Total-machine reducibility and randomness notions

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Outline

- Motivation - review of basics of randomness
- Total-machine reducibility
- Schnorr reducibility
- Summary

Motivation

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- Understand the statement that a sequence A is more random than another sequence B.

Randomness hierarchy

2-randomness

$\Downarrow$

Martin-Löf randomness

$\Downarrow$

Schnorr randomness

$\Downarrow$

Kurtz randomness

Kolmogorov complexity

Definition

$$C(\sigma) = \min\{|\tau| \;:\; V(\tau) = \sigma\},$$
$$K(\sigma) = \min\{|\tau| \;:\; U(\tau) = \sigma\},$$

where V is the universal plain machine and U is the universal prefix-free machine. Then,

$$C(\sigma) \leq |\sigma| + d,$$
$$K(\sigma) \leq |\sigma| + K(|\sigma|) + d,$$

and these are the maximal. .

Prefix-free Kolmogorov complexity

Theorem (Miller 2009)

A set $A \in 2^\omega$ is 2-random iff $K(A \upharpoonright n) > n + K(n) - O(1)$ infinitely often.

Theorem (Levin, Schnorr 1973)

A set $A \in 2^\omega$ is ML-random iff $K(A \upharpoonright n) > n - O(1)$.

Theorem (Theorem 7.4.11 in Nies' book)

There is a computably random set $A \in 2^\omega$ such that $\forall^\infty n \; K(A \upharpoonright n | n) \leq h(n)$ for each computable order function h.

Plain Kolmogorov complexity

Theorem (Nies-Stephan-Terwijn 2005, Miller 2004)

A set $A \in 2^\omega$ is 2-random iff $C(A \upharpoonright n) > n - O(1)$ infinitely often.

Theorem (Miller and Yu 2008)

A set $A \in 2^\omega$ is ML-random iff $C(A \upharpoonright n) > n - K(n) - O(1)$.

Two candidates

Definition

$A \leq_K B$ if, there exists a constant d such that, for all n, we have

$$K(A \upharpoonright n) \leq K(B \upharpoonright n) + d.$$

Definition

$A \leq_C B$ if, there exists a constant d such that, for all n, we have

$$C(A \upharpoonright n) \leq C(B \upharpoonright n) + d.$$

Problems

- K,C-reducibility are not appropriate for studying weaker randomness notions (CR, SR, WR).

Total machines

Why total machines?

- Schnorr version of K is computable measure machines.
- Schnorr version of K-reducibility is Schnorr reducibility.
- Schnorr version of C is total machines.
- Schnorr version of C-reducibility is tm-reducibility.

Total machines

We consider complexity with respect to total machines.

A total machine is a total computable function $M : 2^{<\omega} \to 2^{<\omega}$. Complexity with respect to a total machine M is defined by

$$C_M(\sigma) = \min\{|\tau| \ : \ M(\tau) = \sigma\}.$$

By modifying a total machine, we can assume that $C_M(\sigma) \leq 2|\sigma|$. Then, C_M is a total computable function.

Theorem (M.) X is Schnorr random iff, for every computable measure machine M and every total machine N, we have

$$C_N(X \upharpoonright n) > n - K_M(n) - O(1).$$

Theorem (essentially due to Bienvenu and Merkle 2007) X is Kurtz random iff, for every total machine M and every computable order f, we have

$$C_M(X \upharpoonright n) > n - f(n)$$

infinitely often.

Theorem (Nies, Stephan and Terwijn 2005)

X is 2-random iff

$$C(X \upharpoonright n) > n - O(1)$$

infinitely often.

Then, what is Schnorr version of 2-randomness?

2-random via total machine

Theorem

X is 2-random iff, for every total machine M, we have

$$C_M(X \upharpoonright n) > n - O(1)$$

infinitely often.

Schorr randomness version of 2-randomness is also 2-randomness! I posed this question at CCR 2014 in Singapore. I gave a direct simple proof of this, but actually this is a corollary of a known result.

The following observation is made by Stephan at CCR 2014.

The time-bounded Kolmogorov complexity is defined by

$$C^g(\sigma) = \min\{|\tau| \ : \ U(\tau)[g(|\sigma|)] = \sigma\}.$$

Theorem (Nies, Stephan and Terwijn 2005)
For sufficiently fast growing function g, X is 2-random iff $C^g(X \upharpoonright n) > n - O(1)$ for infinitely often.

Total machines are essentially the same as universal plain Turing machine with computable time bound C^g. This can be formalized.

In their proof, $g(n) = O(n^2)$ is sufficient.

In my proof, the halting time can be bounded by $O(n)$.

Proof idea

For an input $\sigma = \tau\rho$ where $|\tau| < \sqrt{|\sigma|}$, consider

$$(U(\tau)^{\emptyset'_s}[s])\rho$$

where U is the universal prefix-free machine. .

Definition

Definition

$A \leq_{tm} B$ if, for every total machine M, there exists a total machine N such that

$$C_N(A \upharpoonright n) \leq C_M(B \upharpoonright n) + O(1).$$

I would like to claim that this is a natural measure of randomness.

Refinement?

Observation

If X is Schnorr random and $X \leq_{tm} Y$, then Y is Schnorr random. The same statement holds for Kurtz randomness and 2-randomness.

Question

Does this property hold for ML-randomness?

Schnorr reducibility

Theorem (M.)

For a c.m.m. M, there exists a c.m.m. N such that, for every Schnorr random set X,

$$K_M(X \upharpoonright n) \geq n + K_N(n) - O(1)$$

for infinitely many n.

How fast?

- The term K_N(n) should be replaced, but with what?
- It should depend on M, not too fast, not too slow.
- The counting theorem should hold.

$$Q_M(\sigma) = \mu([\![\{\tau \ : \ M(\tau)\downarrow = \sigma\}]\!])$$

Theorem (Coding theorem)

$$K(\sigma) = -\log Q(\sigma).$$

$$Q_M(\in 2^n) = \mu([\![\{\tau \ : \ M(\tau)\downarrow \in 2^n\}]\!])$$

Theorem (M.)

$$K(n) = -\log Q(\in 2^n).$$

Extended counting theorem

Theorem (Counting theorem)

$$|\{\sigma \ : \ |\sigma| = n \wedge K(\sigma) \leq n + K(n) - r\}| \leq 2^{n-r+O(1)}.$$

Theorem (Extended counting theorem, M.)

$$|\{\sigma \ : \ |\sigma| = n \wedge K_M(\sigma) \leq n - \log(Q_M(\in 2^n)) - r\}| \leq 2^{n-r+O(1)}.$$

2-randomness and c.m.m.

Theorem (M.)

A sequence $X \in 2^\omega$ is 2-random iff, for every computable measure machine M,

$$K_M(X \upharpoonright n) \geq n - \log(Q_M(\in 2^n)) - O(1)$$

for infinitely many n.

Corollary

If $X \leq_{Sch} Y$ and X is 2-random, then Y is 2-random.

Summary

Plain machines (C)	MLR	2R	
Prefix-free machines (K)	MLR	2R	
Comp. measure machines	WR	SR	2R
Prefix-free decidable mach.	WR	SR	MLR
Quick process machines	WR	SR	CR
Total machines	WR	SR	2R

Thank you for your listening.