Almost uniform weak n-randomness

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Overview

- Motivation
- uniform relativization
- van Lambalgen's theorem for uniform Kurtz randomness
- almost uniform relativization

Motivation

Question

- What does it mean by saying that

  “two objects are random relative to each other”?

If we say that a set $A \in 2^\omega$ is computable relative to a set $B \in 2^\omega$, then it usually means that

$$A \leq_T B.$$

We can consider many variants:

$$A \leq_{tt} B,\ A \leq_{wtt} B,\ A \leq_m B, \cdots .$$

A natural answer

Theorem (van Lambalgen 1987)

$A \oplus B$ is Martin-Löf random

$\iff$ A is Martin-Löf random

and B is Martin-Löf random relative to A.

$\Rightarrow$: easy direction

$\Leftarrow$: difficult direction

A ML-test is a sequence $\{V_n\}$ of uniformly c.e. open sets such that $\mu(V_n) \leq 2^{-n}$ for all n. A set B is ML-random if $B \notin \bigcap_n V_n$ for each ML-test.

A ML-test relative to A is a sequence $\{V_n\}$ of uniformly A-c.e. open sets such that $\mu(V_n) \leq 2^{-n}$ for all n. A set B is ML-random relative to A if $B \notin \bigcap_n V_n$ for each ML-test relative to A.

Failure of vL-theorem

- “easy direction” does not hold for
- Schnorr randomness or computable randomness (Merkle-Miller-Nies-Reimann-Stephan 2006, Yu 2007)
- Kurtz randomness (Franklin-Stephan 2011)
- weak 2-randomness (Barmpalias-Downey-Ng 2011)

Interpretations

- ML-randomness is more natural than other randomness notions.
- The way of relativization was not appropriate.

Uniform relativization

$A \leq_T B$ if there is a Turing reduction Φ such that $A = \Phi^B$. Note that Φ^Z may not be defined for $Z \neq B$.

$A \leq_{tt} B$ if there is a Turing reduction Φ such that Φ^Z is defined for each $Z \in 2^\omega$ and $A = \Phi^B$.

We know that

$$\leq_{tt} \Rightarrow \leq_T,$$

but the converse does not hold.

A Schnorr test is a sequence $\{V_n\}$ of uniformly c.e. open sets such that $\mu(V_n) = 2^{-n}$ for all n. A set B is Schnorr random if $B \notin \bigcap_n V_n$ for each Schnorr test.

A Schnorr test can be identified with a computable function from ω to τ where τ is the class of open sets.

Uniform relativization

Definition

A uniform Schnorr test is a computable function $f : 2^\omega \times \omega \to \tau$ such that $\mu(f(X,n)) = 2^{-n}$.

We call $\{f(A,n)\}$ a Schnorr test uniformly relative to A.

B is Schnorr random uniformly relative to A if B passes all Schnorr tests uniformly relative to A.

Theorem (M. 2011 and M.-Rute 2013)

$A \oplus B$ is Schnorr random

$\iff$ A is Schnorr random

and B is Schnorr random uniformly relative to A.

$A \oplus B$ is computably random

$\iff$ A is computably random uniformly relative to B

and B is computably random uniformly relative to A.

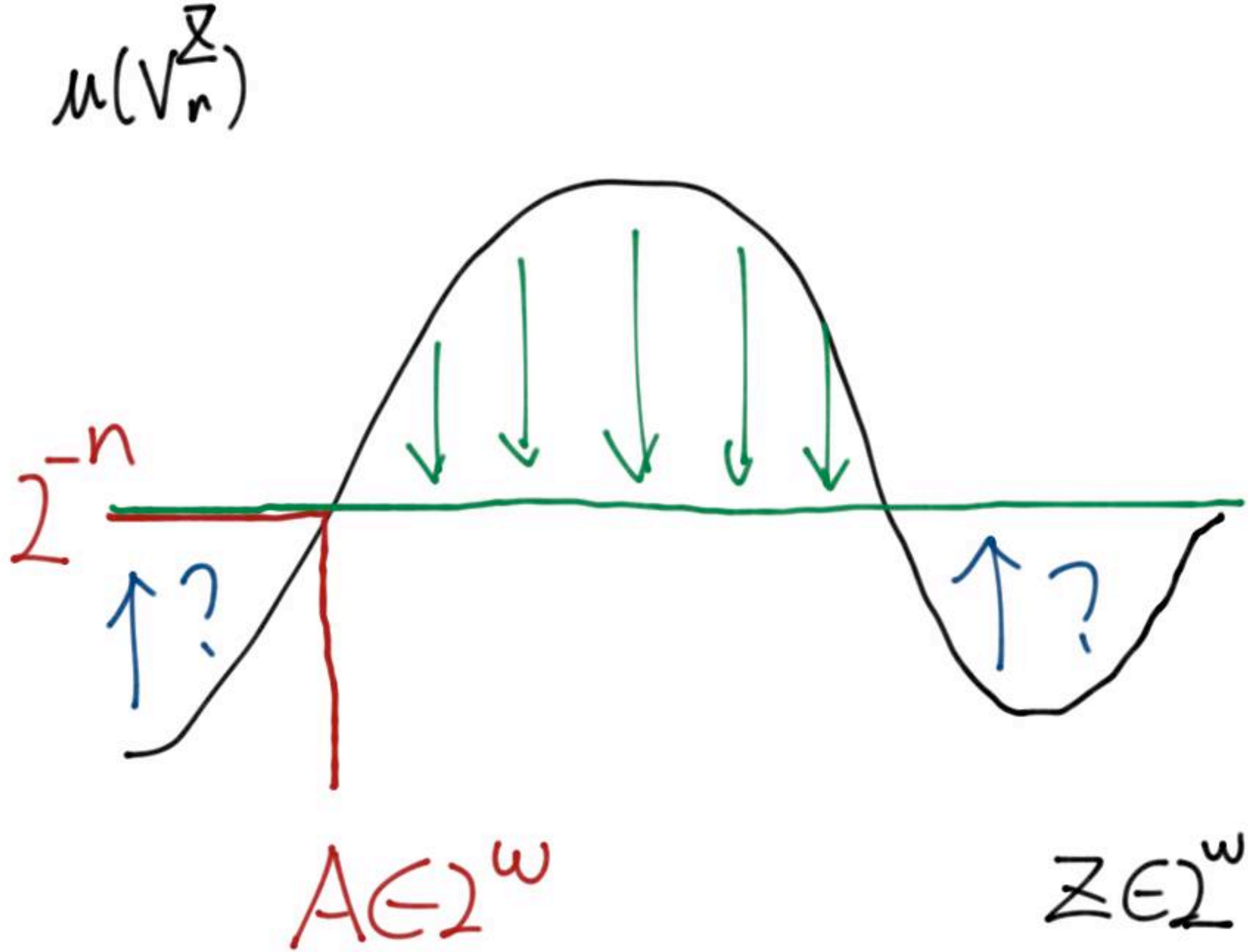
Definition

A Demuth test is a sequence of c.e. open sets $\{V_n\}$ such that $\mu(V_n) \leq 2^{-n}$ for all n, and there is an ω-c.e. function f such that $V_n = [\![W_{f(n)}]\!]$.

A $\text{Demuth}_{\text{BLR}}$ test is a Demuth test relative to A where f is ω-c.e. by A, that is, the approximation is A-computable but the bound on the number of changes is computable.

Theorem (Diamondstone-Greenberg-Turetsky)

Van Lambalgen's theorem holds for $\text{Demuth}_{\text{BLR}}$ randomness.


$\mu(V_n^Z)$
2^{-n}
?
?
$A \in 2^\omega$
$Z \in 2^\omega$

Another relativization

- B is Schnorr random relative to A
  => B is Schnorr random uniformly relative to A
- There exists A such that the converse does not hold.
- Suppose that A is computable.
  Then B is Schnorr random
  iff B is Schnorr random relative to A
  iff B is Schnorr random uniformly relative to A.

Unusual usage of terminology

- The usual way to see is that,
  "we define tests and randomness notions, and then relativize them".
- We need to talk about reduction to distinguish tt and T or usual relativization and uniform relativization.
- Uniform Schnorr randomness means Schnorr randomness with uniform relativization.

Uniform Kurtz randomness

Kurtz randomness

Theorem(Franklin-Stephan '11)

- If A is Kurtz random and B is A-Kurtz random, then $A \oplus B$ is Kurtz random.
- There exists a pair A, B such that $A \oplus B$ is Kurtz random and neither A nor B is Kurtz random relative to the other.

The "difficult direction" holds but the "easy direction" does not hold.

Definition

A uniform Kurtz test is a total computable function $f : 2^\omega \to \tau$ such that $\mu(f(Z)) = 1$ for all $Z \in 2^\omega$.

A set B is called Kurtz random uniformly relative to A if $B \in f(A)$ for each uniform Kurtz test f.

easy direction

Theorem (M.-Kihara)

If $A \oplus B$ is Kurtz random,

then B is Kurtz random uniformly relative to A.

Corollary

There is a pair $A, B \in 2^\omega$ such that B is Kurtz random uniformly relative to A and not Kurtz random relative to A.

difficult direction

Theorem (M.-Kihara)

There is a pair A, B such that A and B are mutually uniformly Kurtz random and $A \oplus B$ is not Kurtz random.

So, the "easy direction" does hold but the "difficult direction" does not hold!!

Lemma

If $A(n) = 0$ or $B(n) = 0$ for all n, then $A \oplus B$ is not Kurtz random.

Proof

Let $\{f_i\}$ be an enumeration of all uniform Kurtz tests. At stage s, we define $\alpha_s \prec A$ and $\beta_s \prec B$ such that $|\alpha_s| = |\beta_s|$.

At stage $s = 2i$, search $\beta \succeq \beta_s$ and m such that

$$[\![\beta]\!] \subseteq f_i(\alpha_s 0^m).$$

Such β and m always exist. We assume $|\alpha_s 0^m| \geq |\beta|$. Define

$$\alpha_{s+1} = \alpha_s 0^m,\ \beta_{s+1} = \beta 0^{|\alpha_s|+m-|\beta|}.$$

At stage $s = 2i + 1$, define α_{s+1} and β_{s+1} similarly by replacing α and β.

Almost uniform relativization

- The usual relativization is too strong for the easy direction to hold.

- The uniform relativization may be too weak for the difficult direction to hold

Theorem (Frankline and Stephan '11)

If A is Kurtz random and B is A-Kurtz random, then $A \oplus B$ is Kurtz random.

Proof

Let A be a Kurtz-random set and U be an arbitrary c.e. open set U with measure 1. For each rational $r < 1$, let

$$U_r = \{P \ : \ \mu(\{Q \ : \ P \oplus Q \in U\}) > r\}.$$

Then U_r is a c.e. open set.

For each r, we have $\mu(U_r) = 1$.

Since A is Kurtz random, $A \in U_r$ for each r. Let

$$T = \{Q \ : \ A \oplus Q \in U\}.$$

Then T is a A-c.e. open set with measure 1. Since B is A-Kurtz random, we have $B \in T$. Hence $A \oplus B \in U$. Since U is arbitrary, $A \oplus B$ is Kurtz random.

Definition

A almost uniform (a.u.) Kurtz test is a computable function $f : 2^\omega \to \tau$ such that $\mu(f(Z)) = 1$ for almost every $Z \in 2^\omega$. A set B is Kurtz random a.u. relative to A if $B \in f(A)$ for each a.u. Kurtz test f such that $\mu(f(A)) = 1$.

random $\Rightarrow$ a.u. random $\Rightarrow$ uniformly random

Theorem (M.)

$A \oplus B$ is Kurtz random iff A is Kurtz random and B is Kurtz random a.u. relative to A.

Definition

An a.u. weak n-test is a computable function $f : 2^\omega \to \Sigma^0_n$ such that $\mu(f(Z)) = 1$ for almost every $Z \in 2^\omega$. A set B is weakly n-random a.u. relative to A if $B \in f(A)$ for each a.u. weak n-test f such that $\mu(f(A)) = 1$.

Definition (Brattka 2005)

Let (X, d, α) be a separable metric space. We define representations $\delta_{\Sigma_k^0(X)}$ of $\Sigma_k^0(X)$, $\delta_{\Pi_k^0(X)}$ of $\Pi_k^0(X)$ for $k \geq 1$ as follows:

- $\delta_{\Sigma_1^0(X)}(p) := \bigcup_{(i,j)\ll(p)} B(\alpha(i), \bar{j})$,
- $\delta_{\Pi_k^0(X)}(p) := X \setminus \delta_{\Sigma_k^0(X)}(p)$,
- $\delta_{\Sigma_{k+1}^0(X)}\langle p_0, p_1, p_2, \cdots \rangle := \bigcup_{i=0}^{\infty} \delta_{\Pi_k^0(X)}(p_i)$,

for all $p, p_i \in \omega^\omega$.

Theorem (M.)

$A \oplus B$ is weak n-random iff A is weak n-random and B is weak n-random a.u. relative to A.

van Lambalgen's theorem

		a.u.	uniform
Demuth	Fail	?	Hold
weak 2	Fail	Hold	?
ML	Hold	Hold	Hold
computable	Fail	?	Hold in a weak sense
Schnorr	Fail	Hold	Hold
Kurtz	Fail	Hold	Fail

Lowness

		a.u.	uniform
Demuth	studied	?	studied
weak 2	K-trivial	K-trivial	K-trivial
ML	K-trivial	K-trivial	K-trivial
computable	computable	?	?
Schnorr	Low(SR)	?	Schnorr trivial
Kurtz	studied	?	studied