Variants of Layerwise Computability

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- Effective version of measurability
  - many applications
- Omega operator
- Demuth randomness implies GL_1.

Layerwise computability

Measurability

Let X, Y be spaces and B_X, B_Y be the set of Borel sets on these spaces. $f : X \to Y$ is measurable if

$$f^{-1}(E) \in B_X \text{ for every } E \in B_Y.$$

Effective versions of this notion have been studied in computable analysis, especially by Brattka.

Another Measurability

Let μ^* be an outer measure on a set X. A subset $S \subset X$ is called μ^*-measurable if

$$\mu^*(A) = \mu^*(A \cap S) + \mu^*(A \setminus S)$$

for every $A \subseteq X$. (This condition is called Carathéodory condition.)

Then, we have the measure μ on the measurable sets. Similarly, we can define μ-measurable functions. In this talk, we only talk about this measurability.

Problem

- The definition is far from constructive.
- How do we effectivize?

Lusin's Theorem

The theorem says that measurability means that one can approximate by continuous functions.

Theorem

A function $f : [0,1] \to \mathbb{R}$ is measurable if and only if, for every $\epsilon > 0$, there is a compact set K and a continuous function g such that

(i) $\mu(K) > 1 - \epsilon$,

(ii) $f|_K = g|_K$.

Computable Measurable Function

A function $f : [0,1] \to \mathbb{R}$ is computably measurable if, for every $n \in \omega$, one can uniformly compute a co-c.e. closed set K_n and a computable function g_n such that

(i) $\mu(K_n) \geq 1 - 2^{-n}$

(ii) $\mu(K_n)$ is uniformly computable,

(iii) $f|_{K_n} = g_n|_{K_n}$.

Schnorr Layerwise Computability

Definition (M. after Hoyrup and Rojas)

A function $f : [0,1] \to \mathbb{R}$ is Schnorr layerwise computable if and only if there is a Schnorr test $\{U_n\}$ and a sequence $\{g_n\}$ of uniformly computable functions such that

$$f|_{K_n} = g_n|_{K_n}$$

where $K_n = [0,1] \setminus U_n$.

Rute defined computable measurability by functions approximated via the distance

$$\int d_{meas}(f(x), g(x))d\lambda$$

where

$$d_{meas}(x, y) = \max\{1, |x - y|\}.$$

M. defined computable measurability by functions such that the preimages of every computable measurable sets are computable measurable sets. They are all equivalent.

Many applications

- Ask Bienvenu or Shen.

Observation by Bienvenu

The Omega operator Ω is layerwise computable relative to $\emptyset'$. This implies that every 2-random set is GL_1.

Question

Can we improve this so that the improved theorem implies that every Demuth random set is GL_1?

The Chaitin's Ω is defined by

$$\Omega = \sum_{\sigma \in \mathrm{dom}(U)} 2^{-|\sigma|}$$

where U is a universal Turing machine.

Proposition

- $0 < \Omega < 1$.
- Ω is a left-c.e. real.
- Ω is ML-random.
- $\Omega \equiv_T \emptyset'$.

The Omega operator Ω is defined by

$$\Omega^X = \sum_{\sigma \in \mathrm{dom}(U^X)} 2^{-|\sigma|}$$

where U^X is a universal Turing machine relative to X.

Proposition

- $\Omega : [0,1] \to [0,1]$.
- Ω is a lower semicomputable function.
- Ω^X is ML-random relative to X for every X.
- $\Omega^X \oplus X \equiv_T X'$.

- 2-randomness
- Demuth randomness
- ML-randomness
- Schnorr randomness

$A \in 2^\omega$ is GL_1 if

$$A' \leq A \oplus \emptyset'.$$

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Variants of computability of reals and functions

Variants of computability of reals

- effectively computable reals
- weakly computable reals
- divergence bounded computable reals
- computably approximable

Computable Reals

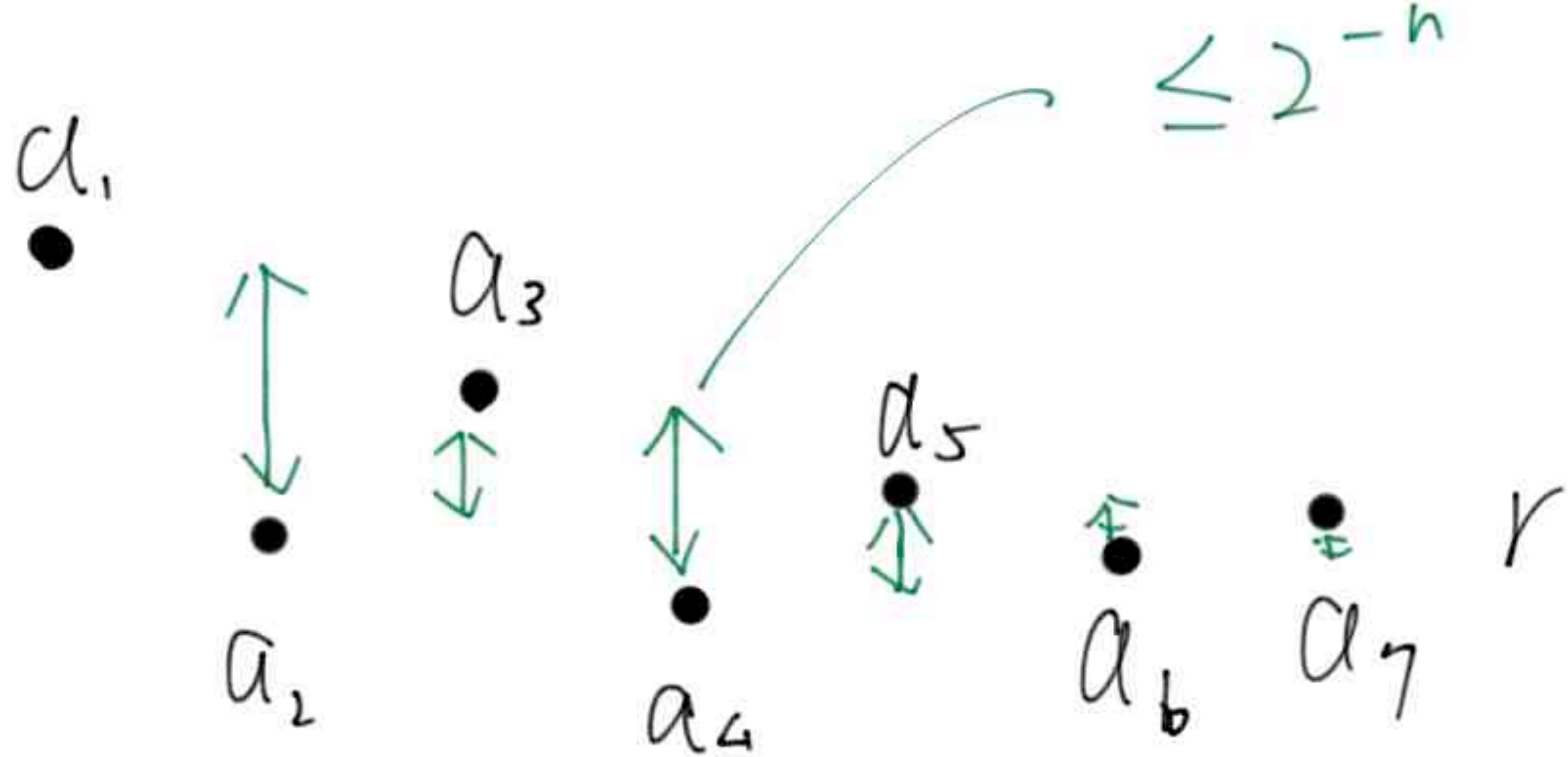
Definition (Turing '36, Robinson '51)

A real $x \in [0, 1]$ is computable if there is a computable sequence $\{r_n\}$ of rationals such that

(i) the sequence converges to x,

(ii) $|r_n - r_{n-1}| \leq 2^{-n}$ for every n.

We say that the convergence is effective.


a_1
a_2
a_3
a_4
a_5
a_6
a_7
r
$\leq 2^{-n}$

Computable Approximation

Definition (Weihrauch and Zheng '98)

A real $x \in \mathbb{R}$ is computably approximable if there is a computable sequence $\{r_n\}$ of rationals such that

$$x = \lim_n r_n.$$

Theorem (Ho '99)

A real x is c.a. if and only if it is $\emptyset'$-computable.

Weak computability

Definition (Ambos-Spies et al. 2000)

A real $x \in \mathbb{R}$ is weakly computable if there is a computable sequence $\{r_n\}$ of rationals such that

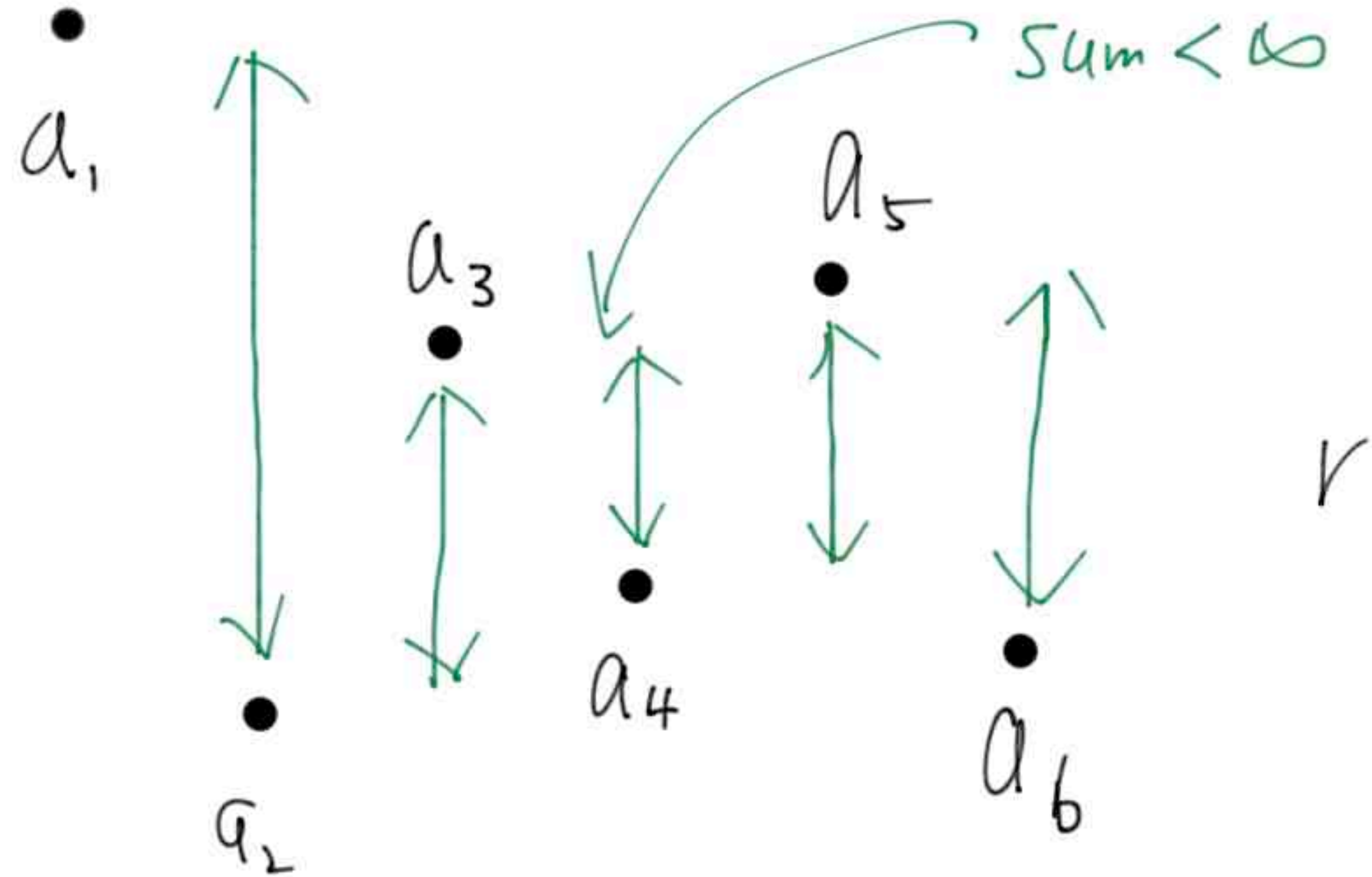
(i) $x = \lim_n r_n$,

(ii) $\sum_n |r_n - r_{n-1}| < \infty$.

Theorem (Ambos-Spies et al. 2000)

A real x is weakly computable if there are two left-c.e. reals α and β such that

$$x = \alpha - \beta.$$


sum < ∞
a_1
a_2
a_3
a_4
a_5
a_6
r

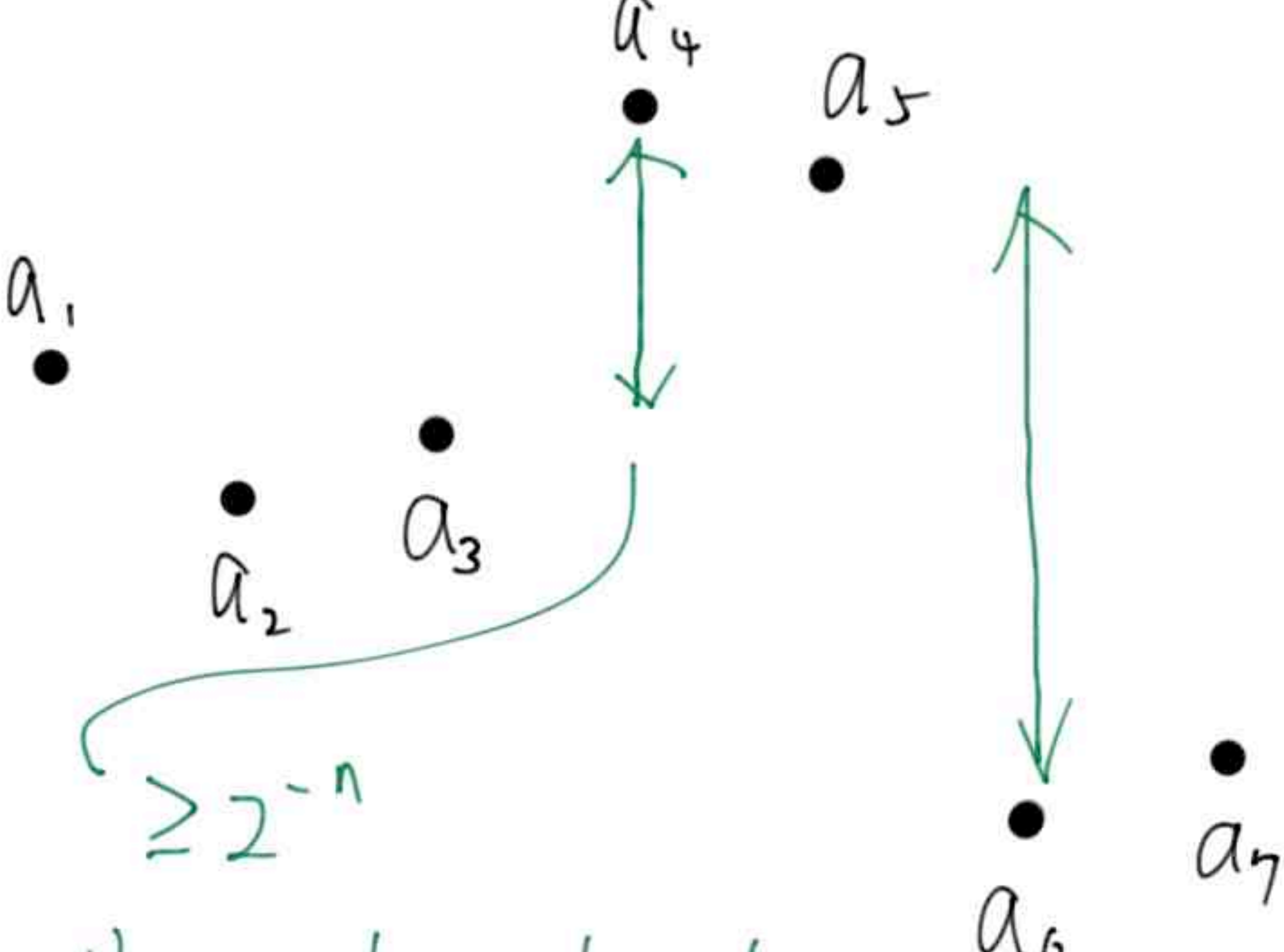
Divergence bounded computability

Definition (Zheng '02)

A real $x \in \mathbb{R}$ is divergence bounded computable if there is a computable sequence $\{r_s\}$ of rationals such that

(i) $x = \lim_n r_n$,

(ii) the number of non-overlapping pairs (i, j) of indices such that $|x_i - x_j| \geq 2^{-n}$ is bounded by a computable function.



the number of such an n
is bounded by a comp. func

A hierarchy of sets

- Delta^0_2
- omega-c.e. sets
- d.c.e. sets
- c.e. sets
- computable sets

A hierarchy of reals

- 0’-computable reals
- divergence bounded computable reals
- weakly computable reals
- left-c.e. reals
- computable reals

Computable Functions

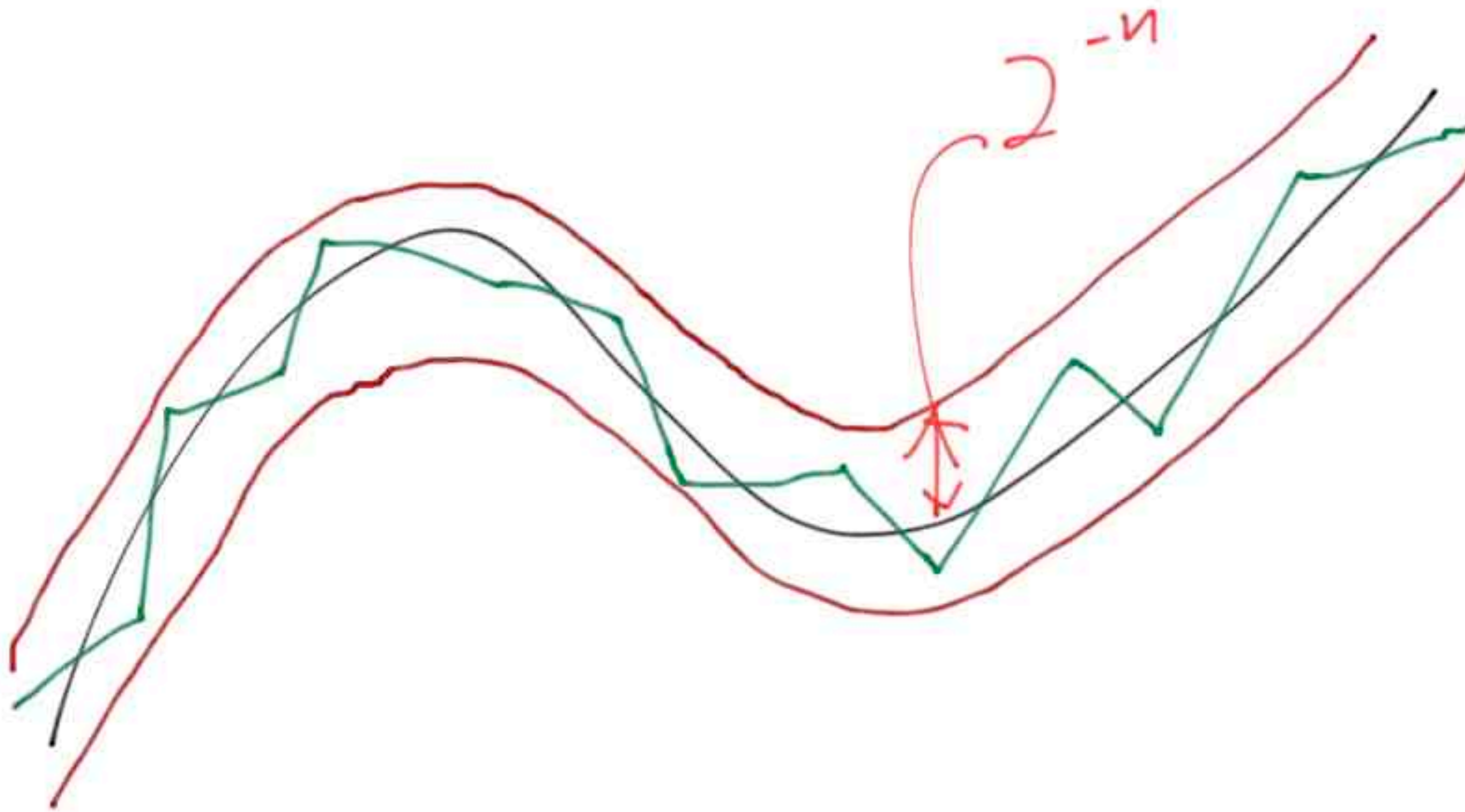
Definition (Pour-El-Richards '89)

A function $f : [0,1] \to \mathbb{R}$ is computable if there is a computable sequence $\{r_n\}$ of rational polygons such that

(i) $f(x) = \lim_n r_n(x)$ for all $x \in [0,1]$.

(ii) $||r_n - r_{n-1}||_\infty \leq 2^{-n}$.

Here, $||f||_\infty = \sup_{x\in[0,1]} |f|$.



2^{-n}

Variants of computability of functions

- (effectively) computable function
- uniformly weakly computable function
- uniformly divergence bounded computable function
- uniformly computably approximable function

Uniformly Weakly Computable Functions

Definition (Bauer-Zheng '10)

A function $f : [0,1] \to \mathbb{R}$ is uniformly weakly computable if there is a computable sequence $\{r_n\}$ of rational polygons such that

(i) $f(x) = \lim_n r_n(x)$ for all $x \in [0,1]$.

(ii) $\sum_n ||r_n - r_{n-1}||_\infty \leq 1$.

Recall that $||f||_\infty = \sup_{x\in[0,1]} |f|$.

Representations

$$\begin{aligned}
&\text{(i)} \quad \rho_{EC} = \rho \\
&\text{(ii)} \quad \rho_{WC} \\
&\text{(iii)} \quad \rho^h_{DBC} \\
&\text{(iv)} \quad \rho_{CA} = \lim \circ \rho
\end{aligned}$$

Variants of computability of functions

(i) (effectively) computable = (ρ, ρ)-computable

(ii) uniformly WC = (ρ, ρ_{WC})-computable

(iii) uniformly DBC = (ρ, ρ^h_{DBC})-computable for some computable function h

(iv) uniformly CA = (ρ, ρ_{CA})-computable ?

Remark

Let f be a uniformly DBC function. If $f(x)$ is defined, then

$$f(x) \leq_T x \oplus \emptyset'.$$

Variants of layerwise computability

Real or Function	Test
Computable	Schnorr
Weakly Computable	ML
Divergence Bounded Computable	Demuth
Computably Approximable	Weak 2?

Demuth randomness

Definition

A Demuth test is a sequence $\{U_n\}$ of c.e. open sets such that $\mu(U_n) \leq 2^{-n}$, and there is an ω-c.e. function f such that $U_n = [W_{f(n)}]$. A set X is Demuth random if $Z \notin U_n$ for almost all n.

Demuth-layerwise DBC

Definition

A function $f : [0,1] \to \mathbb{R}$ is Demuth-layerwise divergence bounded computable if there are

- a Demuth test $\{U_n\}$
- a uniform sequence $\{f_n\}$ of uniformly DBC functions

such that

$$f|_{K_n} = f_n|_{K_n} \text{ where } K_n = [0,1] \setminus \bigcup_{k=n}^{\infty} U_k.$$

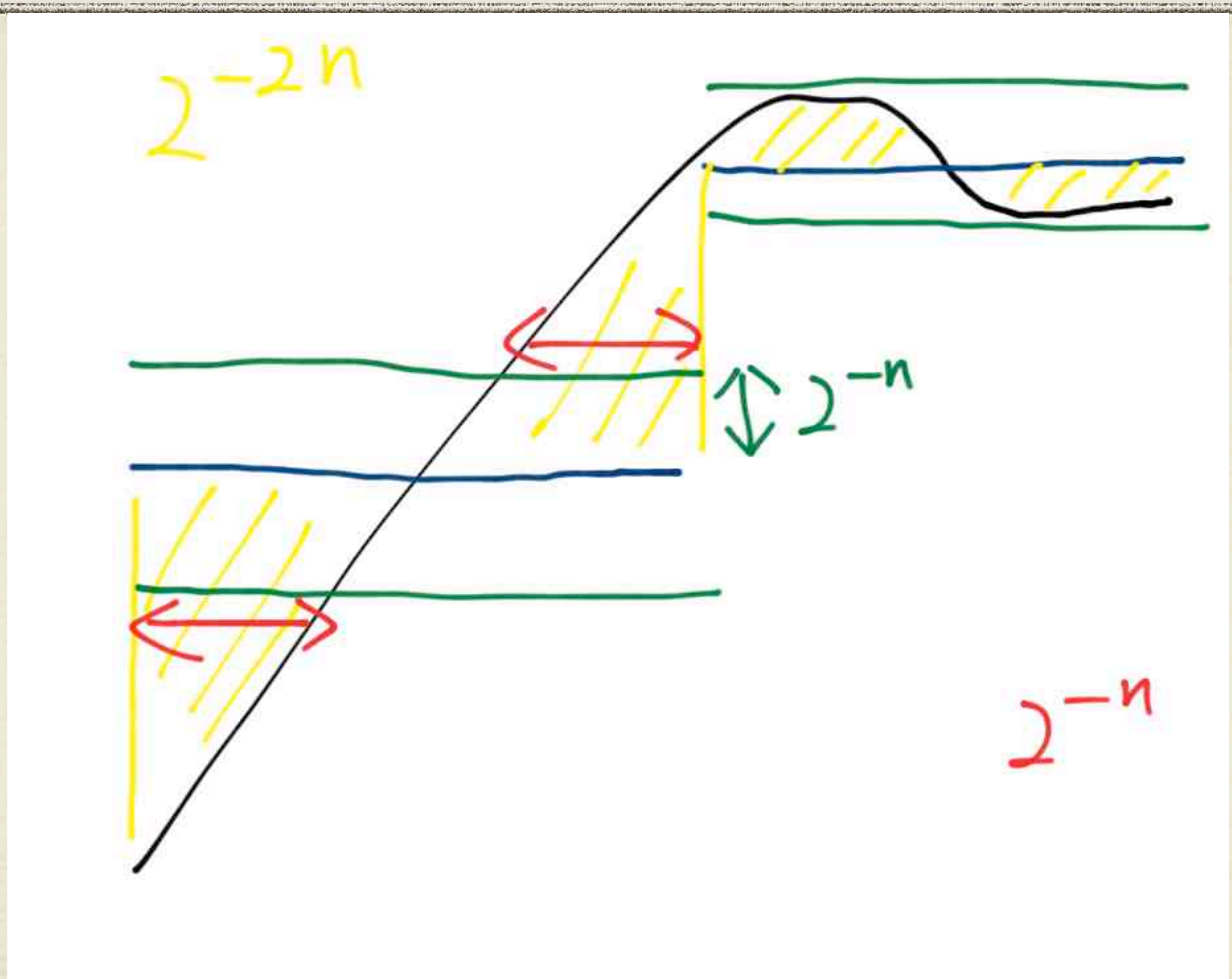
Main theorem of this talk

Theorem

Every integrable lower semicomputable function is Demuth-layerwise DBC.

Corollary

In particular, the omega operator Ω is Demuth-layerwise DBC. Hence, every Demuth random set is GL_1.



2^-2n
2^-n
2^-n

ML-layerwise WC

Definition

A function $f : [0,1] \to \mathbb{R}$ is ML-layerwise weakly computable if there are

(i) a Solovay test $\{U_n\}$

(ii) a sequence $\{f_n\}$ of uniformly weakly computable functions

such that

$$f|_{K_n} = f_n|_{K_n} \text{ where } K_n = [0,1] \setminus \bigcup_{k=n}^{\infty} U_k.$$

p-weakly computability

Definition

For $0 < p \leq 1$, a real x is p-weakly computable if there is a computable sequence $\{r_n\}$ of rationals

(i) $x = \lim_n r_n$

(ii) $\sum_n |r_n - r_{n-1}|^p < \infty$.

If $p \leq q$, then p-weakly computability implies q-weakly computability.

Theorem

Every 1/2-weakly L^1-computable function is ML-layerwise weakly computable.

Remark

- The omega operator is not ML-layerwise weakly computable.
- These two theorems also can be seen as effectivizations of one direction of Lusin's theorem.

Summary

- We have hierarchies of reals, continuous functions and layerwise continuous functions.
- The omega operator is Demuth-layerwise DBC.
- Are there any other interesting measurable functions that is computable in some sense?
- How about the converse?

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