Variants of layerwise computability

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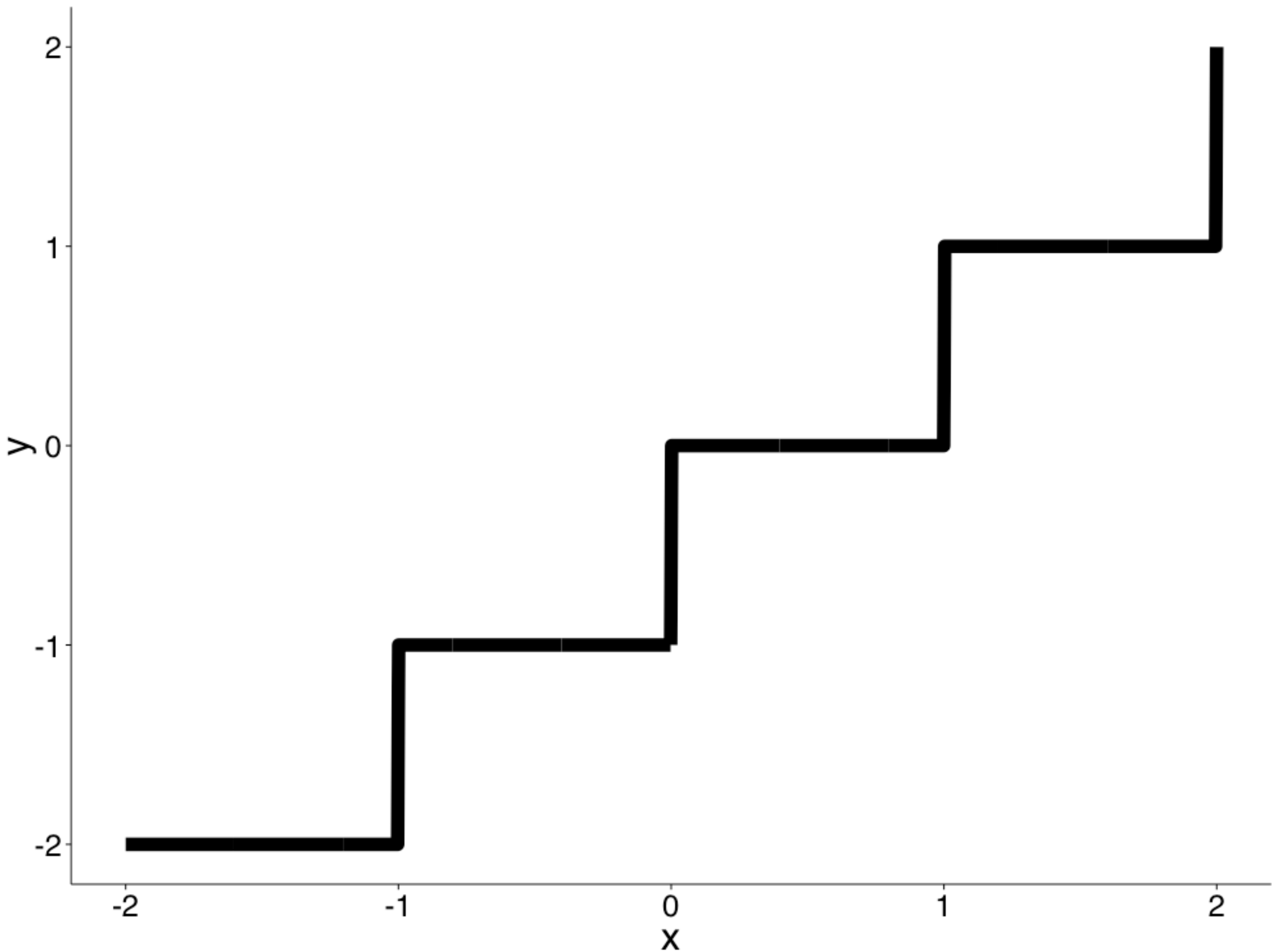
CCA2016@Faro, Portugal

15-17 June 2016

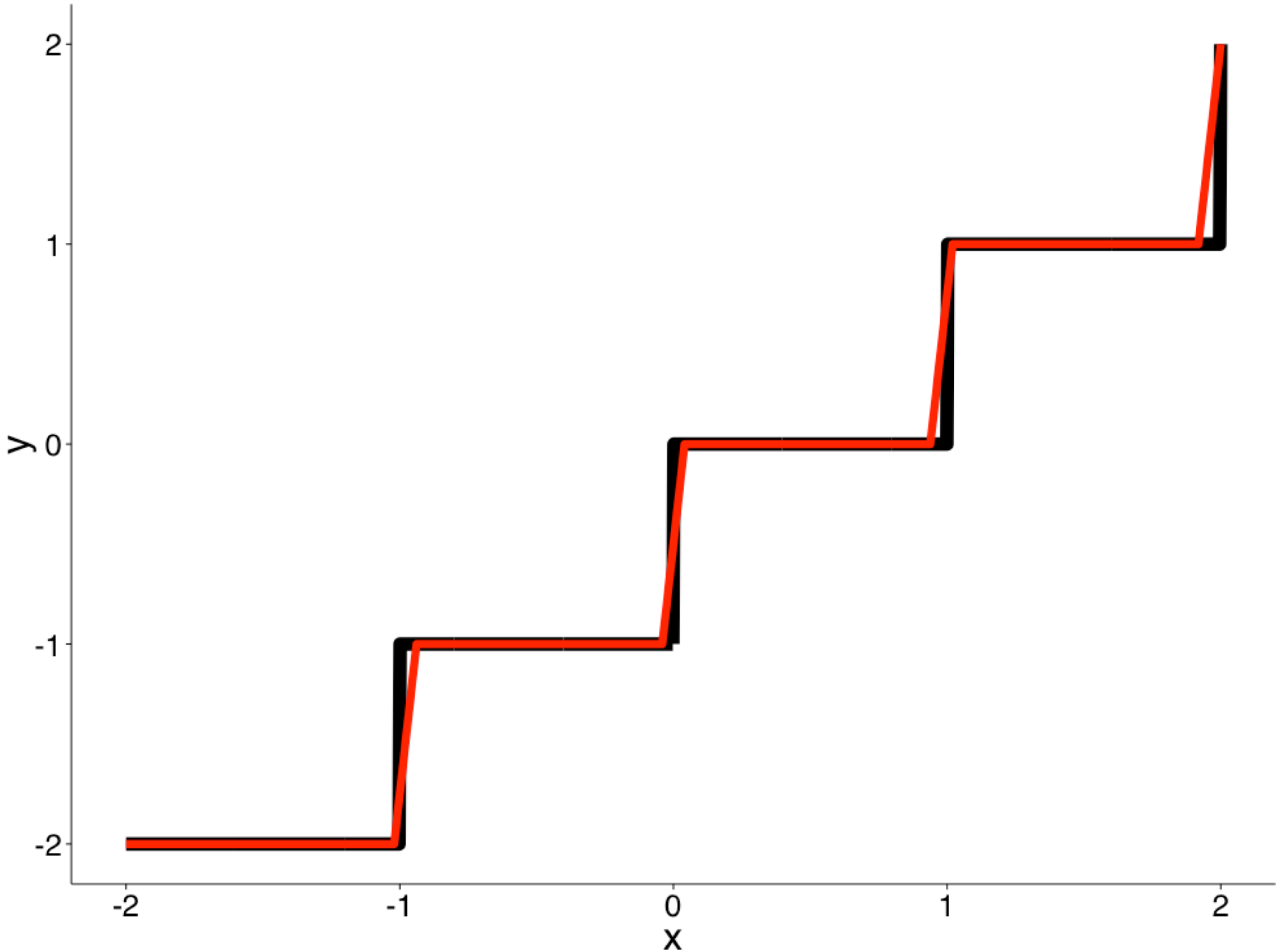
Introduction

Definition (roughly speaking)**.**
A function $f : \mathbb{R} \to \mathbb{R}$ is computable if, for each input and positive error bound, each output can be computably approximated within the bound.

Proposition.
Every computable function is continuous.



2
1
0
-1
-2
y
-2
-1
0
1
2
x



2
1
0
-1
-2
y
-2
-1
0
1
2
x

- The floor function is not computable because it is not continuous.
  The floor function is "almost computable" in some sense.

- (1) It is computable at almost all points.
  => layerwise computability

- (2) It can be approximated by simple functions as accurate as possible.
  => L^p-computability

The main claim

- “Layerwise computability” and “L^p-computability” are essentially the same notion.

Computability

Definition.
A real x is computable if there exists a computable sequence $\{r_n\}$ of rationals such that

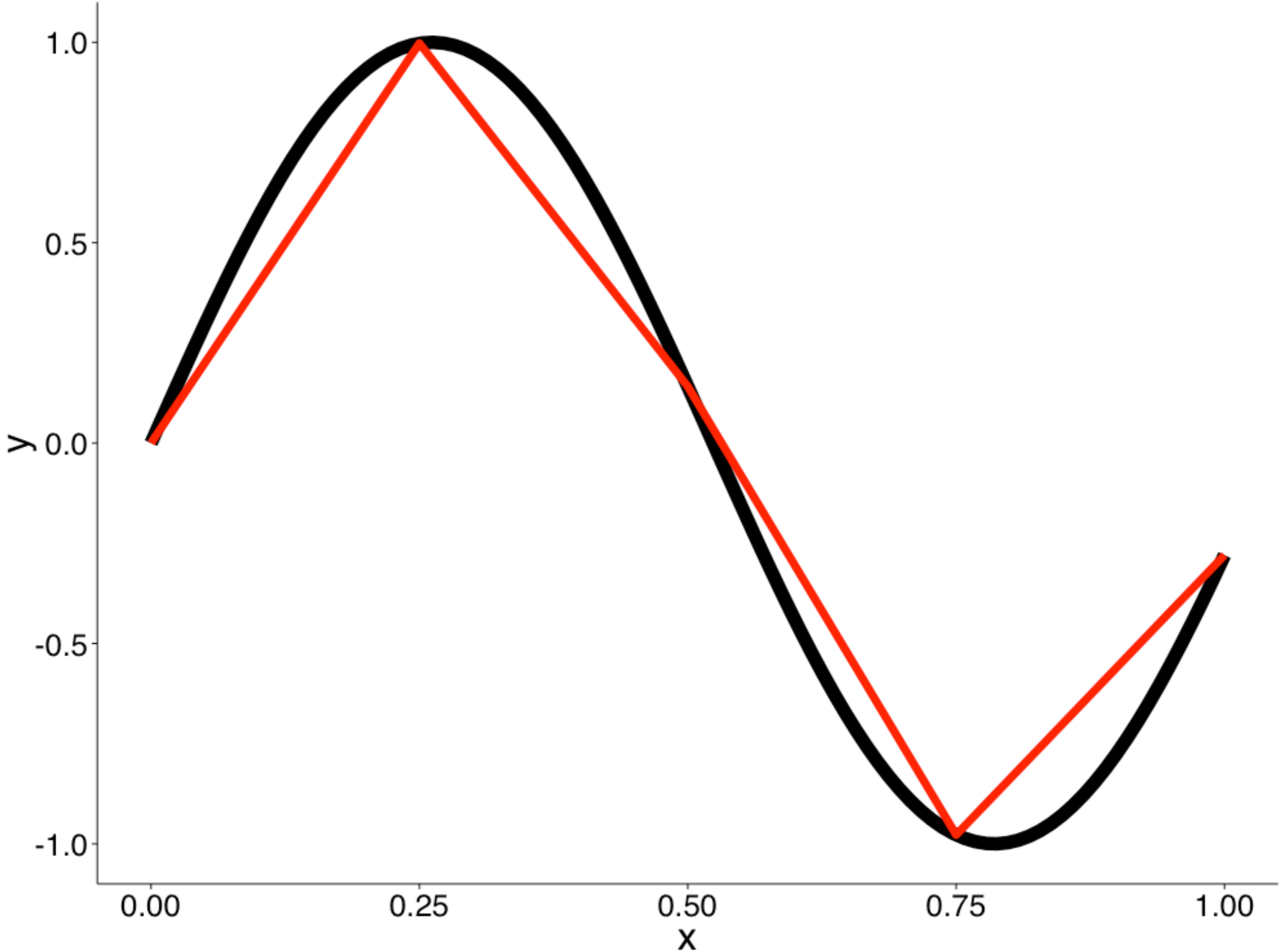
$$|r_n - r_{n-1}| \leq 2^{-n}$$

for all n and $\lim_n r_n = x$.

Definition.
A function $f : [0,1] \to \mathbb{R}$ is computable if there exists a computable sequence $\{r_n\}$ of polylines such that

$$\sup |r_n - r_{n-1}| \leq 2^{-n}$$

for all n and $\lim_n r_n(x) = f(x)$ for all x.



1.0
0.5
0.0
-0.5
-1.0
y
0.00
0.25
0.50
0.75
1.00
x

Computability

- Both definitions of computability have similar forms.
- Different norms (distance functions) induce different notions of computability.

L^p-norm

Definition (L^p-norm)**.**
For a function $f : [0, 1] \to \mathbb{R}$,

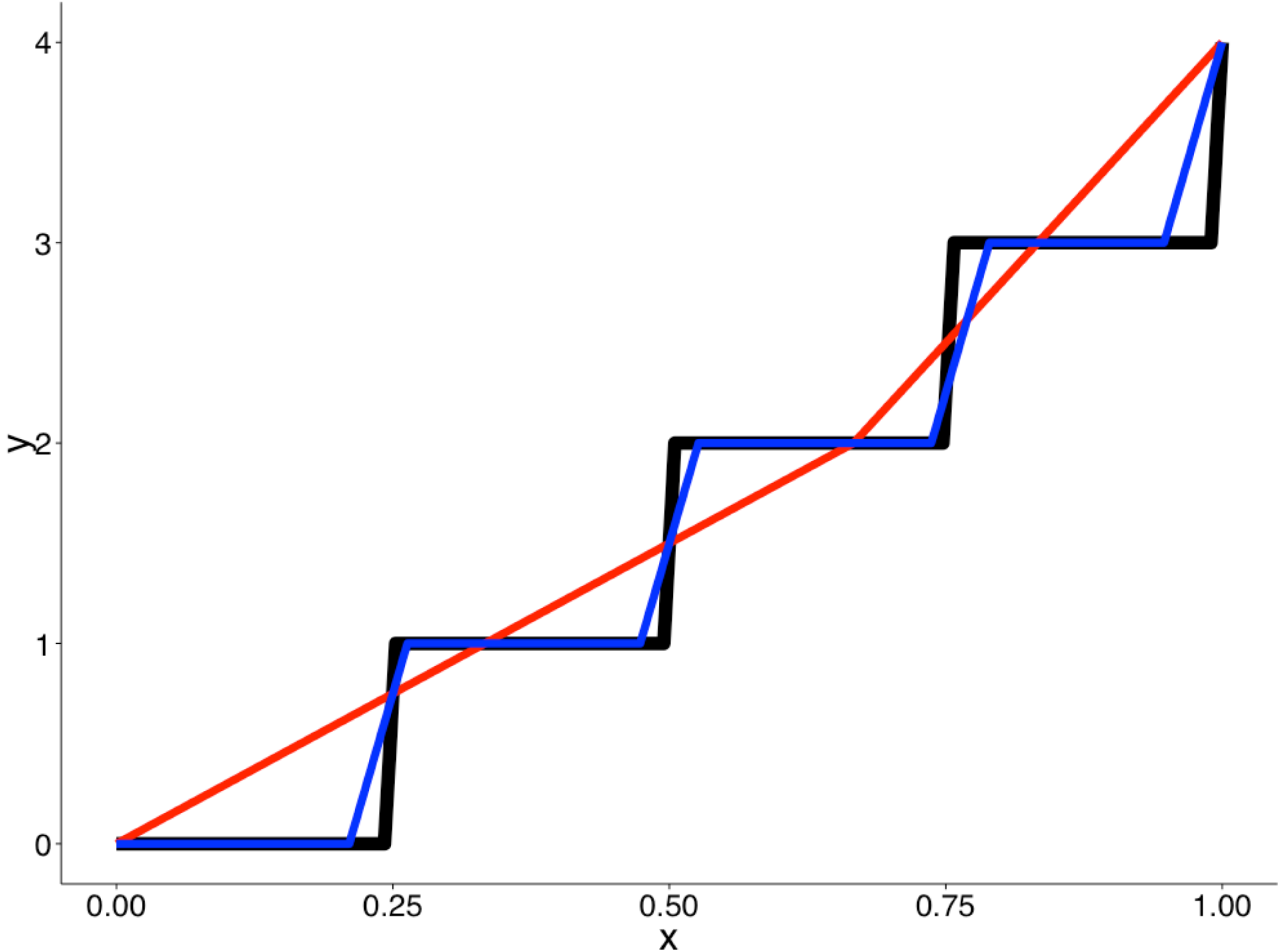
$$||f||_1 = \int |f| \; d\lambda$$

where λ is the Lebesgue measure.

For $p \geq 1$ and a function $f : [0, 1] \to \mathbb{R}$,

$$||f||_p = \left(\int |f|^p \; d\lambda \right)^{1/p}$$

.



4
3
2
1
0
y
0.00
0.25
0.50
0.75
1.00
x

L^p-computability

Definition (L^p-computability)**.**
A function $f :\subseteq [0,1] \to \mathbb{R}$ is called L^p-computable if there exists a computable sequence $\{r_n\}$ of polylines such that

$$||r_n - r_{n-1}||_p \leq 2^{-n}$$

for all n and $\lim_n r_n(x) = f(x)$ for all x.

Remark

- Historically, a different definition has been used for L^p-computability, but it turned out that this more effective version is needed for some application.
- In the definition, the function should be partial because lim r_n(x) may not be defined for some x.

Schnorr null

Definition.
A Schnorr test is a sequence $\{U_n\}$ of uniformly c.e. open sets such that $\mu(U_n) \leq 2^{-n}$ for all n and they are uniformly computable. A class $A \subseteq [0,1]$ is called Schnorr null if there exists a Schnorr test $\{U_n\}$ such that $A \subseteq \bigcap_n U_n$.

Theorem.
The set of undefined points of an L^1-computable function is Schnorr null. Moreover, a real x is Schnorr random if and only if $f(x)$ is defined for every L^1-computable function.

Theorem (Lusin's theorem, see Theorem 2.2.10 in Bogachev). *A function $f : [0,1] \to \mathbb{R}$ is measurable precisely when for each $\epsilon > 0$, there exist a continuous function f_ϵ and a compact set K_ϵ such that $\lambda(I \setminus K_\epsilon) < \epsilon$ and $f = f_\epsilon$ on K_ϵ.*

Definition.
A function $f : [0,1] \to \mathbb{R}$ is Schnorr layerwise computable if there exist a sequence $\{f_n\}$ of uniformly computable functions and a Schnorr test $\{U_n\}$ such that $f = f_n$ on $[0,1] \setminus U_n$.

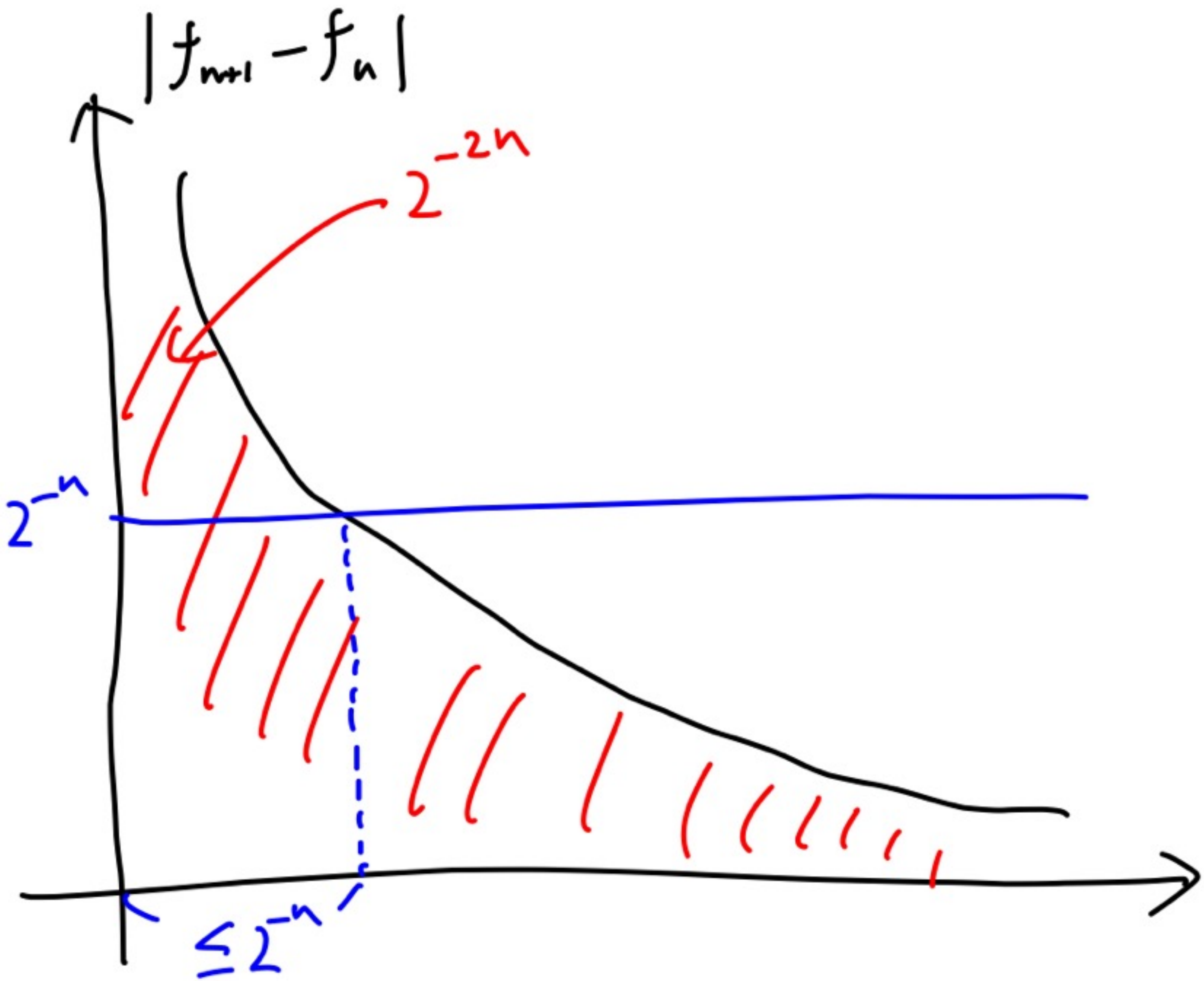
Theorem.
A function $f : [0,1] \to \mathbb{R}$ *is* L^1*-computable function if and only if it is Schnorr layerwise computable and* $\int |f| \, d\lambda$ *is computable.*

Proof Sketch. For an L^1-computable function,

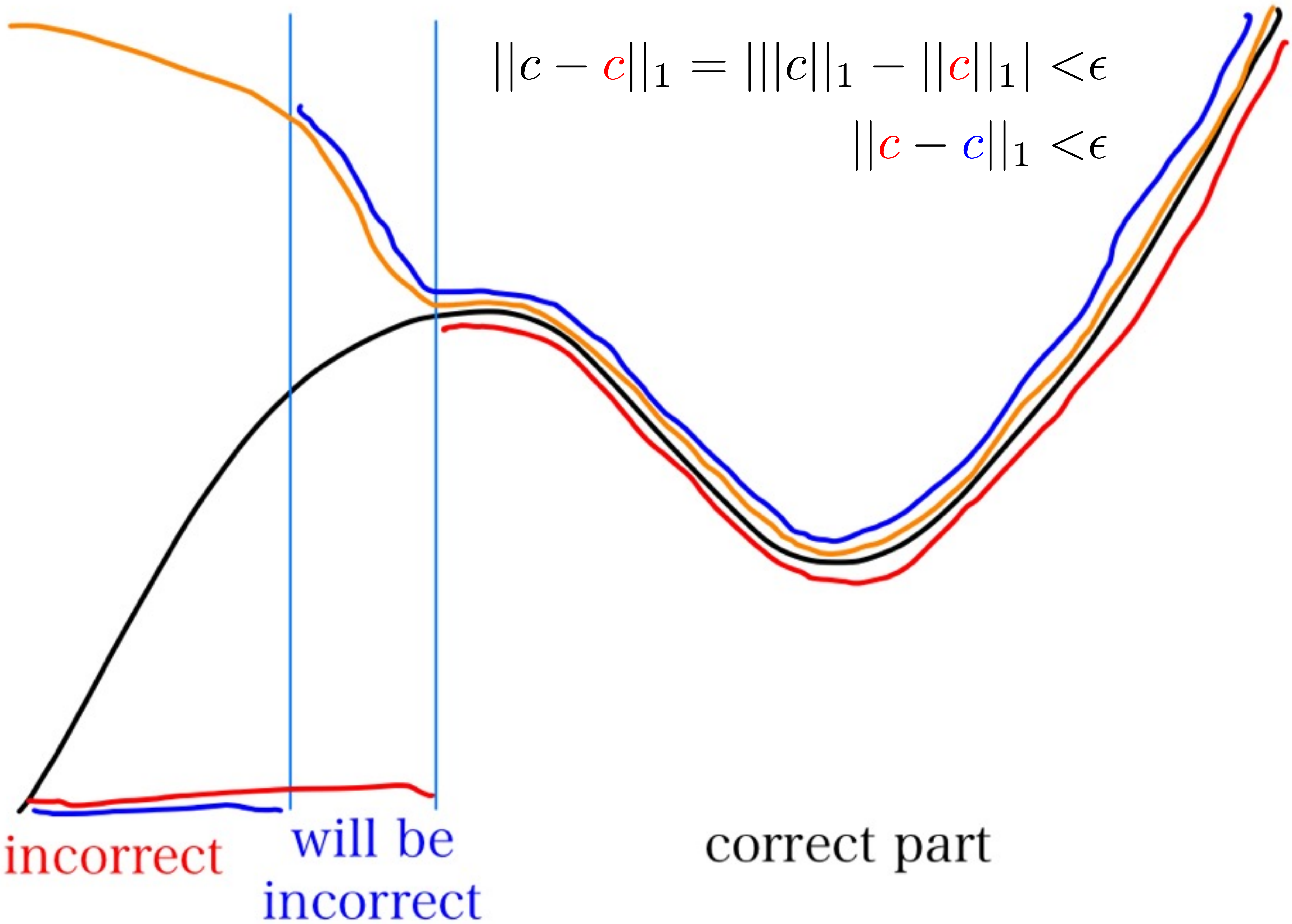
$$\int |f - f_k| d\lambda \leq \sum_{n \geq k} ||f_{n+1} - f_n||_1 \leq 2^{-k}$$

and $\int |f| d\lambda$ is computable. If $||f_{n+1} - f_n||_1$ is small, then $|f_{n+1}(x) - f_n(x)|$ is small on large part, thus computable on the part. The area of exception can be converted to a Schnorr test.

For the other direction, we can make $||f - f_n||_1$ as small as possible by taking sufficiently large n because of computability of $\int |f| d\lambda$. Furthermore, we also know the most incorrect part because of computability of $\{U_n\}$. $\square$



$|f_{n+1} - f_n|$
2^{-2n}
2^{-n}
$\leq 2^{-n}$


$||c - c||_1 = |||c||_1 - ||c||_1| < \epsilon$
$||c - c||_1 < \epsilon$
incorrect
will be incorrect
correct part

Summary of former half

- "Approximable by simple functions"
  = "computable outside small exception"

Motivation

- Turing degrees, Weihrauch degrees, reverse math have studied similar lattices in some sense. The hierarchy in the theory of algorithmic randomness has a little different flavor.
- The study of characterizations of the randomness notions via differentiability clarifies the correspondence between the hierarchy of the randomness notions and the hierarchy of the class of functions.
- Can algorithmic randomness join them? How related?

Computable reals

Definition.
A real $x \in [0,1]$ is computable if there exists a computable sequence of rationals $\{r_n\}$ such that

$$|r_n - r_{n-1}| \leq 2^{-n}$$

for all n and $\lim_n r_n = x$.

Definition.
A real $x \in [0,1]$ is left-c.e. if there exists a non-decreasing computable sequence of rationals $\{r_n\}$ such that $\lim_n r_n = x$.

Weakly comp. reals

Definition.
A real $x \in [0,1]$ is d.c.e if $x = \alpha - \beta$ for some left-c.e. reals α, β.

A real $x \in [0,1]$ is weakly computable (w.c.) if there exists a computable sequence $\{r_n\}$ of rationals such that $\sum_n |r_n - r_{n-1}| < \infty$ and $\lim_n r_n = x$.

Theorem.
A real is d.c.e. if and only if it is w.c.

Weakly comp. functions

Definition. A function $f : [0,1] \to \mathbb{R}$ is w.c. if there exists a computable sequence $\{r_n\}$ of polylines such that

$$\sum_n \sup |r_n - r_{n-1}| < \infty$$

for all n and $\lim_n r_n(x) = f(x)$ for all x.

A function $f :\subseteq [0,1] \to \mathbb{R}$ is L^p-w.c. if there exists a computable sequence $\{r_n\}$ of polylines such that

$$\sum_n ||r_n - r_{n-1}||_p < \infty$$

for all n and $\lim_n r_n(x) = f(x)$ for all x.

Question

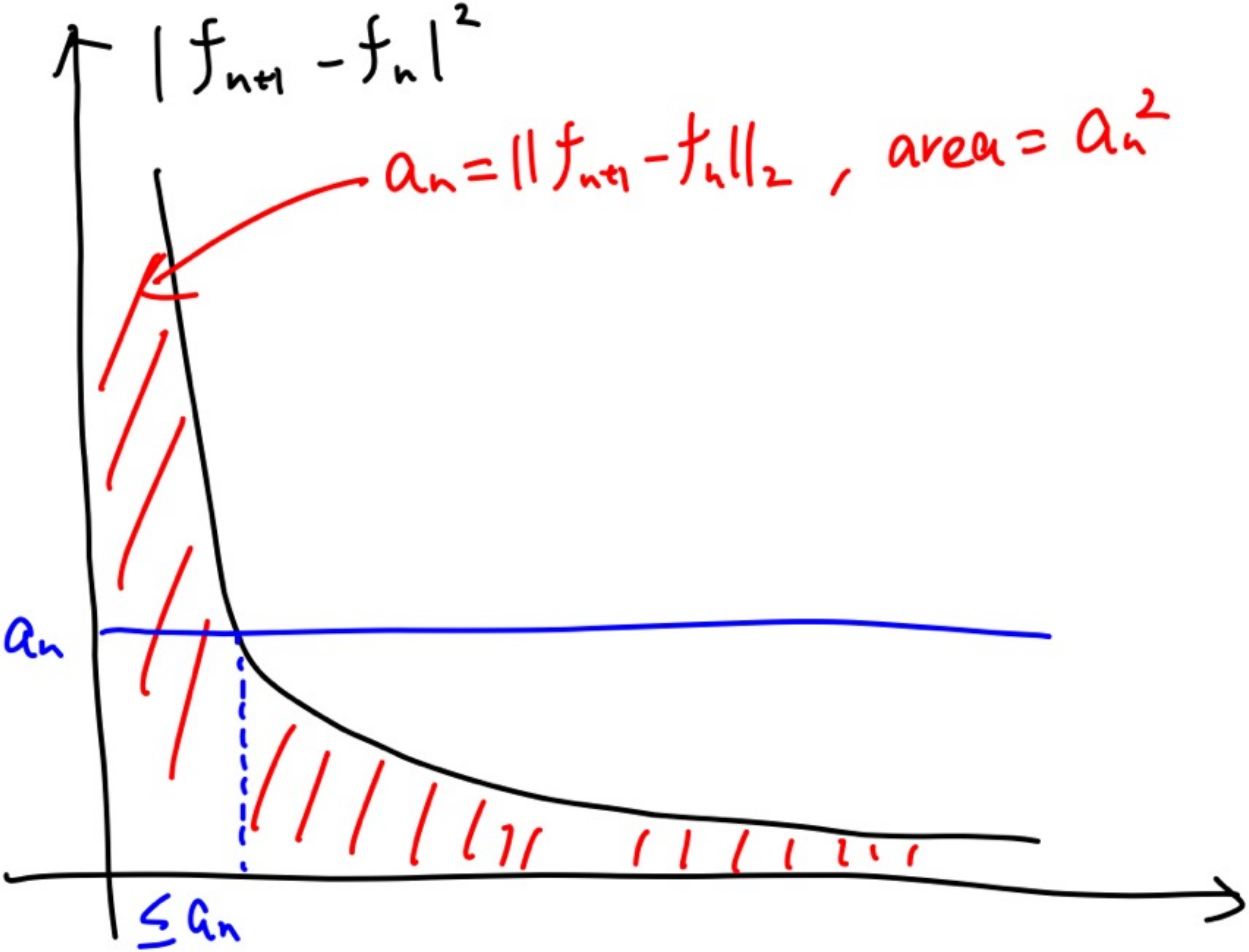
- L^p-comp. = Schnorr l.w. comp. + comp. L^p-norm
- L^p-w.c. = ? l.w. p-w.c. + w.c. L^p-norm

Theorem.
Let f be an L^2-w.c. function. Then, f is Solovay-layerwise 2-w.c. and $||f||_2$ is w.c.

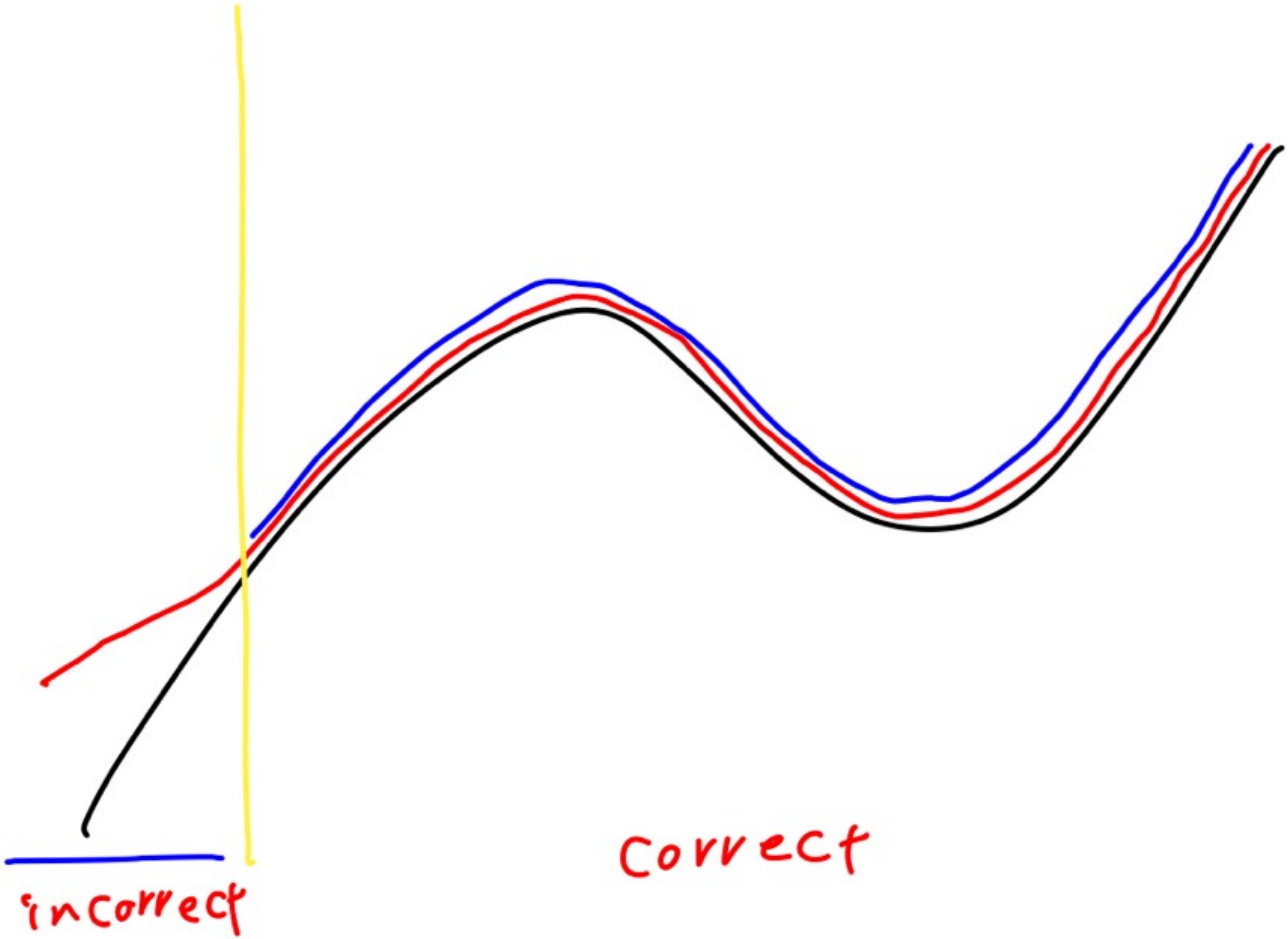
For a bounded function, the converse also holds.

Definition.
A function $f :\subseteq [0,1] \to \mathbb{R}$ is Solovay-layerwise computable if there exists a Solovay test $\{U_n\}$ and a sequence $\{f_n\}$ of uniformly computable functions such that $f = f_n$ on $[0,1] \setminus \bigcup_{k=n}^{\infty} U_k$.

Definition. A real x is p-weakly computable (p-w.c.) if there exists a computable sequence $\{r_n\}$ of rationals such that $\sum_n |r_{n+1} - r_n|^p < \infty$ and $\lim_n r_n = x$.



$|f_{n+1} - f_n|^2$
$a_n = \|f_{n+1} - f_n\|_2$, area $= a_n^2$
a_n
$\leq a_n$


correct
incorrect

Summary

- L^p-comp. = Schnorr l.w. comp. + comp. L^p-norm
- L^2-w.c. = Solovay l.w. 2-w.c. + w.c. L^p-norm
  if bounded
- divergence bounded computability <=> Demuth?
- Thank you for your attention.