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Reducibilities relating to Schnorr randomness

Kenshi Miyabe

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Abstract Some measures of randomness have been introduced for Martin-Löf randomness such as K-reducibility, C-reducibility and vL-reducibility. In this paper we study Schnorr-randomness versions of these reducibilities. In particular, we characterize the computably-traceable reducibility via relative Schnorr randomness, which was asked in Nies' book [22, Problem 8.4.22]. We also show that Schnorr reducibility implies uniform-Schnorr-randomness version of vL-reducibility, which is the Schnorr-randomness version of the result that K-reducibility implies vL-reducibility.

Keywords Algorithmic randomness, Schnorr randomness, van Lambalgen reducibility, Schnorr reducibility, integral test

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1 Introduction

1.1 K-reducibility as a measure of randomness

The theory of algorithmic randomness clarifies the meaning of randomness of infinite binary sequences from the computability point of view. For instance, a set $X \in 2^\omega$ is ML-random if and only if $K(X \upharpoonright n) > n - O(1)$ where K is the prefix-free Kolmogorov complexity. Roughly saying, a sequence is random if the complexities of the initial segments of the sequence is high.

The main theme of this paper is to understand the assertion that a sequence is "more random" than another sequence. By the above result, it is natural to define it via the complexities of the initial segments. One such partial order is

Kenshi Miyabe
Graduate school of Information Science and Technology, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, JAPAN.
Tel: +81-3-5841-6943

K-reducibility defined by $X \leq_K Y$ if and only if $K(X \upharpoonright n) \leq K(Y \upharpoonright n) + O(1)$. We can similarly C-reducibility where C is the plain Kolmogorov complexity.

1.2 LR-reducibility

One direction to analyse K-reducibility is the one via lowness. The researchers have studied the class of the sets X satisfying $X \leq_K \emptyset$. Such a set is called *K-trivial*. Surprisingly, this class has a characterization via relative randomness. A set X is called *low for ML-randomness* if each ML-random is ML-random relative to X. The intuition behind this is that the set can not derandomize any random set. In fact, a set is K-trivial if and only if it is low for ML-randomness [21]. This result says that being far from random is equivalent to computationally weakness. This is an interesting relation between the theory of randomness and computability theory.

Nies [21] introduced *low-for-random reduciblity (LR-reduciblity)*, which is defined by $X \leq_{LR} Y$ if and only if every ML-random set relative to Y is ML-random relative to X. We also consider *LK-reducibility* defined by $X \leq_{LK} Y$ if and only if $K^Y(n) \leq K^X(n) + O(1)$. In particular, a set is called low for K if $X \leq_{LK} \emptyset$. Nies [21] also showed that lowness for ML-randomness is equivalent to lowness for K. Kjos-Hanssen et al. [12] strengthned this result to that LK-reducibility is actually equivalent to LR-reducibility.

1.3 Van Lambalgen reducibility

A very useful tool to analyse K-reducibility is *van Lambalgen reducibility*, which was introduced in Miller and Yu [16]. We say that $X \leq_{vL} Y$ if and only if, for every ML-random Z, if $X \oplus Z$ is ML-random, then $Y \oplus Z$ is ML-random. Notice that this ia a measure of randomness for ML-random sets. This is (as the name indicates) inspired by van Lambalgen's theorem, which says that $X \oplus Y$ is ML-random if and only if X is ML-random and Y is ML-random relative to X. Notice that vL-reducibility is the inverse of LR-reducibility for ML-random sets by van Lambalgen's theorem. Many properties are easy to prove for vL-reducibility because of close connection to Turing reducibility. Furthermore, Miller and Yu [16] showed that K-reducibility implies vL-reducibility, and C-reducibility implies vL-reducibility. By this result, we can deduce many properties for K-reducibilities and the interaction with Turing reducibility.

1.4 Schnorr-randomness versions

As already stated, the main theme of this paper is to understand a measure of randomness (more random than another), but, not in the sense of Martin-Löf randomness but in the sense of Schnorr randomness.

Downey and Griffiths [6] showed that $X \in 2^\omega$ is Schnorr random if and only if $K_M(X \upharpoonright n) > n - O(1)$ for every computable measure machine M.

Hence, the natural measure of randomness in the sense of Schnorr randomness is *Schnorr reducibility*, which is defined by $X \leq_{Sch} Y$ if for every computable measure machine M there is a computable measure machine N such that $K_N(X \upharpoonright n) \leq K_M(Y \upharpoonright n) + O(1)$.

Here, natural questions arise:

(A) What is a Schnorr-randomness version of C-reducibility?
(B) What is a Schnorr-randomness version of LR-reducibility?
(C) Is Schnorr-randomness version of vL-reducibility useful to analyse Schnorr reduciblity?

For an answer to Question (A), we claim in Section 3.2 that total machines play a part of plain machines in the study of Schnorr randomness.

On Question (B), unlike the case of ML-randomness, Schnorr triviality is not equivalent to lowness for Schnorr randomness. Franklin and Stephan [9], instead, showed that Schnorr triviality is equivalent to truth-table version of lowness for Schnorr randomness. Miyabe [18] noted that we can see this as a different way to relativize and Miyabe and Rute [20] call this *uniform relativization*.

With the usual relativization, the following are equivalent for a set A:

(i) A is low for Schnorr randomness,
(ii) A is low for Schnorr tests,
(iii) A is computably traceable,
(iv) A is low for computable measure machines,

which is shown by Terwijn and Zambella [25], Kjos-Hanssen et al. [13] and Downey et al. [5]. Computable traceability is a notion inspired by a one in set theory. It is an open question whether K-triviality has a characterization via traceability. Notice that the equivalence between (i) and (iv) is a Schnorr-randomness version of $X \leq_{LR} \emptyset \iff X \leq_{LK} \emptyset$. Thus, it is a natural question whether the reducibility versions are also equivalent. In particular, Nies [22, Problem 8.4.22] asked whether the reducibility versions of (i) and (iii) are equivalent. We give the affirmative answer to this question in Section 4.

With uniform relativization, Miyabe [17] showed the equivalence among the reducibility versions of lowness for uniform Schnorr randomness, lowness for uniform Schnorr tests and lowness for uniformly computable measure machines.

We also give characterizations of some reducibilities via integral tests in Section 4.

On Question (C), the key in defining vL-reducibility is van Lambalgen's theorem. For Schnorr randomness, van Lambalgen's theorem does not hold with the usual relativization while it does hold with uniform relativiation [18, 20]. By this result, we define a Schnorr-randomness version of vL-reducibility and study the relation with Schnorr reducibility in Section 6. In particular, we show that Schnorr reducibility implies the Schnorr-randomness version of vL-reducibility.

In Section 5, we strengthen the Ample Excess Lemma, which is a key lemma in the study of vL-reducibility. This result will be used in the next section, Section 6. We also study the property of maximality up to a constant infinitely often for Schnorr randomness with independent interest.

2 Preliminaries

In this section, we fix notations and review definitions and results we use in later sections. This includes the results on the reducibilities relating to ML-randomness and some useful tools used to study Schnorr randomness such as some results on computable analysis, uniform relativization and Schnorr integral tests.

2.1 Notation

A string $\sigma \in 2^{<\omega}$ sometimes denotes the natural number n such that σ is the n-th lexicographical order. For instance, the empty string represents the natural number 0 and the string 0 represents the natural number 1. For the avoidance of confusion, we sometimes use the notation $\overline{\sigma}$. For instance, $\overline{1} = 2$. For a string σ with length n, we have $2^n - 1 \leq \overline{\sigma} \leq 2^{n+1} - 2$.

2.2 ML and Schnorr randomness

For background on the theory of algorithmic randomness, we refer to Downey and Hirschfeldt [7] and Nies [22].

We consider Cantor space 2^ω with the uniform measure μ.

A *ML-test* is a sequence of uniformly c.e. open sets $\{U_n\}$ such that $\mu(U_n) \leq 2^{-n}$ for every n. A set $X \in 2^\omega$ is *ML-random* if $X \notin \bigcap_n U_n$ for every ML-test $\{U_n\}$. There is a *universal ML-test* $\{V_n\}$ in the sense that X is ML-random if and only if $X \notin \bigcap_n V_n$.

A *Schnorr test* is a ML-test $\{U_n\}$ such that $\mu(U_n)$ is uniformly computable. A set X is *Schnorr random* if $X \notin \bigcap_n U_n$ for every Schnorr test $\{U_n\}$. It is known that there is no universal Schnorr test.

A *Schnorr Solovay test* is a sequence of uniformly c.e. open sets $\{U_n\}$ such that $\sum_n \mu(U_n)$ is a computable real. A set X is Schnorr random if and only if, for every Schnorr Solovay test, $X \in U_n$ for at most finitely many n.

The Levin-Schnorr theorem says that a set X is ML-random if and only if $K(X \upharpoonright n) > n - O(1)$ where K is the universal prefix-free Kolmogorov complexity and $X \upharpoonright n$ denotes the initial segment of X with the length n. For a prefix-free machine M, the *measure* of M is

$$\Omega_M = \sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|}.$$

The measure of M is left-c.e. in general. The machine is called a *computable measure machine* (c.m.m. for short) if the measure of M is computable. A set X is Schnorr random if and only if $K_M(X \upharpoonright n) > n - O(1)$ for every computable measure machine.

2.3 Computable analysis

For background on computable analysis, see [26,3]. An *integral test* is a lower semicomputable function $t : 2^\omega \to \overline{\mathbb{R}}^+$ with $\int t \, d\mu < \infty$. A set X is ML-random if and only if $t(X) < \infty$ for every integral test. A *Schnorr integral test* is a lower semicomputable function $t : 2^\omega \to \overline{\mathbb{R}}^+$ such that $\int t \, d\mu$ is a computable real. A set X is Schnorr random if and only if $t(X) < \infty$ for every Schnorr integral test [19, Theorem 3.5].

2.4 Uniform relativization

For details of uniform relativization, see [20] or [11]. Uniform relativization is a relativization different from the usual way. For some $X \in 2^\omega$, Schnorr randomness relative to X is different from Schnorr randomness uniformlly relative to X. The following is one characterization of Schnorr randomness with uniform relativization. A *uniformly computable measure machine* is an oracle prefix-free machine M such that the function $X \mapsto \Omega_{M^X}$ is computable. A set X is *Schnorr random uniformly relative to* Y if and only if $K_{M^Y}(X \upharpoonright n) > n - O(1)$ for every uniformly computable measure machine M ([18, Theorem 4.14] and [17, Theorem 2.4]).

With uniform relativization, lowness notions behave well for Schnorr randomness [17]. Furthermore, van Lambalgen's theorem for Schnorr randomness holds as follows.

Theorem 1 (Miyabe [18], Miyabe and Rute [20]) *A set $X \oplus Y$ is Schnorr random if and only if X is Schnorr random and Y is Schnorr random uniformly relative to X.*

Note that van Lambalgen's theorem for Schnorr randomness with the usual relativization does not hold [14,27]. See also [22, Remark 3.5.22].

3 Some kinds of machines

In this section, we show some results on some kinds of machines that are used to characterize Schnorr randomness. These facts are often used throughout the paper.

3.1 The variant of halting probability

For a universal prefix-free machine U, the value $\Omega_U = \sum_{\sigma\in\mathrm{dom}(U)} 2^{-|\sigma|}$ is called *halting probability* and has studied extensively (see [7, Chapter 9]).

Chaitin [4] also defined $\widehat{\Omega} = \sum_{\sigma\in 2^{<\omega}} 2^{-K(\sigma)}$. Notice that this value depends on the universal Turing machine used to define K. Extending this definition, we use the following notation.

Definition 1 For a prefix-free machine M, the value $\widehat{\Omega}_M$ is defined by

$$\widehat{\Omega}_M = \sum\{2^{-K_M(\sigma)} \ : \ \sigma \in 2^{<\omega},\ K_M(\sigma) < \infty\}.$$

The machine M is not universal in general, thus $K_M(\sigma)$ may be infinity. For simplicity, we sometimes write $2^{-K_M(\sigma)}$ including the case of $K_M(\sigma) = \infty$. Notice that $\widehat{\Omega}_M$ is a left-c.e. real for every machine M. If U is a prefix-free universal machine, then $\widehat{\Omega}_U$ is ML-random [7, p.229].

We will show that, if M is a computable measure machine, then $\widehat{\Omega}_M$ is also computable. To state a stronger statement, we recall Solovay reducibility. The following is not the original definition by Solovay [24] but is a characterization by [8]. Let α and β be left-c.e. reals. Then, a real α is *Solovay reducible* to a real β (written $\alpha \le_S \beta$) if and only if there are a constant d and a left-c.e. real γ such that $d\beta = \alpha + \gamma$.

Proposition 1 *Let M be a prefix-free machine. Then,*

$$\widehat{\Omega}_M \le_S \Omega_M.$$

In particular, if M is a computable measure machine, then $\widehat{\Omega}_M$ is computable.

Proof For each $\tau \in 2^{<\omega}$, let

$$\alpha_\tau = \sum\{2^{-|\sigma|} \ : \ M(\sigma) = \tau\} \text{ and } \beta_\tau = 2^{-K(\tau)}.$$

Then, α_τ and β_τ are left-c.e. reals uniformly in τ. Furthermore, $\gamma_\tau = \alpha_\tau - \beta_\tau$ is also a left-c.e. real uniformly in τ. Notice that

$$\Omega_M = \sum_{\tau\in 2^{<\omega}} \alpha_\tau = \sum_{\tau\in 2^{<\omega}} (\beta_\tau + \gamma_\tau) = \widehat{\Omega}_M + \sum_{\tau\in 2^{<\omega}} \gamma_\tau.$$

Thus, $\widehat{\Omega}_M \le_S \Omega_M$. □

The relation $\Omega_M \le_S \widehat{\Omega}_M$ does not hold in general. Let A be a c.e. set of natural numbers such that $\sum_{n\in A} 2^{-n}$ is not computable and $0 \in A$. We define a prefix-free machine M by

$$M(\sigma) = \begin{cases} \epsilon & \text{if } \sigma = 0^n 1 \text{ for } n \in A \\ \uparrow & \text{otherwise} \end{cases}$$

where ϵ denotes the empty string. Then, $\Omega_M = \sum_{n\in A} 2^{-(n+1)}$ is not computable while $\widehat{\Omega}_M = 2^{-K(\epsilon)} = 1/2$ is computable.

3.2 Total machines

ML-randomness has a characterization via prefix-free Kolmogorov complexity. Its Schnorr-randomness version is the characterization via computable measure machines. ML-randomness also has a characterization by plain Kolmogorov complexity as follows.

Theorem 2 (Miller and Yu [16]) *The following are equivalent for a set $X \in 2^\omega$:*

(i) *X is ML-random.*
(ii) *$C(X \upharpoonright n) \geq n - K(n) - O(1)$.*

Then, what is the counterpart of this result in the study of Schnorr randomness? Recall that a machine is called *decidable* if the domain of the machine is computable, and Bienvenu and Merkle [1] showed that a set X is Schnorr random if and only if for all decidable prefix-free machines M and all computable orders g, we have $K_M(X \upharpoonright n) \geq n - g(n) - O(1)$. Notice that the inequalities are similar. Thus, decidable machines are a first candidate. Notice that decidable machines without the requirement of prefix-freeness are essentially the same as total machines. A machine is called *total* if the domain of the machine is total, and total machines are used to characterize Schnorr triviality in Hölzl and Merkle [10, Theorem 26]. In fact, we have a characterization of Schnorr randomness via total machines.

Theorem 3 *The following are equivalent for a set X:*

(i) *X is Schnorr random.*
(ii) *For every total machine N and every computable order g, we have*

$$C_N(X \upharpoonright g(n)) \geq g(n) - n - O(1).$$

(iii) *For every computable measure machine M and every total machine N, we have*

$$C_N(X \upharpoonright n) \geq n - K_M(n) - O(1).$$

Remark 1 We can replace "total machine" in the statement with "decidable machine".

Proof (i)⇒(iii) Let M be a computable measure machine and N be a total machine. For $k \in \omega$, let

$$V_{k,n} = \{\sigma \in 2^n \ : \ C_N(\sigma) < n - K_M(n) - k\}.$$

Each $\sigma \in V_{k,n}$ should have a string τ such that $|\tau| < n - K_M(n) - k$ and $N(\tau_\sigma) = \sigma$. Since the number of strings with the length less than m is $2^m - 1$, we have

$$\#V_{k,n} \leq 2^{n-K_M(n)-k}.$$

Let

$$G_k = \{Z \in 2^\omega \ : \ C_N(Z \upharpoonright n) < n - K_M(n) - k \text{ for some } n\}.$$

Notice that $G_k = \bigcup_n [\![V_{k,n}]\!]$. We claim that $\{G_k\}$ is a Schnorr test. Clearly, G_k is a uniformly c.e. open set. The measure of G_k is

$$\mu(G_k) \le \sum_n 2^{-n} \cdot 2^{n-K_M(n)-k} \le 2^{-k}.$$

Since M and N are decidable machines, the functions C_N and K_M are computable and $V_{k,n}$ is uniformly computable. Then, $\bigcup_{n\le N}[\![V_{k,n}]\!]$ is uniformly computable. Furthermore, we have

$$\mu\left(\bigcup_{n>N}[\![V_{k,n}]\!]\right) \le \sum_{n>N} 2^{-K_M(n)-k}.$$

Since M is a computable measure machine, the value $\sum_n 2^{-K_M(n)}$ is a computable real by Proposition 1. Thus, G_k is uniformly computable. Hence, $\{G_k\}$ is a Schnorr test.

If X is a Schnorr random, then Z passes the test $\{G_k\}$ and $C_N(X \upharpoonright n) \ge n - K_M(n) - O(1)$.

(iii)⇒(ii) Suppose that (ii) does not hold. Let N and g be the pair of the witness. We define a computable measure machine M by

$$M(0^n1) = g(n).$$

Then, $K_M(g(n)) = n+1$. Hence, for every k there exists n such that

$$C_N(X \upharpoonright g(n)) < g(n) - n - k = g(n) - K_M(g(n)) - (k-1).$$

Thus, (iii) does not hold. □

Our proof of (ii)⇒(i) uses the following characterization of Schnorr randomness.

Lemma 1 *The following are equivalent for a set X:*

(i) *X is Schnorr random,*
(ii) *For every computable order g and every computable sequence $\{S_n\}$ of sets of strings such that $S_n \subseteq 2^{g(n)}$ and $\#S_n = 2^{g(n)-n}$, we have $X \upharpoonright g(n) \in S_n$ for at most finitely many n.*

Proof (i)⇒(ii) This is because $\{[\![S_n]\!]\}$ is a Schnorr Solovay test.

(ii)⇒(i) Suppose that X is not Schnorr random. Then there exists a Schnorr Solovay test $\{G_n\}$ such that $X \in G_n$ for infinitely many n. We can assume that $G_n = [\sigma_n]$ for a computable sequence $\{\sigma_n\}$ of strings without loss of generality. Let $\alpha = \sum_n 2^{-|\sigma_n|}$. Then, α is a computable real. Hence, there exists a computable order h such that $\sum_{n>h(k)} 2^{-|\sigma_n|} \le 2^{-k}$. Let $g(k)$ be the maximum of the lengths σ_n with $h(k) < n \le h(k+1)$. Then, there exists a computable sequence $\{S_k\}$ of sets of strings such that $S_k \subseteq 2^{g(k)}$ and $\bigcup_{n=h(k)+1}^{h(k+1)} \sigma_n \subseteq [\![S_k]\!]$. Since $\sum_{n=h(k)+1}^{h(k+1)} 2^{-|\sigma_n|} \le 2^{-k}$, we can assume that $\#S_n = 2^{g(n)-n}$. Since $X \in [\sigma_n]$ for infinitely many n, we have $X \upharpoonright g(n) \in S_n$ for infinitely many n. □

Proof (of (ii)⇒(i) of Theorem 3) Suppose that X is not Schnorr random. Then, there exist a computable order g and a computable sequence $\{S_n\}$ of sets of strings such that $S_n \subseteq 2^{g(n)}$, $\#S_n = 2^{g(n)-2n}$ and $X \upharpoonright g(n) \in S_n$ for infinitely many n. We can assume that $h(n) = g(n) - 2n$ is strictly increasing.

We define a total machine N as follows: Assume that N receives σ as an input. If $|\sigma| = g(n) - 2n$ for some n, let $k \in \omega$ be such that σ is the k-th string in $2^{g(n)-2n}$. Then the machine N outputs k-th string in S_n. If no n satisfy with $|\sigma| = g(n) - 2n$, then N outputs the empty string. Then, $C_N(X \upharpoonright g(n)) \leq g(n) - 2n$ for infinitely many n. □

4 Reducibilities relating to lowness notions

In this section we characterize some reducibilities.

4.1 Computable traceable reducibility

Many traceability notions have been considered in the literature [10]. A *trace* is a sequence $\{T_n\}$ of sets. A trace is a *trace for* a partial function f, if $f(n) \in T_n$ holds for all n such that $f(n)$ is defined. Nies [22, Excercise 8.4.21] defined the following reducibility.

Definition 2 Let $A, B \in 2^\omega$. We say that $A \leq_{CT} B$ if there is a computable order h such that for each $f \leq_T A$ there exists $p \leq_T B$ such that for all n we have $f(n) \in D_{p(n)}$ and $|D_{p(n)}| \leq h(n)$.

Here, D_k is the k-th finite set usually of strings. In other words, $A \leq_{CT} B$ if there is a computable order h such that every A-computable function is traced by a h-bounded B-computable trace.

As is in the excercise, $\leq_{CT}$ is transitive. Clearly, $\leq_{CT}$ is reflexive. It is easy to see that $\leq_T$ implies $\leq_{CT}$.

Remark 2 Notice that $A \leq_{CT} \emptyset$ if and only if A is computably traceable. Terwijn and Zambella [25] observed that c.e. traceability is equivalent to c.e. traceability via every computable order. As is in Hölzl and Merkle [10, Remark 4], many variants have this property. The reducibility $\leq_{CT}$ also has this property, that is, $A \leq_{CT} B$ if and only if every A-computable function is traced by a B-computable trace via every computable order.

Recall that A is computably traceable if and only if A is low for Schnorr randomness [25,13]. We strengthen this equivalence as follows, answering the question of Nies [22, Problem 8.4.22] affirmatively.

Theorem 4 *The following are equivalent for $A, B \in 2^\omega$:*

(i) $A \leq_{CT} B$,
(ii) *Every Schnorr random set relative to B is Schnorr random relative to A.*

4.1.1 The proof of (i)⇒(ii)

We first prove the easy direction, (i)⇒(ii), by giving the following lemma.

Lemma 2 *Suppose that $A \leq_{CT} B$. Then, every A-Schnorr test is covered by a B-Schnorr test.*

We use the standard method to prove the implication from traceability to a lowness notions such as Theorem 2 in [25]. The essential idea is that we construct an open set which is a union of open sets with all possible oracles. By the property of traceability, possible oracles are few and the measure of the union can be small enough.

Proof Let $\{U_n\}$ be an A-Schnorr test. Then, there exists an A-computable function $f : \omega \to (2^{<\omega})^{<\omega}$ such that

$$\bigcup_m [\![f(\langle n, m\rangle)]\!] = U_n \text{ and } \mu([\![f(\langle n, m\rangle)]\!]) \leq 2^{-n-m-1}.$$

Then, there exists a B-computable function p such that for all n we have $f(\langle n, m\rangle) \in D_{p(\langle n,m\rangle)}$ and $|D_{p(\langle n,m\rangle)}| \leq n + m$ by Remark 2. We assume that, for each $S \in D_{p(\langle n,m\rangle)}$, we have $\mu([\![S]\!]) \leq 2^{-n-m-1}$. For each n, we define an open set V_n by

$$V_n = \bigcup_{m\in\omega} \bigcup_{S\in D_{p(\langle 2n+c,m\rangle)}} [\![S]\!]$$

where c will be defined later. Then, $\{V_n\}$ is a sequence of uniformly B-c.e. open sets. For each $S \in D_{p(\langle n,m\rangle)}$, the measure $\mu([\![S]\!])$ is computable. Furthermore,

$$\mu(\bigcup_{m\geq M} \bigcup_{S\in D_{p(\langle 2n+c,m\rangle)}} [\![S]\!]) \leq \sum_{m\geq M} 2^{-2n-c-m-1} \times (n + m).$$

Thus, the measure $\mu(V_n)$ is uniformly computable. By taking c sufficiently large, we have $\mu(V_n) \leq 2^{-n}$. Hence, g is a B-Schnorr test.

Finally, we cliam that $U_{2n+c} \subseteq V_n$ for all n. Suppose $Z \in U_{2n+c}$. Then there exists m such that $Z \in [\![f(\langle 2n + c, m\rangle)]\!]$. Thus, $f(\langle 2n + c, m\rangle) \in D_{p(\langle 2n+c,m\rangle)}$. Hence $Z \in V_n$. □

Next we give a proof of the other direction, (ii)⇒(i) of Theorem 4 by giving a series of lemmas.

4.1.2 The open covering method

First, we use the open covering method developed in [2]. This method is very powerful and has been used to show some similar results such as [17, 11].

An open set $U \subseteq 2^\omega$ is *bounded* if $\mu(U) < 1$. We say that an open set U is an A-Schnorr open set if U is A-c.e. open and the measure $\mu(U)$ is A-computable.

Lemma 3 *Suppose that every Schnorr random set relative to B is Schnorr random relative to A. Then, every bounded A-Schnorr open set is covered by a bounded B-Schnorr open set.*

Our proof of this lemma uses the following notation and the proposition. For a set $W \subseteq 2^{<\omega}$, we denote by W^ω the set of all sets of the form $\sigma_0\sigma_1\sigma_2\ldots$ such that $\sigma_i \in W$ for every $i \in \omega$. A *test* is a non-increasing sequence $\{U_n\}$ of open sets such that $\bigcap_n U_n$ has measure 0.

Proposition 2 (Bienvenu and Miller [2, Theorem 9]) *The following are equivalent for a set $X \in 2^\omega$:*

(i) *X is not Schnorr random.*
(ii) *$X \in U^\omega$ for some bounded Schnorr prefix-free subset U of $2^{<\omega}$.*

The following is a rewritten version of Lemma 12 and Proposition 13 of Bienvenu and Miller [2].

Proposition 3 (Bienvenu and Miller [2]) *Let $\mathcal{C}$ be the class of bounded Schnorr open sets and T^e be the family of Schnorr tests. Let W be a prefix-free subset of $2^{<\omega}$ such that $[\![W]\!]$ cannot be covered by any set $U \in \mathcal{C}$. Then there exists $X \in W^\omega$ that passes all tests T^e.*

Proof (of Lemma 3) We show the contrapositive. Let $\mathcal{C}$ be the class of B-bounded Schnorr open sets and T^e be the family of B-Schnorr tests. Let V be a bounded A-Schnorr open set that is not covered by any bounded B-Schnorr open set. Let W be a A-c.e. prefix-free subset such that $[\![W]\!] = V$. By relativizing Proposition 3, there exists $Z \in W^\omega$ that passes all test T^e. Then, Z is B-Schnorr random, but not A-Schnorr random by relativizing Proposition 2. □

4.1.3 Summable functions

Next we show a property of summable functions from the open covering property with the method used in [12].

A function $g : \omega \to \mathbb{R}^+$ is called *summable* if $\sum_n g(n) < \infty$.

Lemma 4 *Suppose that every bounded A-Schnorr open set is covered by a bounded B-Schnorr open set. For every A-computable function $f : \omega \to \mathbb{R}^+$ such that $\sum_n f(n)$ is A-computable, there exists a B-left-c.e. function $g : \omega \to \mathbb{R}^+$ such that $\sum_n f(n)$ is B-left-c.e. and $f(n) \leq g(n)$ for all n.*

The proof is almost identical to that of Proposition 5.1 in the revised version of [2] in arXiv. See also Section 5.2 in [17].

4.1.4 KC Theorem

Finally, we show $\leq_{CT}$ from the property of summable functions via the KC Theorem.

Lemma 5 *Suppose that, for every A-left-c.e. function $f : \omega \to \mathbb{R}^+$ such that $\sum_n f(n)$ is A-computable, there exists a B-left-c.e. function $g : \omega \to \mathbb{R}^+$ such that $\sum_n f(n)$ is B-computable and $f(n) \leq g(n)$ for all n. Then, for every A-computable measure machine M, there exists a B-computable measure machine N such that $K_N(\sigma) \leq K_M(\sigma) + O(1)$.*

The proof is almost identical to that of Proposition 27 in [2]. See also Section 5.3 in [17].

Lemma 6 *Suppose that, for every A-computable measure machine M, there exists a B-computable measure machine N such that $K_N(\sigma) \leq K_M(\sigma) + O(1)$. Then, we have $A \leq_{CT} B$.*

Proof Let $f \leq_T A$. We can assume that f is a function from ω to $2^{<\omega}$. Let M be the A-c.m.m. defined by $M(0^n 1) = f(n)$. Then there exists a B-c.m.m. N and a constant c such that $K_N(\sigma) = K_M(\sigma) + c$ for all σ. Consider the function p such that

$$D_{p(n)} = \{N(\tau) \ : \ |\tau| \leq n + c + 1\}.$$

Since N is a B-c.m.m., the domain of N is computable from B, thus so is the set $D_{p(n)}$ as a finite set and we have $p \leq_T B$. Furthermore, for each n, $f(n) = M(0^n 1)$ and

$$K_N(f(n)) \leq K_M(f(n)) + c \leq n + c + 1.$$

Thus, $f(n) \in D_{p(n)}$. Finally, notice that $|D_{p(n)}| \leq 2^{n+c+2}$. □

Now, we have finished the proof of Theorem 4.

4.2 Characterizations via integral tests

Summable functions are very similar to integral tests. Here, we characterize some reducibilities via integral tests.

We say that a function $f : 2^\omega \to \overline{\mathbb{R}}^+$ is *dominated by* a function $g : 2^\omega \to \overline{\mathbb{R}}^+$ if $f(X) \leq g(X)$ for every $X \in 2^\omega$.

Theorem 5 *$A \leq_{CT} B$ if and only if every A-Schnorr integral test is dominated by a B-Schnorr integral test.*

Proof (The "if" direction)

Suppose that every A-Schnorr integral test is dominated by a B-Schnorr integral test. Let Z be a set that is not Schnorr random relative to A. Then, there exists an A Schnorr integral test f such that $f(Z) = \infty$. By the assumption, there exists a B-Schnorr integral test g that dominates f. Then, $g(Z) = \infty$ and Z is not Schnorr random relative to B. Since Z is arbitrary, every Schnorr random set relative to B is Schnorr random relative to A. By Theorem 4, we have $A \leq_{CT} B$.

(The "only if" direction)

Suppose that $A \leq_{CT} B$. Let f be an A-Schnorr integral test. Then, f can be written as

$$f = \sum_n p_n \mathbf{1}_{[\sigma_n]}$$

where p_n is an A-computable sequence of positive rationals and σ_n is an A-computable sequence of strings. Let

$$\hat{f}(k) = \sum \{p_n \ : \ n \in \omega, \ \overline{\sigma_n} = k\}.$$

Then, $\hat{f}$ is an A-left c.e. function such that $\sum_n \hat{f}(n)$ is A-computable. Thus, there exists a B-left-c.e. function $\hat{g} : \omega \to \mathbb{R}^+$ such that $\sum_n \hat{g}(n)$ is B-computable. Let

$$g = \sum_k \hat{g}(k) \mathbf{1}_{s(k)}$$

where $s(k)$ is the k-th binary string in lexicographical order. Then, g is an B-integral test and $f \leq g$. □

We can prove the following theorem by a similar way.

Theorem 6 *For sets $A, B \in 2^\omega$, $A \leq_{LR} B$ if and only if every A-integral test is dominated by a B-integral test.*

For the proof, notice that Theorem 22 in [2] can be relativized.

5 Extended Ample Excess Lemma

It is well known that a set $A \in 2^\omega$ is ML-random if and only if $K(A \upharpoonright n) > n - O(1)$. The following Ample Excess Lemma says that $K(A \upharpoonright n)$ is larger than n if A is ML-random.

Theorem 7 (Miller and Yu [16]) *A set $A \in 2^\omega$ is ML-random if and only if $\sum_n 2^{n-K(A\upharpoonright n)} < \infty$.*

We would like to have a Schnorr-randomness version of this result. We start from the following observation.

Theorem 8 (Extended Ample Excess Lemma) *For a machine M, let $f_M : 2^\omega \to \mathbb{R}$ be a function such that*

$$f_M(X) = \sum_{n=0}^{\infty} 2^{n-K_M(X\upharpoonright n)}.$$

Then, we have

$$\int f_M(X)\ d\mu = \widehat{\Omega}_M.$$

Proof This is because

$$\int f_M(X)\ d\mu = \int \sum_{n=0}^{\infty} 2^{n-K_M(X\upharpoonright n)}\ d\mu = \sum_{n=0}^{\infty} \int 2^{n-K_M(X\upharpoonright n)}\ d\mu$$
$$= \sum_{n=0}^{\infty} \sum_{\sigma\in 2^n} 2^{n-K_M(\sigma)} \cdot 2^{-n} = \sum_{\sigma\in 2^{<\omega}} 2^{-K_M(\sigma)} = \widehat{\Omega}_M.$$

□

Notice that M can be a universal prefix-free machine and a computable measure machine. Since f_M is a lower semicomputable function, f_M is an integral test. Thus, Theorem 8 implies the "only if" direction of Theorem 7. Recall that the "if" direction of Theorem 7 follows from the Levin-Schnorr theorem.

In exactly the same way, we obtain a Schnorr-randomness version.

Corollary 1 *A set $A \in 2^\omega$ is Schnorr random if and only if $\sum_n 2^{n-K_M(A\upharpoonright n)} < \infty$ for every computable measure machine M.*

Proof If A is not Schnorr random, then there is a computable measure machine M such that $K_M(A \upharpoonright n) < n$ for infintely many n, thus $\sum_n 2^{n-K_M(A\upharpoonright n)} = \infty$.

Suppose that A is Schnorr random. Let M be a computable measure machine. Then, f_M is Schnorr integral test. Hence, $f_M(A) < \infty$. □

There are many interesting theorems the proof of which uses the Ample Excess Lemma. One of them is the following characterization of 2-randomness via prefix-free Kolmogorov complexity.

Theorem 9 ([15]) *A set $A \in 2^\omega$ is 2-random (ML-random relative to $\emptyset'$) if and only if $K(A \upharpoonright n) \geq n + K(n) - O(1)$ for infinitely many n.*

The proof of the "only if" direction goes like this. A corollary of the Ample Excess Lemma says that, if A is ML-random, then $K(A \upharpoonright n) \geq n + K^A(n) - O(1)$ [16]. Nies, Stephan and Terwijn [23] and Miller [15] showed that, for a ML-random set A, A is 2-random if and only if A is weakly low for K, which means that $K(n) \leq K^A(n) + O(1)$ for infinitely many n. Thus, the result follows.

Now we consider the Schnorr-randomness version of this result. The situation is quite different from the case of ML-randomness.

Proposition 4 *Let A be a Schnorr random set. For every computable measure machine M, there exists a uniformly computable measure machine N such that*

$$K_M(A \upharpoonright n) \geq n + K_{N^A}(n) - O(1).$$

First notice that N depends on M in the statement. Such dependency does not occur in the case of ML-randomness because of universality, but this happens for Schnorr randomness because of lack of universality.

The proof is not simple because of the uniformity. One may try the following proof idea first. Let $d \in \omega$ be such that $f_M(A) < 2^d$. Then construct a KC set to induce a uniformly computable measure machine N from M and d with the desired property. We wish that the following set is the KC set:

$$S^X = \{(-n + K_M(X \upharpoonright n) + d,\ n)\ :\ n \in \omega\}.$$

The weight $w(X)$ of this set is

$$w(X) = \sum_n 2^{n - K_M(X\upharpoonright n) - d} = 2^{-d} f_M(X).$$

Here, $w(A) \leq 1$, but the inequality $w(X) \leq 1$ may not be true for some X. Thus, we should be careful when enumerating the pairs.

The key lemma of the proof is the following.

Lemma 7 (Lemma 2.1 in [20]) *Let t be a Schnorr integral test. Then, there is a uniformly computable sequence $\{h_n\}$ of total computable functions $h_n : 2^\omega \to [0, \infty)$ such that $h_n \leq t$ everywhere and if A is Schnorr random, then there is some n such that $h_n(A) = t(A)$.*

Proof (of Proposition 4) Let $d \in \omega$ be such that $f_M(A) < 2^d$. By the Extended Ample Excess Lemma (Lemma 8), $2^{-d} f_M$ is a Schnorr integral test. Thus, by Lemma 7, there is a total computable function $h_0 : 2^\omega \to [0, \infty)$ such that $h_0 \leq 2^{-d} f_M$ everywhere and $h_0(A) = 2^{-d} f_M(A)$.

Let $h = \min\{h_0, 1\}$. Then, $h \leq 1$ everywhere, h is a total computable function, $h \leq 2^{-d} f_M$ everywhere and $h(A) = 2^{-d} f_M(A)$.

With using this h, we construct a KC set S^X as follows. At stage n, we construct $S^X_n \supseteq S^X_{n-1}$. By $w_n(X)$, we denote the weight of S^X_n. We set $S^X = \lim_n S^X_n$. Let S^X_{-1} be the empty set. Thus, $w_{-1}(X) = 0$. For each n, exactly one of the following holds:

$$h(X) > w_{n-1}(X) + 2^{n - K_M(X\upharpoonright n) - d} \tag{1}$$

or

$$h(X) < w_{n-1}(X) + 2^{n - K_M(X\upharpoonright n) - d} + 2^{n+1-K_M(X\upharpoonright (n+1)) - d}. \tag{2}$$

If (1) holds, then set $S^X_n = S^X_{n-1} \cup (-n + K_M(X \upharpoonright n) + d,\ n)$. If (2), then stop enumerating and make S^X satisfy $w(X) = h(X)$.

We claim that $w(X) = h(X)$ for every $X \in 2^\omega$. If $h(X) = 2^{-d} f_M(X)$, then (1) always holds and $w(X) = 2^{-d} f_M(X) = h(X)$. If $h(X) < 2^{-d} f_M(X)$, then (2) holds for some n and $w(X) = h(X)$.

Let N be the uniformly computable measure machine constructed from the KC set S via the KC Theorem. By construction, $K_{N^A}(n) \le -n + K_M(A \upharpoonright n) + d$ for every n. □

Since Schnorr-randomness version of 2-randomness is not clear, we consider a Schnorr-randomness version of weak lowness for K.

Definition 3 A set A is called *weakly low for computable measure machines* if, for every uniformly computable measure machine M, there exists a computable measure machine N such that

$$K_N(n) \le K_{M^A}(n) + O(1)$$

for infinitely many n.

This class is, however, too large in the following sense.

Proposition 5 *Every set is weakly low for computable measure machines.*

Proof Let A be a set and M be a uniformly computable measure machine. Let

$$m(n) = \min\{K_{M^X}(k) \ : \ k \ge n, \ X \in 2^\omega\}.$$

Since M is a decidable machine, m is a computable function. Clearly, m is non-decreasing. Consider the set W defined by

$$W = \{(m(n), n) \ : \ 1 \le m(n) \ne m(n+1)\}.$$

Then, the weight of W is computable and ≤ 1. Let N be a computable measure machine constructed from this KC set. Then, for every $(m(n), n) \in W$, we have

$$K_N(n) \le m(n) \le K_{M^A}(n).$$

Since W is infinite, the proposition follows. □

Corollary 2 *Let A be a Schnorr random set. For every computable measure machine M, there exists a computable measure machine N such that*

$$K_M(A \upharpoonright n) \ge n + K_N(n) - O(1)$$

for infinitely many n.

Proof Let A be a Schnorr random set and M be a computable measure machine. By Proposition 4, there exists a uniformly computable measure machine N such that $K_M(A \upharpoonright n) \ge n + K_{N^A}(n) - O(1)$. By the proposition above, there exists a computable measure machine L such that $K_L(n) \le K_{N^A}(n)$ for infinitely many n. Thus,

$$K_M(A \upharpoonright n) \ge n + K_L(n) - O(1)$$

for infinitely many n. □

Now we consider the "if" direction of Theorem 9, which follows from a result in Solovay [24] (Corollary 4.3.3 in [7]) and Theorem 2.8 in [23]. This result says that, if A is not 2-random, $\lim_n(n + K(n) - K(A \upharpoonright n)) = \infty$. For the case of computable measure machines, this property holds, again, for every set.

Proposition 6 *Let A be a set. Then for every computable measure machine N, there exists a computable measure machine M such that*

$$\lim_n (n + K_N(n) - K_M(A \upharpoonright n)) = \infty.$$

Lemma 8 *For every computable measure machine N, there exists a computable measure machine L and a computable order f such that*

$$K_L(n) \leq K_N(n) - f(n).$$

Proof We identify the machine N with a computable list $\{\langle \sigma_n, \tau_n \rangle\}$ of pairs of strings. Since N is a decidable machine, we can assume that $\tau_n = n$. Since N is a computable measure machine, there exists a computable order g such that $\sum_{n>g(k)} 2^{-|\sigma_n|} < 2^{-k}$. Without loss of generality, we can assume that

$$\sum_{n=g(k)+1}^{g(k+1)} 2^{-|\sigma_n|} = 2^{-k}.$$

Take a computable order h such that $\sum_k 2^{h(k)-k}$ is a computable real, which is less than 2^c where $c \in \omega$. The sequence of pairs of strings

$$\langle c - h(k) + |\sigma_n|, n \rangle$$

where $h(k) < n \leq h(k+1)$ is a KC set because

$$\sum_k \sum_{n=h(k)+1}^{h(k+1)} 2^{-c+h(k)-|\sigma_n|} = 2^{-c} \sum_k 2^{h(k)-k}.$$

Since the weight is computable, the machine L constructed by this KC set is a computable measure machine. Furthermore,

$$K_L(n) \leq K_N(n) - h(k) + c$$

if $h(k) < n \leq h(k+1)$. □

Proof (of Proposition 6) Let N be a computable measure machine and L be a computable measure machine constructed in the lemma above. We define a machine M by $M(\sigma_0\sigma_1) = \sigma_1$ if $L(\sigma_0) = n$. Since L is a computable measure machine, so is M. Furthermore, for any $\tau \in 2^n$, we have $K_M(\tau) \leq n + K_L(n)$. Thus, the proposition follows. □

One may try to understand that Proposition 4 says that, for every Schnorr random set, the complexities of its initial segments are infinitely often maximal and Proposition 6 says that, for every set, the complexities of its initial segments can not be infinitely often maximal. These intepretations, however, contradict with each other. This is because it is not clear what is "maximal complexity" in the case of Schnorr randomness. Also notice that the order of quantifiers of the propositions are different.

6 vLS-reducibility

In this section we study a Schnorr-randomness version of van Lambalgen reducibility. Almost all proofs are very similar to corresponding results in [16], but again we should be careful about the order of quantifiers.

6.1 The definition of vLS-reducibility

Miller and Yu [16] defined the following reducibility. We say that X is *van Lambalgen reducible to* Y, or simply *vL-reducible* to Y, and write $X \leq_{vL} Y$, if for all Z, if $X \oplus Z$ is ML-random, then $Y \oplus Z$ is ML-random.

We consider its Schnorr-randomness version.

Definition 4 We define $X \leq_{vLS} Y$ by the following statement: for all $Z \in 2^\omega$, if $X \oplus Z$ is Schnorr random, then $Y \oplus Z$ is Schnorr random.

The least vL-degree is $\mathbf{0}_{vL} = \{X : X \text{ is not ML-random}\}$. Similarly, the least vLS-degree is $\mathbf{0}_{vLS} = \{X : X \text{ is not Schnorr random}\}$.

Recall that $\leq_{vL}$ is the converse of $\leq_{LR}$ for Martin-Löf random sets. The Schnorr-randomness version of $\leq_{LR}$ is $\leq_{LUS}$ defined in [17, after Theorem 5.1] by $X \leq_{LUS} Y$ if and only if every set that is Schnorr random uniformly relative to Y is Schnorr random uniformly relative to X. For Schnorr random sets X and Y, we have $X \leq_{vLS} Y$ if and only if $Y \leq_{LUS} X$, by van Lambalgen's Theorem for Schnorr randomness (Theorem 1).

The following facts are almost immediate.

Proposition 7 (i) *There is no join in the vLS-degree.*
(ii) *If $Y \leq_{tt} X$ and Y is Schnorr random, then $X \leq_{vLS} Y$.*
(iii) *There is no maximal vLS-degree.*
(iv) *There is no minimal Schnorr-random vLS-degree.*

6.2 Schnorr reducibility implies vLS-reducibility

One interesting property of vL-reducibility is that $\leq_K$ implies $\leq_{vL}$. This is because a set $X \oplus Z$ is ML-random iff $K(X \upharpoonright (Z \upharpoonright n)) \geq (Z \upharpoonright n) + n - O(1)$, which is shown by Miller and Yu [16]. Similarly, Schnorr reducibility implies vLS-reducibility.

Definition 5 (Miller and Yu [16]) Given $X = x_0x_1\cdots$ and $Z = z_0z_1\cdots$, let $X\widehat{\oplus}Z$ be

$$z_0x_0x_1z_1x_2x_3x_4x_5z_2\cdots z_{n-1}x_{2^n-2}\cdots x_{2^{n+1}-3}z_n\cdots.$$

Theorem 10 *A set $X \oplus Z \in 2^\omega$ is Schnorr random iff*

$$K_M(X \upharpoonright (Z \upharpoonright n)) \geq (Z \upharpoonright n) + n - O(1)$$

for every computable measure machine M.

The proof is just a modification of the correspoinding result in Miller and Yu [16], but we need to be careful about the order of the quantifier.

Proof Assume that $X \oplus Z$ is Schnorr random. Then, $X\widehat{\oplus}Z$ is also Schnorr random. Let $\sigma = Z \upharpoonright n$ and $\sigma' = Z \upharpoonright (n+1)$. Notice that

$$(X\widehat{\oplus}Z) \upharpoonright (\sigma + n + 1) = (X \upharpoonright \sigma)\widehat{\oplus}\sigma'.$$

Let M be a computable measure machine. Then, there exists a computable measure machine N such that

$$K_M(X \upharpoonright \sigma) \geq K_N((X \upharpoonright \sigma)\widehat{\oplus}\sigma') = K_N((X\widehat{\oplus}Z) \upharpoonright (\sigma + n + 1)) \geq \sigma + n - O(1).$$

For the other direction, suppose that $X \oplus Z$ is not Schnorr random. Then, there exists a computable measure machine M such that, for every $k \in \omega$ there exists $m \in \omega$ such that

$$K_M((X\widehat{\oplus}Z) \upharpoonright m) \leq m - k. \tag{3}$$

Thus, for each $\eta_1, \eta_2 \in 2^{<\omega}$ such that $\eta_1\widehat{\oplus}\eta_2 = (X\widehat{\oplus}Z) \upharpoonright m$, there exists $\tau_1 \in 2^{<\omega}$ such that $M(\tau_1) = \eta_1\widehat{\oplus}\eta_2$ and $|\tau_1| \leq m - k$.

We define a computable measure machine N as follows. Suppose that N receives τ as an input. Then, look for τ_1, τ_2, η_1 and η_2 such that $\tau = \tau_1\tau_2$, $M(\tau_1) = \eta_1\widehat{\oplus}\eta_2$ and $|\eta_1\tau_2| = \eta_2$. If these are found, define $N(\tau) = \eta_1\tau_2$.

Clearly, N is a partial computable function, that is, a machine.

We show that N is prefix-free. For each $\tau \in 2^{<\omega}$, if $N(\tau)$ halts, then the separation $\tau = \tau_1\tau_2$ is unique, because M is prefix-free. Furthermore, the length of τ_2 is determined by the equality $|\eta_1\tau_2| = \eta_2$, thus by η_1 and η_2, which is again determined by τ_1. Thus, N is prefix-free.

We show that the measure of N is computable because

$$\begin{aligned}&\sum\{2^{-|\tau_1\tau_2|} \ : \ \tau_1 \in \mathrm{dom}(M),\ M(\tau_1) = \eta_1\widehat{\oplus}\eta_2 \text{ and } |\tau_2| = \eta_2 - |\eta_1|\}\\ =&\sum\{2^{-\tau_1} \ : \ \tau_1 \in \mathrm{dom}(M)\},\end{aligned}$$

and M is a computable measure machine.

Take k and m satisfying the inequality (3). Let η_1, η_2 be such that $\eta_1\widehat{\oplus}\eta_2 = (X\widehat{\oplus}Z) \upharpoonright m$. Then there exists $\tau_1 \in 2^{<\omega}$ such that $M(\tau_1) = \eta_1\widehat{\oplus}\eta_2$ and

$|\tau_1| \leq m - k$. For $n = |\eta_2|$, we have $|\eta_1| \leq 2^n - 2$ and $\eta_2 \geq 2^n - 1$, so there is a string τ_2 such that $\eta_1\tau_2 = X \upharpoonright \eta_2$. Hence, $N(\tau_1\tau_2) = X \upharpoonright \eta_2$. Therefore,

$$\begin{aligned} K_N(X \upharpoonright (Z \upharpoonright n)) =& K_N(X \upharpoonright \eta_2) \leq |\tau_1\tau_2| \leq m - k + |\tau_2| \\ =& |\eta_1\eta_2| - k + |\tau_2| = |\eta_1\tau_2| + |\eta_2| - k = \eta_2 + |\eta_2| - k \\ =& Z \upharpoonright n + n - k. \end{aligned}$$

□

Corollary 3 *If $X \leq_{Sch} Y$, then $X \leq_{vLS} Y$.*

6.3 Total machine reducibility implies vLS-reducibility

Miller and Yu [16] also showed that C-reducibility implies vL-reducibility. The key theorem is the following. Let Z be a ML-random set. Then, $X \oplus Z$ is ML-random if and only if $C(X \upharpoonright n) \geq n - K^Z(n) - O(1)$ if and only if $C(X \upharpoonright n) + K(Z \upharpoonright n) \geq 2n - O(1)$.

As we have seen in Section 3.2, total machines play the role of plain machines for Schnorr randomness. Thus, a Schnorr-randomness version of C-reducibility can be defined as follows.

Definition 6 Let $X, Y \in 2^\omega$ be sets. We write $X \leq_{tm} Y$ if for every total machine M there exists a total machine N such that

$$C_N(X \upharpoonright n) \leq C_M(Y \upharpoonright n) + O(1).$$

The key theorem is the following.

Theorem 11 *Let Z be a Schnorr random set. The following are equivalent.*

(i) *$X \oplus Z$ is Schnorr random.*
(ii) *$C_N(X \upharpoonright n) \geq n - K_{M^Z}(n) - O(1)$ for every uniformly computable measure machine M and every total machine N.*
(iii) *$C_N(X \upharpoonright n) + K_M(X \upharpoonright n) \geq 2n - O(1)$ for every computable measure machine M and every total machine N.*

Note that the statement (ii) has a form similar to the statement (iii) of Theorem 3.

Corollary 4 *If $X \leq_{tm} Y$, then $X \leq_{vLS} Y$.*

We begin from proving the following simple fact.

Lemma 9 *There exists a total machine L such that*

$$C_L(X \upharpoonright (X \upharpoonright n)) \leq (X \upharpoonright n) - n + 1 \tag{4}$$

for every $X \in 2^\omega$.

The proof idea is as follows. For $X \in 2^\omega$, let $\sigma = X \upharpoonright n$ and $\sigma\tau = X \upharpoonright (X \upharpoonright n) = X \upharpoonright \overline{\sigma}$. Note that the length of τ is $\overline{\sigma} - |\sigma|$. If n can be determined by the length of τ, then σ is determined by τ. Note that, for $\sigma \in 2^n$,

$$2^n - 1 - n \leq \overline{\sigma} - |\sigma| \leq 2^{n+1} - 2 - n.$$

Notice that

$$2^{n+1} - 2 - n = 2^{n+1} - 1 - (n+1).$$

Then only in this case we can not determine the length of σ. To avoid this, we use the last bit.

Proof Let $a_n = 2^{n+1} - 2 - n$. Then, $a_0 = 0$, $a_1 = 1$, $a_2 = 4$ and so on.

The machine L behaves as follows. Suppose that L receives τi as an input where $\tau \in 2^{<\omega}$ and $i \in \{0,1\}$. If $|\tau| = a_n$ for some $n \in \omega$, then L outputs $0^{n+1}\tau$ if $i = 0$ and $1^n\tau$ if $i = 1$. Otherwise, let n be such that $a_{n-1} < |\tau| < a_n$, and $\sigma \in 2^n$ be such that $\overline{\sigma} = |\tau| + n$. Then L outputs $\sigma\tau$. Notice that such an σ always exists and is unique because $2^n - 1 < |\tau| + n < 2^{n+1} - 2$. Clearly, L is a total machine.

Finally, we claim (4). Let $\sigma' = X \upharpoonright n$ and $\sigma'\tau' = X \upharpoonright (X \upharpoonright n)$. Note that $|\tau'| = X \upharpoonright n - n = \overline{\sigma'} - |\sigma'|$. If $|\tau'| = 2^{n+1} - 2 - n$ for some n, then $\sigma' \in \{1^n, 0^{n+1}\}$. Then, $\sigma'\tau' \in \{L(\tau'0), L(\tau'1)\}$. Hence, $C_L(X \upharpoonright (X \upharpoonright n)) = C_L(\sigma'\tau') \leq |\tau'| + 1 = X \upharpoonright n - n + 1$. □

Proof (of Theorem 11) (i)⇒(ii). Suppose that $X \oplus Z$ is Schnorr random. Then, X is Schnorr random uniformly relative to Z. Let M be a uniformly computable measure machine and N be a total machine. Note that N can be seen as an oracle total machine By the uniform relativization of Theorem 3, we have

$$C_N(X \upharpoonright n) = C_{N^Z}(X \upharpoonright n) \geq n - K_{M^Z}(n) - O(1).$$

(ii)⇒(iii). Assume (ii). Let M be a computable measure machine and N be a total machine. Recall that Z is Schnorr random. By Proposition 4, for the computable measure machine M, there exists a uniformly computable measure machine L such that

$$K_M(Z \upharpoonright n) \geq n + K_{L^Z}(n) - O(1).$$

Hence,

$$C_N(X \upharpoonright n) \geq n - K_{L^Z}(n) - O(1) \geq 2n - K_M(Z \upharpoonright n) - O(1).$$

(iii)⇒(i). Assume (iii). Let M be a computable measure machine and L be the total machine in Lemma 4. Then,

$$\begin{aligned} K_M(Z \upharpoonright (X \upharpoonright n)) \geq& 2(X \upharpoonright n) - C_L(X \upharpoonright n) - O(1) \\ \geq& 2(X \upharpoonright n) - (X \upharpoonright n) + n - O(1) \\ =& (X \upharpoonright n) + n - O(1). \end{aligned}$$

Hence, $X \oplus Z$ is Schnorr random by Theorem 10. □

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