Being a Lebesgue point for integral tests

ALC2013 at 中山大学（Sun Yat-sen University）

16-20 Sep 2013

宮部賢志（Kenshi Miyabe）

学振PD（東京大学）

JSPS Research Fellow at The University of Tokyo

Theorem (Lebesgue 1904)

Every nondecreasing function $f : [0,1] \to \mathbb{R}$ is differentiable almost everywhere.

Theorem (Brattka-Miller-Nies 20xx)

A real $x \in [0,1]$ is computably random if and only if every nondecreasing computable function $f : [0,1] \to \mathbb{R}$ is differentiable at x.

randomness notions	class of functions
weak 2-randomness	differentiable a.e. (Brattka-Miller-Nies)
Martin-Löf randomness	bounded variation (Demuth 1976, BMN)
computable randomness	nondecreasing (BMN) or Lipschitz (Freer-Kjos-Hanssen-Nies-Stephan)

Theorem (Lebesgue 1910)

Let $f : [0, 1] \to \mathbb{R}$ be an integrable function. Then,

$$\lim_{\epsilon \to 0} \frac{1}{\lambda(B(x, \epsilon))} \int_{B(x,\epsilon)} f \, d\lambda = f(x)$$

almost everywhere. Here, λ is the Lebesgue measure.

Such a point x is called a Lebesgue point for f. The Lebesuge density theorem is a corollary of this result. (Consider a $\{0, 1\}$-valued function f.) We will show some effectivizations of this result.

randomness notions	class of functions
?	integral tests
Martin-Löf randomness	?
Schnorr randomness	effectively L^1-computable (Pathak-Rojas-Simpson)
Kurtz randomness	a.e. computable (M. 2013)

Let $f : [0, 1] \to \overline{\mathbb{R}}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x < \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Then, f is an integral test, but Ω is not a Lebesgue point for f.

Density randomness

Theorem (M.)

The following are equivalent for a real $z \in [0, 1]$:

(i) z is density random.

(ii) z is a dyadic Lebesgue point for each integral test.

(iii) z is a Lebesgue point for each integral test.

Definition

An integral test is an integrable lower semicomputable function $f : [0, 1] \to \overline{\mathbb{R}}^+$.

Theorem

A real $z \in [0, 1]$ is ML-random if and only if $f(z)$ is finite for each integral test.

Definition

The density of a class $C \subseteq \mathbb{R}$ at a point z is

$$\rho(C|z) = \liminf_{\gamma,\delta \to 0^+} \frac{\lambda([z-\gamma, z+\delta] \cap C)}{\gamma + \delta}.$$

Theorem (Lebesgue density theorem)

Let C be a measurable set. Then $\rho(C|z) = 1$ for almost every $z \in C$.

Definition

A real $z \in [0,1]$ is called density random if it is ML-random and every Π^0_1-class containing z has density one at z.

Let $f : [0,1] \to \mathbb{R}$ be an integrable function. A real x is a Lebesgue point for f if

$$\lim_{\epsilon \to 0} \frac{1}{\lambda(B(x,\epsilon))} \int_{B(x,\epsilon)} f \, d\lambda = f(x).$$

A real x is a dyadic Lebesgue point for f if

$$\lim_{n \to 0} \frac{1}{\lambda([x \upharpoonright n])} \int_{[x \upharpoonright n]} f \, d\lambda = f(x).$$

If the left-hand side converges, a real x is called a (dyadic) weak Lebesgue point.

Theorem (M.)

The following are equivalent for a real $z \in [0,1]$:

(i) z is density random.

(ii) z is a dyadic Lebesgue point for each integral test.

(iii) z is a Lebesgue point for each integral test.

The direction from (ii) to (i) is not difficult.

The equivalence between (ii) and (iii) follows by combining

(i) a proposition in Logic Blog 2013,
(ii) a fact in Brattka-Miller-Nies' paper,
(iii) a lemma in Demuth-Denjoy-Density paper.

See Logic Blog 2013 for details.

Theorem (Andrews, Cai, Diamondstone, Lempp and Miller)

A real z is density random if and only if every left-c.e. martingale converges along Z, where $0.Z$ is the binary expansion of z.

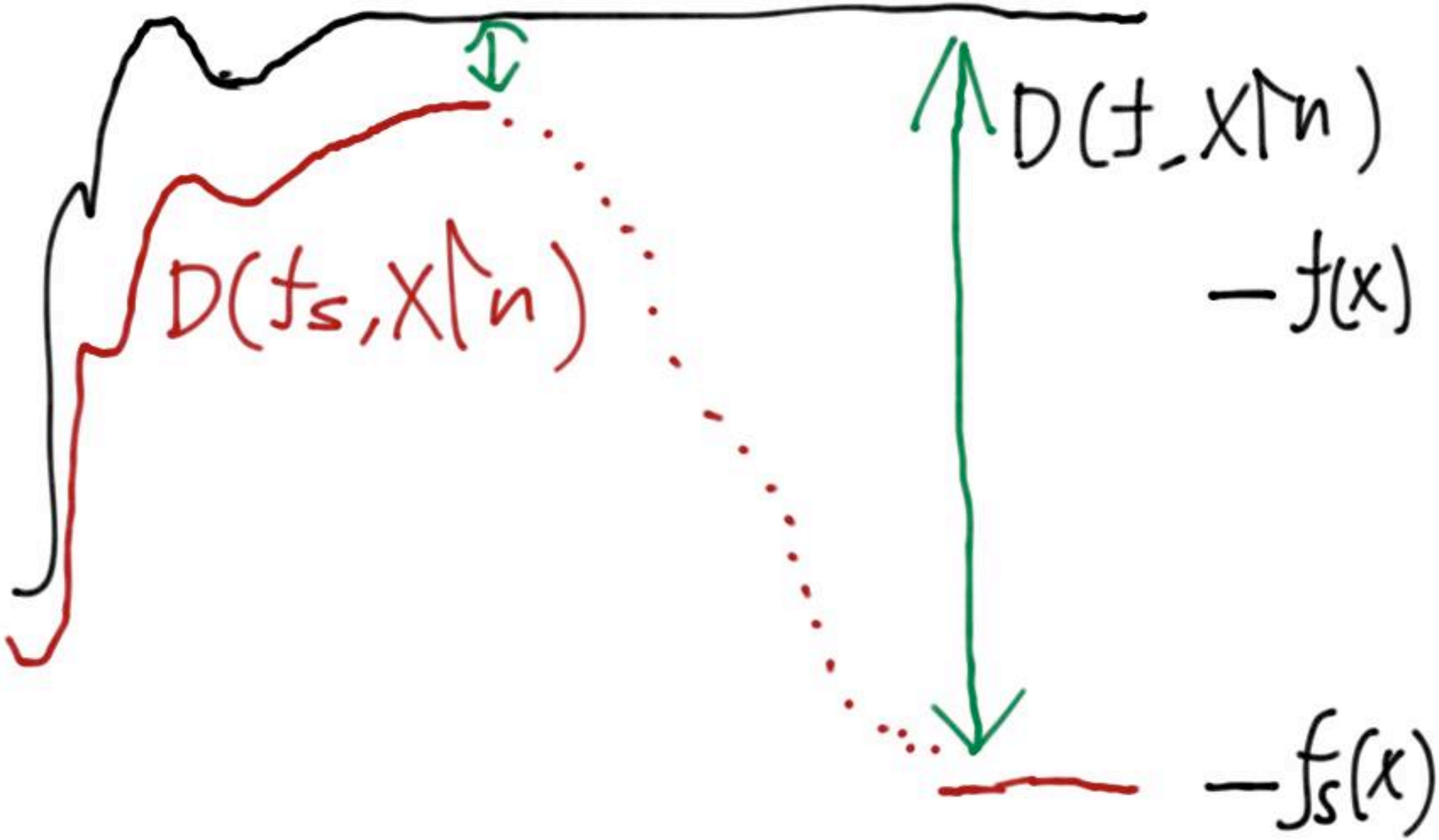
Lemma (M.)

If an ML-random set z is a dyadic weak Lebesgue point for an integral test f, then z is a dyadic full Lebesgue point for f.

Let

$$D(f,\sigma) = \frac{1}{2^{-|\sigma|}} \int_{[\sigma]} f \; d\mu.$$

If f is an integral test, then $D(f,-)$ is a left-c.e. martingale.

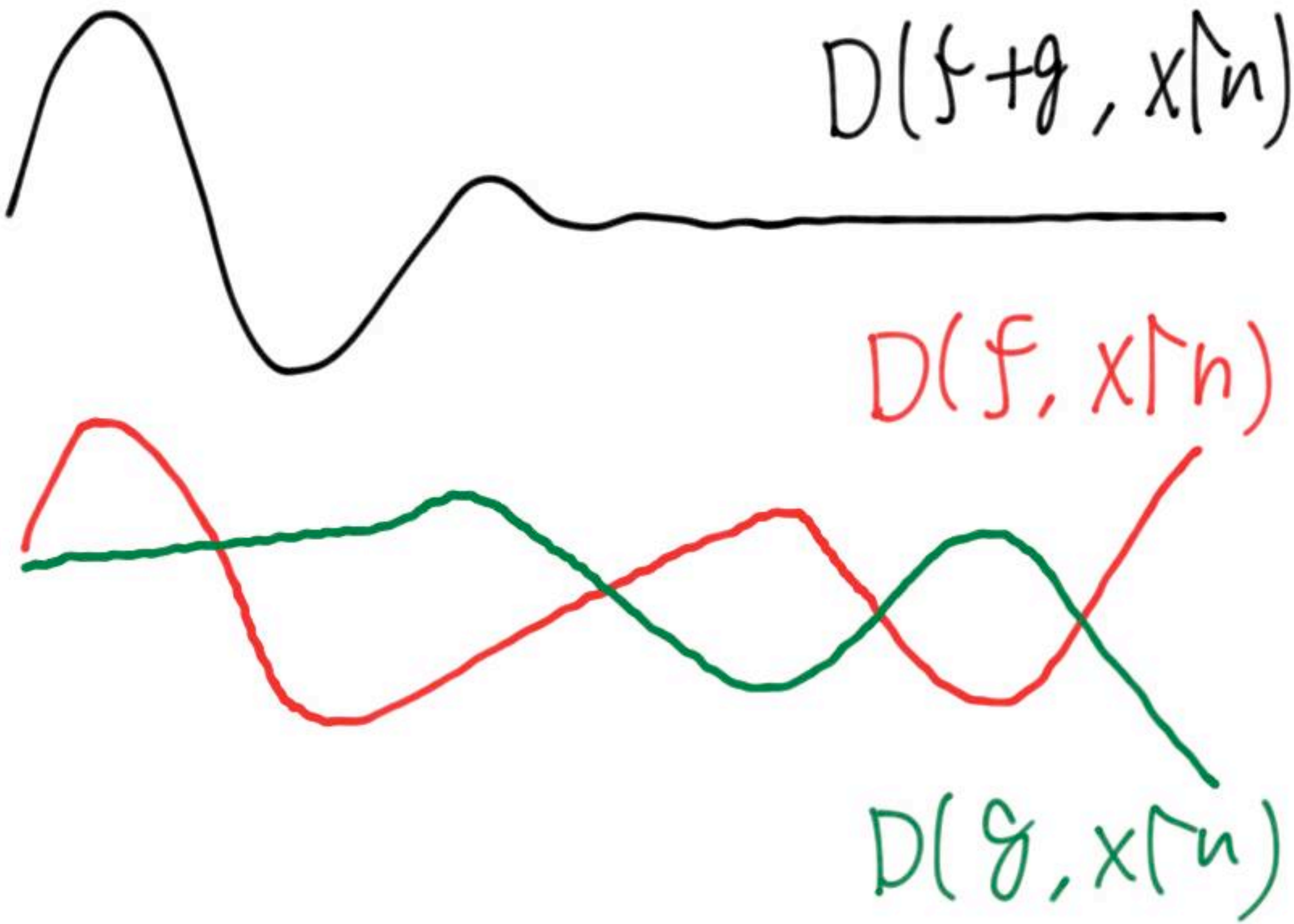

D(f, X↾n)
D(f_s, X↾n)
−f(x)
−f_s(x)

Theorem (M.)

There is an integral test f such that x is density random if and only if x is a Lebesgue point for f.

Lemma

Let f, g be integral tests. If an ML-random real x is a dyadic weak Lebesgue point for $f + g$, then x is a dyadic weak Lebesgue point for f and g.


D(f+g, x↾n)
D(f, x↾n)
D(g, x↾n)

Characterization via Lebesgue points

randomness notions	class of functions
density randomness	integral test
Martin-Löf randomness	?
Schnorr randomness	effectively L^1-computable
Kurtz randomness	a.e. computable

太谢谢您了