnorr null.

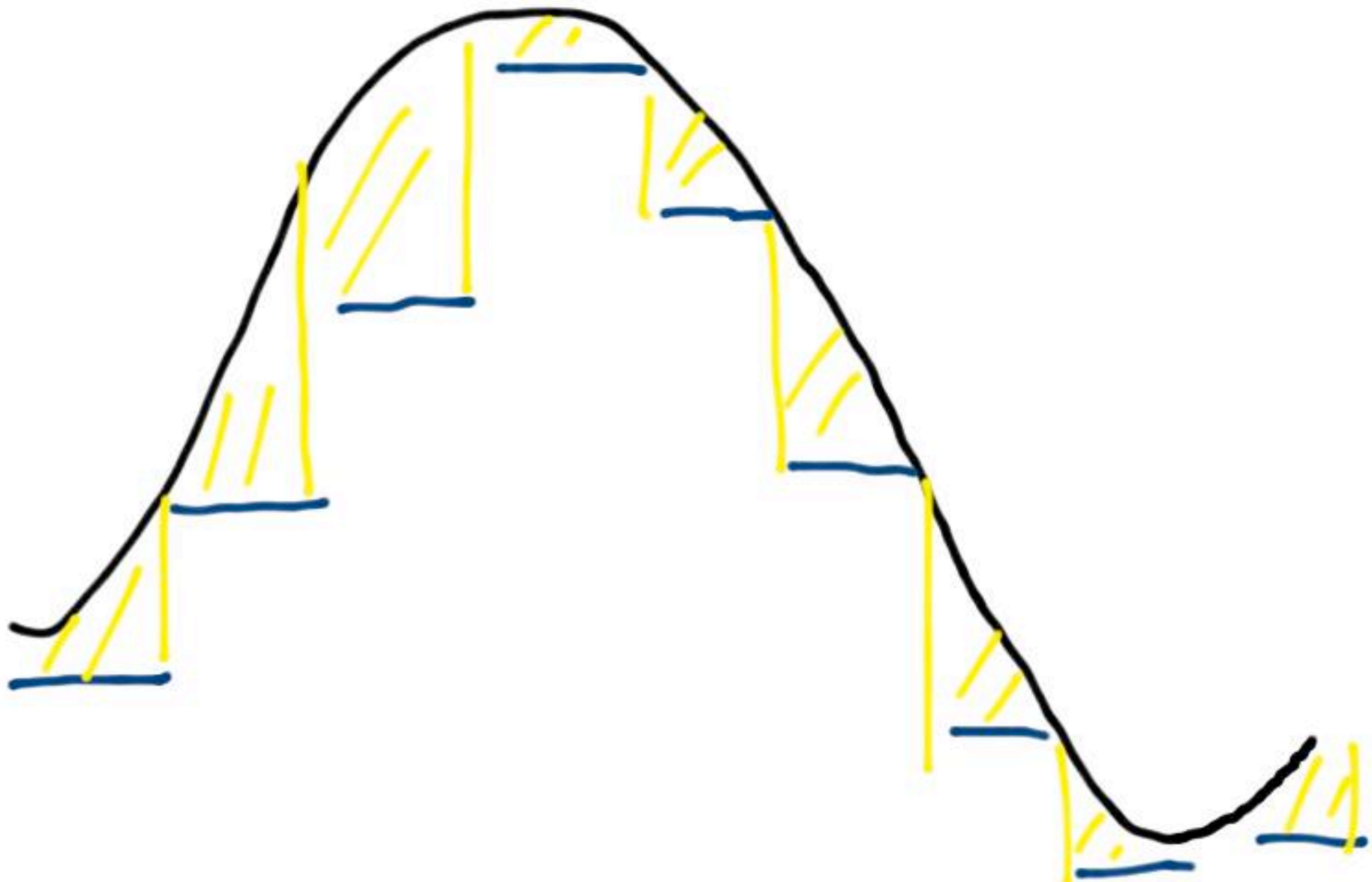
Notice that

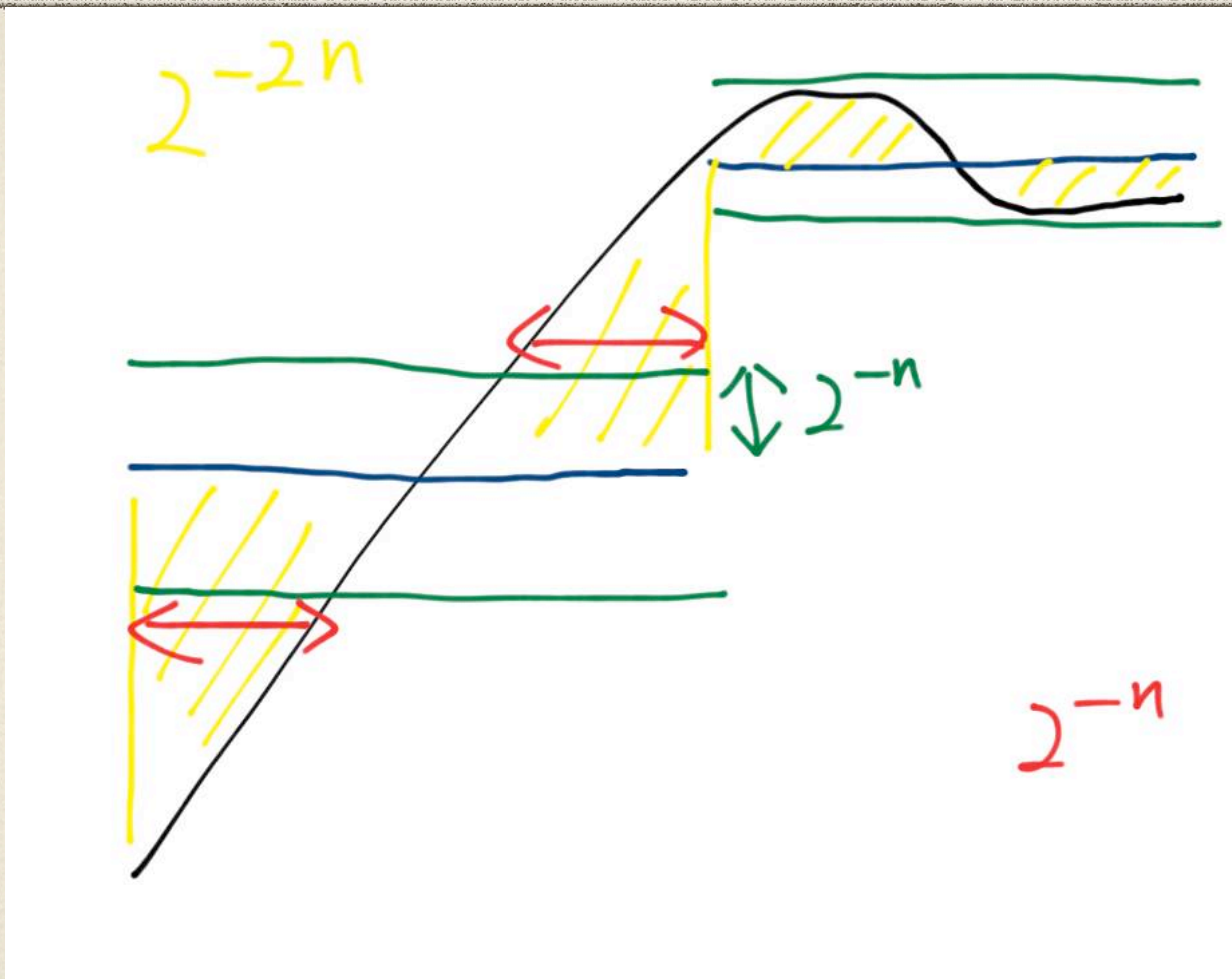
$$||s_{n+1} - s_n|| \leq ||f - s_{n+1}|| + ||f - s_n|| < 2^{-n+1}.$$

Split $s_{n+1} - s_n$ into the positive part t_n^+ and the negative part t_n^-. Then,

$$f(x) = \sum_n (s_{n+1}(x) - s_n(x)) = \sum_n t_n^+(x) - \sum_n t_n^+(x),$$

if $\sum_n t_n^{\pm}(x)$ is defined. Note that $||t^+|| + ||t^-|| < 2^{-n+1}$ and $\sum_n t_n^{\pm}(x)$ are computable.





2^{-2n}
2^{-n}
2^{-n}

Proof Sketch of effective LDT.

Let

$$g_n(x) = \frac{\int_{[x \restriction n]} f \, d\mu}{2^{-n}}.$$

Notice that

(i) $g_n(x)$ converges $f(x)$ a.e.,

(ii) g_n is a rational step function,

(iii) the convergence is also effective,

(iv) $h(x \restriction n) = g_n(x)$ is a martingale,

(v) we can "speed up" the convergence.

Use Marayne-Solecki trick to get the theorem. .

Let μ be a measure with L^1-computable density w.r.t. λ, that is, $\mu(A) = \int_A f \; d\lambda$ for an L^1-computable function. Then, f is the Radon-Nikodym derivative of μ w.r.t. λ.

No more "up to null"!

Theorem (Hoyrup and Rojas)

Let μ, λ be such that $\mu \ll \lambda$ and μ is computably norm able relative to λ. Then the Radon-Nikodym derivative $\frac{d\mu}{d\lambda}$ can be computed as an element of $L^1(\lambda)$ from μ and λ.

Corollary

If we further assume that μ and λ are computable, then the Radon-Nikodym derivative is unique up to Schnorr null.

Let λ be the uniform measure on 2^ω. Let μ be a computable measure on 2^ω such that $\mu \ll \lambda$. Then,

$$g(x) = \lim_n \frac{\mu(x \upharpoonright n)}{\lambda(x \upharpoonright n)}$$

is defined for each computably random point x, and g is a Radon-Nikodym derivative of μ w.r.t. λ.

Schnorr layerwise computability

Let f be an effective $L^1(\lambda)$-computable function on $[0,1]$. Then,

$$f(x) \leq_T x.$$

Definition (M.; Hoyrup-Rojas)

A function $f : [0,1] \to \mathbb{R}$ is Schnorr layerwise computable if there is a Schnorr test $\{U_n\}$ such that $f|_{X \setminus U_n}$ is uniformly computable.

(i) Computable correctly with high probability.

(ii) Computable with advice.

(iii) Computable in the limit.

Theorem (M.)

Each effective L^1-computable function is Schnorr layerwise computable. Each Schnorr layerwise computable function with computable integral is equivalent to an effective L^1-computable function up to Schnorr null.

Theorem (M. & Rute)

Let f be an effective L^1-computable function. Then, there are a Schnorr test $\{U_n\}$ and a sequence $\{f_n\}$ of uniformly computable total functions such that

$$f|_{X\setminus U_n} = f_n|_{X\setminus U_n}.$$

The fact in the previous page was used to show van Lambalgen's theorem for Schnorr randomness.

Theorem (M. & Rute)
$X \oplus Y$ is Schnorr random if and only if X is Schnorr random and Y is Schnorr random uniformly relative to X.

Theorem (Lusin's Theorem)

Let $f : [0,1] \to \mathbb{R}$. Then, f is measureble if and only if, for each $\epsilon > 0$, there are a continuous function g and a compact set K such that

$$\mu([0,1]\backslash K) < \epsilon,$$
$$f(x) = g(x) \text{ for } x \in K.$$

Thus, Schnorr layerwise computability is an effective version of measurability. Furthermore, we can formalize this idea.

We define a computable measurable set similarly to an effective L^1-computable function.

Theorem

A function $f :\subseteq [0, 1] \to \mathbb{R}$ is Schnorr layerwise computable if and only if $f^{-1}(B_i)$ is a computable measurable set uniformly in i, where $\{B_i\}$ is a computable sequence of basic open sets.

Thus, we also call it an effectively measurable function.

Summary

- L^1-computability and Schnorr layerwise computability seem important notions.
- How about variants?
- How related to
  - computation with advice,
  - randomized algorithm
  - learnability?