Randomness notions in Muchnik and Medvedev degrees

Kenshi Miyabe

Meiji University, Japan,
research@kenshi.miyabe.name

The main question of this talk is whether one can construct a more random set from a random set. This question can be formalized by mass problems. We view the elements of a class $P \subseteq 2^\omega$ as solutions to some problem. If a solution of a problem P can be constructed (or computable) from each solution of another problem Q, then we would say that P is more difficult than Q. We have two ways to formalize this notion. The difference between the two notions is uniformity.

Definition 1 (Muchnik [5], Medvedev [4]). *Let $P, Q \subseteq 2^\omega$.*

1. *We say that P is* Muchnik reducible (or weakly reducible) *to Q, denoted by $P \leq_w Q$, if, for every $f \in Q$, there is an element $g \leq_T f$ in P.*
2. *We say that P is* Medvedev reducible (or strongly reducible) *to Q, denoted by $P \leq_s Q$, if there is a Turing functional Φ such that $\Phi^f \in P$ for every $f \in Q$.*

Simpson [7] has already pointed out the importance of the class of ML-random sets in Muchnik degrees of Π^0_1 classes. We straightforwardly study some randomness notions in Muchnik and Medvedev degrees. The randomness notions we consider are ML-randomness, difference randomness, Demuth randomness, weakly 2-randomness, 2-randomness, computable randomness, Schnorr randomness, and Kurtz randomness. Each class is denoted by MLR, DiffR, DemR, W2R, 2R, CR, SR, and WR, respectively.

The obvious reductions $\leq_w$ and $\leq_s$ follows from inclusion of a randomness notion in another. Thus, our interest is only strictness, which can be interpreted as impossibility of (uniform) construction of a more random set from a random set.

For Muchnik degrees, we have the following result.

Theorem 1.

$$\mathrm{WR} <_w \mathrm{SR} \equiv_w \mathrm{CR} <_w \mathrm{MLR} \equiv_w \mathrm{DiffR} \begin{matrix} <_w \ \mathrm{W2R} \ <_w \\ <_w \ \mathrm{DemR} \ <_w \end{matrix} \mathrm{2R}.$$

The proofs of the strictness are given by finding a degree with some property. For instance, CR $<_w$ MLR follows from the existence of a high minimal degree because each high degree contains a computably random set while each set Turing below a minimal degree cannot be ML-random by van Lambalgen's theorem.

The equivalence MLR $\equiv_w$ DiffR in Muchnik degrees is shown very interestingly. For every $A \oplus B \in$ MLR, at least one of A and B should be difference random because, if $A \geq_T \mathbf{0}'$, then B is 2-random, and thus difference random. However, we do not know which is difference random. In fact, we can not do this uniformly as the following result shows.

Theorem 2.

$$\text{MLR} <_s \text{DiffR}.$$

The proof uses the Levin-Kautz theorem on computable continuous measures [8, 3] and no-randomness-from-nothing for ML-randomness [2].

For the last part, we could separate SR and CR in Medvedev degrees.

Theorem 3.

$$\text{SR} <_s \text{CR}.$$

The proof uses the method to separate SR and CR [6], some results on randomness for computable measures [1], and Lévy's zero-one law from probability theory. This type of interaction between computability theory and analysis seems interesting to me.

Finally we remark that we can ask similar questions in reverse mathematics and Weihrauch degrees. One of the major differences between reverse mathematics and Weihrauch degrees is the uniformity. Thus, this work may help to clarify the difference.

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