Schnorr randomness versions of K, C, LR, vL-reducibilities

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Conference on Computability, Complexity and Randomness 2014
13 June 2014

Motivation

- Give Schnorr randomness versions of theorems for ML-randomness.
- Why important or interesting?
  -> Deep understanding of the theorems.
  -> Unexpected findings.

Question 1

- Question 1.
  ML-randomness has characterizations by K and C.
  Schnorr randomness by K_M where M is a computable measure machine.
  What is a Schnorr randomness version of the one by C?

- Answer.
  Schnorr randomness has a characterization by C_M where M is a total machine.

Question 2

- Question 2.
  Computably-traceable-reducibilitiy can be characterized by relative Schnorr randomness?
  (Problem 8.4.22 in Nies' book)

- Answer.
  Yes.

Question 3

- Question 3.
  2-randomness can be characterized by infinitely-often maximality of complexity.
  Is there a Schnorr randomness version?

- Partial answer.
  No.

Schnorr version of C

ML-randomness

Definition (Martin-Löf 1966)

$X \in 2^\omega$ is ML-random if $x \notin \bigcap_n U_n$ for each ML-test $\{U_n\}$, i.e., U_n is a uniformly c.e. open set with $\mu(U_n) \leq 2^{-n}$.

Theorem

X is ML-random iff $K(X \upharpoonright n) > n - O(1)$. (Levin, Schnorr 1973)

X is ML-random iff $C(X \upharpoonright n) > n - K(n) - O(1)$. (Miller-Yu 2008)

Schnorr randomness

Definition (Schnorr 1971)

$X \in 2^\omega$ is Schnorr random if $x \notin \bigcap_n U_n$ for each Schnorr test $\{U_n\}$, i.e., $\{U_n\}$ is a ML-test and $\mu(U_n)$ is uniformly computable.

Theorem (Downey-Griffiths 2004)

$X \in 2^\omega$ is Schnorr random iff $K_M(X \upharpoonright n) > n - O(1)$ for every computable measure machine M, i.e., M is a prefix-free machine and $\sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|}$ is computable.

Schnorr version of C

Theorem (M.)

X is Schnorr random iff, for every computable measure machine M and every total machine N, we have

$$C_N(X \upharpoonright n) > n - K_M(n) - O(1).$$

Related results

Theorem (Bienvenu-Merkle 2007)

X is Schnorr random iff, for every decidable prefix-free machine M and every computable order g, we have

$$K_M(X \upharpoonright n) > n - g(n) - O(1).$$

Theorem (Hölzl-Merkle 2010)

A is Schnorr trivial iff, for every computable order g, there exists a total machine M such that

$$K_M(A \upharpoonright g(n)) \leq n + O(1).$$

Schnorr version of C-reducibility

$X \leq_C Y$ if

$$C(X \upharpoonright n) \leq C(Y \upharpoonright n) + O(1).$$

Definition

$X \leq_{tm} Y$ if, for every total machine M, there exists a total machine N such that

$$C_N(X \upharpoonright n) \leq C_M(Y \upharpoonright n) + O(1).$$

We come back to this notion later.

Schnorr version of LR

LR-reducibility

A is *low for MLR* if every A-ML-random set is ML-random.
A is *low for K* if $K(n) \leq K^A(n) + O(1)$. These notions are equivalent.

Theorem (Kjos-Hanssen-Miller-Solomon 2012)
The following are equivalent for $X, Y \in 2^\omega$:

(i) Every X-ML-random set is Y-ML-random. ($X \leq_{LR} Y$)

(ii) $K^Y(n) \leq K^X(n) + O(1)$. ($X \leq_{LK} Y$)

Schnorr version of LR?

The following are equivalent for $A \in 2^\omega$:

(i) A is *low for Schnorr Randomness.*

(ii) A is *computably traceable.*

(iii) A is *low for computable measure machines.*

(Terwijn-Zambella 2001, Kjos-Hanssen-Nies-Stephan 2005, Downey-Greenberg-Mihailovic-Nies 2008)

Nies (Problem 8.4.22 in his book) asked whether the reducibility version of the equivalence between (i) and (ii) holds.

Definition (Nies)

$A \leq_{CT} B$ if there is a computable order h such that for each $f \leq_T A$ there exists $p \leq_T B$ such that $f(n) \in D_{p(n)}$ and $|D_{p(n)}| \leq h(n)$ for every n.

Theorem (M.)

The following are equivalent for $A, B \in 2^\omega$:

(i) $A \leq_{CT} B$.

(ii) Every Schnorr random set relative to B is Schnorr random relative to A.

Techniques

- low for tests = low for random (open covering method)
  by Bienvenu-Miller 2012
- LR = LK
  by Kjos-Hanssen-Miller-Solomon 2012
- low for Schnorr tests = low for c.m.m.
  by Bienvenu in arXiv.
- Combine and relativize them, then you get the result!

Schnorr version of vL

K-reducibility

Definition

$X \leq_K Y$ if

$$K(X \upharpoonright n) \leq K(Y \upharpoonright n) + O(1).$$

.

vL-reducibility

Definition (Miller-Yu 2008)

$X \leq_{vL} Y$ if, for every Z,

$$X \oplus Z \in \mathrm{MLR} \Rightarrow Y \oplus Z \in \mathrm{MLR}.$$

For $X, Y \in \mathrm{MLR}$,

$$X \leq_{vL} Y \iff Y \leq_{LR} X.$$

C,K implies vL

Theorem (Miller-Yu 2008)

(i) $X \leq_K Y$ implies $X \leq_{vL} Y$.

(ii) $X \leq_C Y$ implies $X \leq_{vL} Y$.

.

Schnorr versions

Definition (Downey-Griffiths 2004)

$X \leq_{Sch} Y$ if, for every c.m.m. M, there exists a c.m.m. N such that

$$K_N(X \upharpoonright n) \leq K_M(X \upharpoonright n) + O(1).$$

Theorem (M. 2011, M.-Rute 2013)

$X \oplus Y$ is Schnorr random iff X is Schnorr random and Y is Schnorr random uniformly relative to X.

Definition

$X \leq_{vLS} Y$ if, for every Z,

$$X \oplus Z \in \mathrm{SR} \Rightarrow Y \oplus Z \in \mathrm{SR}.$$

Schnorr versions hold

Theorem (M.)

(i) $X \leq_{Sch} Y$ impllies $X \leq_{vLS} Y$.

(ii) $X \leq_{tm} Y$ implies $X \leq_{vLS} Y$.

.

Technique

- An extension of Ample Excess Lemma

Theorem (Ample Excess Lemma; Miller-Yu 2008)

(i) X is ML-random iff $\sum_n 2^{n-K(X\restriction n)} < \infty$.

(ii) If X is ML-random, then

$$K(X \restriction n) \geq n + K^X(n) - O(1).$$

.

An extension of AEL

Observation

For a machine M, we define a function $f_M : 2^\omega \to \mathbb{R}$ by

$$f_M(X) = \sum_{n=0}^{\infty} 2^{n - K_M(X \upharpoonright n)}.$$

Then, we have

$$\int f_M(X) d\mu = \widehat{\Omega}_M = \sum \{2^{-K_M(\sigma)} : \sigma \in 2^{<\omega},\ K_M(\sigma) < \infty\}.$$

If U is a universal prefix-free machine, then

$$\Omega_U = \sum_{\sigma \in \mathrm{dom}(U)} 2^{-|\sigma|}$$

and $\widehat{\Omega}_U$ are ML-random.

If M is a computable measure machine, then Ω_M and $\widehat{\Omega}_M$ are computable.

In general, we have

$$\widehat{\Omega}_M \leq_S \Omega_M$$

where $\leq_S$ is Solovay reducibility. The converse does not hold in general.

Corollary

X is Schnorr random iff $\sum_n 2^{n-K_M(X\upharpoonright n)} < \infty$ for every computable measure machine M.

Recall that

$$f_M(X) = \sum_{n=0}^{\infty} 2^{n-K_M(X\restriction n)}.$$

Then

$$\begin{aligned}
\int f_M(X)\ d\mu &= \int \sum_{n=0}^{\infty} 2^{n-K_M(X\restriction n)}\ d\mu \\
&= \sum_{n=0}^{\infty} \sum_{\sigma\in 2^n} 2^{n-K_M(\sigma)} \cdot 2^{-n} \\
&= \sum_{\sigma\in 2^{<\omega}} 2^{-K_M(\sigma)} \\
&= \widehat{\Omega}_M
\end{aligned}$$

.

Proposition (M.)

Let X be a Schnorr random set. For every computable measure machine M, there exists a uniformly computable measure machine N such that

$$K_M(X \upharpoonright n) \geq n + K_{N^X}(n) - O(1).$$

Theorem (Miller 2009)

X is 2-random if and only if

$$K(X \upharpoonright n) \geq n + K(n) - O(1)$$

for infinitely many n.

(i) Ample Excess Lemma

(ii) Ω is X-ML-random (low for Ω) iff $K(n) \leq K^X(n)$ for infinitely many n (weakly low for K)

Definition

A set A is weakly low for c.m.m. if, for every u.c.m.m. M, there exists a c.m.m. N, such that

$$K_N(n) \leq K_{M^A}(n) + O(1)$$

for infinitely many n.

Proposition

Every set is weakly low for c.m.m.

Theorem (M.)

For a c.m.m. M, there exists a c.m.m. N such that, for every Schnorr random set X,

$$K_M(X \upharpoonright n) \geq n + K_N(n) - O(1)$$

for infinitely many n.

Open question

Is the following notion equivalent to Schnorr randomness?

For every total machine M,

$$C_M(X \upharpoonright n) \geq n - O(1)$$

for infinitely many n.

Summary

- We looked at Schnorr-randomness versions of some theorems on ML-randomness.
- Because of non-universality, we should take care about the dependency on the machine, which (may) deepen the understanding.
- Hierarchy of Schnorr randomness?

- Thank you for listening.