Randomness notions in Muchnik and Medvedev degrees

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Background

Question

What is computational power with random access?

Can we construct a more random set from a given random set?

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Old answer

Theorem 1 (De Leeuwe, Moore, Shannon, Shapiro (1956), Sacks)**.** *If $A \in 2^\omega$ is not computable, then the class*

$$\{X \in 2^\omega \ : \ A \leq_T X\}$$

has measure 0.

Thus, if A is computable with random access, then A is computable without random access.
How about the case there are many answers?

Mass problems

Definition 2. Let $P, Q \subseteq 2^\omega$.

P is Muchnik reducible to Q ($P \leq_w Q$) if, for every $f \in Q$, there exists $g \in P$ such that $g \leq_T f$.

P is Medvedev reducible to Q ($P \leq_s Q$) if, there exists a Turing functional Φ such that $\Phi^f \in P$ for every $f \in Q$.

The difference is uniformity.

Main results

Randomness hierarchy

$$\mathrm{KR} \supset \mathrm{SR} \supset \mathrm{CR} \supset \mathrm{MLR} \supset \mathrm{DiffR} \begin{array}{ccc} \supset & \mathrm{W2R} & \supset \\ \supset & \mathrm{DemR} & \supset \end{array} \mathrm{2R}$$

Muchnik degrees

$$\mathrm{KR} <_w \mathrm{SR} \equiv_w \mathrm{CR} <_w \mathrm{MLR} \equiv_w \mathrm{DiffR} \begin{array}{ccc} <_w & \mathrm{W2R} & <_w \\ <_w & \mathrm{DemR} & <_w \end{array} \mathrm{2R}$$

Medvedev degrees

$$\mathrm{SR} <_s \mathrm{CR}, \;\; \mathrm{MLR} <_s \mathrm{DiffR}$$

Proof

$\mathrm{KR} <_w \mathrm{SR}$

Let $\mathbf{a}$ be a minimal degree below $\mathbf{0}'$.
$\mathbf{a}$ is hyperimmune.
Every hyperimmune degree contains $X \in \mathrm{KR} \setminus \mathrm{SR}$.
Suppose $Y \leq_T X$ and $Y \in \mathrm{SR}$.
Since $\mathbf{a}$ is minimal, $Y \in \mathbf{a}$.
No minimal degree below $\mathbf{0}'$ can be high (Cooper '73), so Y is not high.
Nonhigh Schnorr random Y should be ML-random.
This contradicts to minimality of $\mathbf{a}$ by van Lambalgen's theorem.

MLR $<_s$ DiffR *(1/2)*

The goal is

$$\forall \Phi \exists X \in \mathrm{MLR}[\Phi(X) \notin \mathrm{DiffR}].$$

We can assume that Φ is almost-everywhere computable.
Let μ be the induced measure from Φ and the fair-coin measureλ, that is $\mu(U) = \lambda(\Phi^{-1}(U))$. Note that μ is computable.
If $\mu(\{Z\}) > 0$, then $\lambda(\Phi^{-1}(\{Z\})) > 0$. Hence, $\Phi^{-1}(\{Z\})$ contains a ML-random set X. However, every atom of a computable measure is computable.

$\mathrm{MLR} <_s \mathrm{DiffR}$ *(2/2)*

Levin-Kautz theorem says that, for a continuous measure ν and $\mathbf{a} > 0$, $\mathbf{a}$ contains ML-random iff $\mathbf{a}$ contains ν-ML-random. Apply this to μ and $\mathbf{0}'$ and get a μ-ML-random $Y \in \mathbf{0}'$.
By no-randomness-from-nothing for ML-randomness, there exists $X \in \mathrm{MLR}$ such that $\Phi(X) = Y$.

$\mathrm{SR} <_s \mathrm{CR}$

The goal is

$$\forall \Phi \exists X \in \mathrm{SR}[\Phi(X) \notin \mathrm{CR}].$$

When $\Phi = \mathrm{id}$, this means

$$X \in \mathrm{SR} \setminus \mathrm{CR}.$$

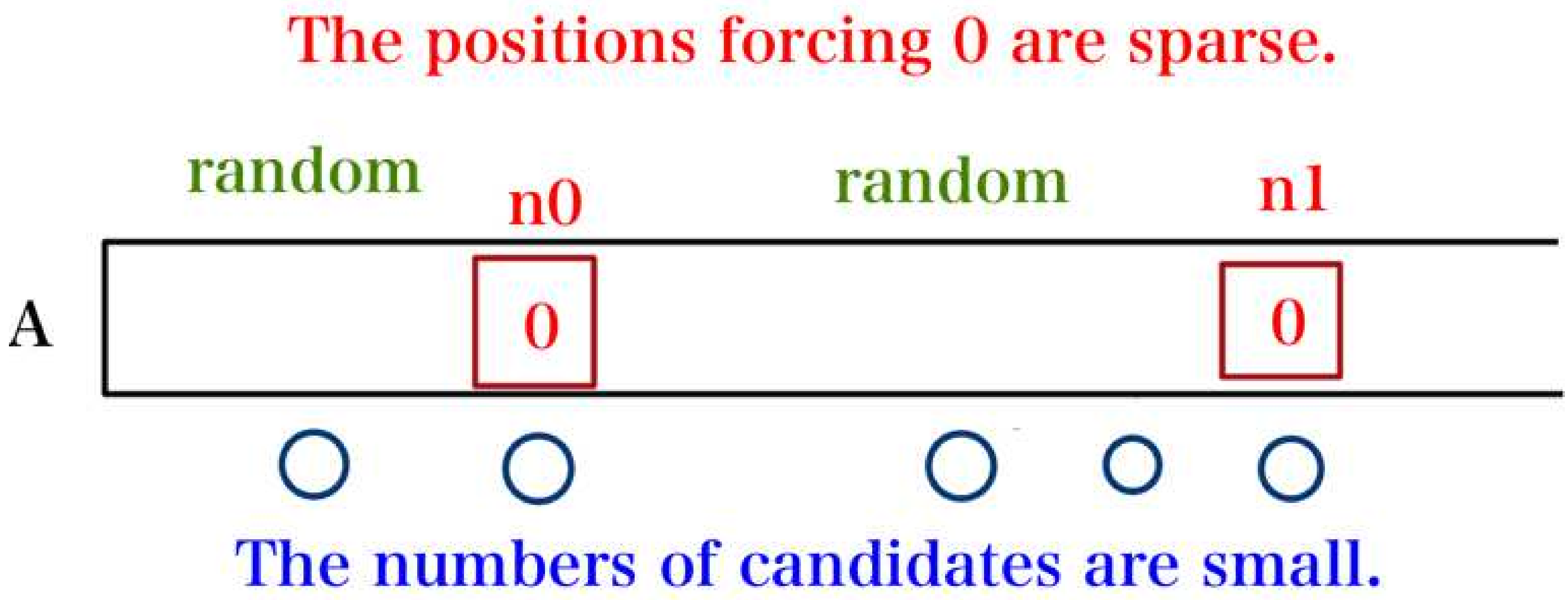
Thus, we extend the method separating SR and CR.

SR $\supsetneq$ CR

id case

- Construct a random set A.
- Forcing $A(n_k) = 0$ in sparse positions
  $\Rightarrow$ too sparse not to be Schnorr random
- The number of candidates of n_k is small
  $\Rightarrow$ so small that some computable martingale succeeds on it.

SR $\supsetneq$ CR



general case

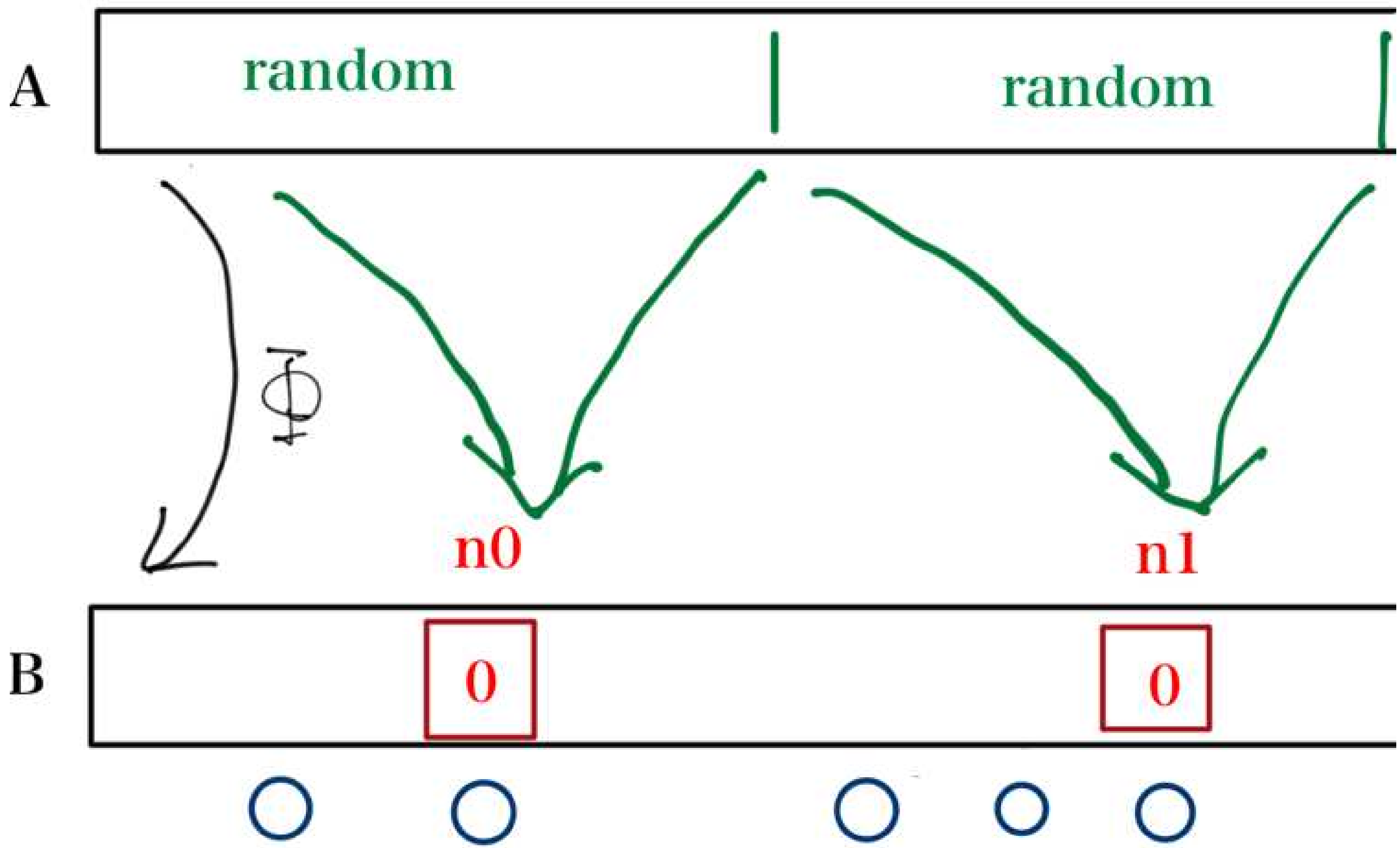
- Construct $A \in SR$ and $B = \Phi(A) \notin \mathrm{CR}$.
- Forcing $B(n_k) = 0$ in some positions (*)
- The number of candidates of n_k should be small
  $\Rightarrow B \notin \mathrm{CR}$.

The requirement (*) may be strong because

$$\lambda(\{X \in 2^\omega \ : \ \Phi(X)(n_k) = 0\})$$

may be too small (even empty).

SR $<_s$ CR



The numbers of candidates are small.

$\mathrm{SR} <_s \mathrm{CR}$

We divide the case into two by the induced measure μ.

- μ is ”close to” uniform measure $(\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda))$
  $\Rightarrow$ the same method can be applied
- μ is ”far from” uniform measure $(\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda))$
  $\Rightarrow$ we can show it by a different reason

$\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda)$

$$\exists Y \in \mathrm{CR}(\mu) \setminus \mathrm{CR}(\lambda)$$

By no-randomness-from-nothing for CR,

$$\exists X \in \mathrm{CR}\, [\Phi(X) = Y].$$

Then, $X \in \mathrm{SR}$ and $\Phi(X) \notin \mathrm{CR}$.

$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$

Lemma 3 (essentially Bienvenu-Merkle)**.** *Let μ, ν be computable measures.*

$$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\nu) \Rightarrow \mathrm{MLR}(\mu) \subseteq \mathrm{MLR}(\nu) \Rightarrow \mu \ll \nu$$

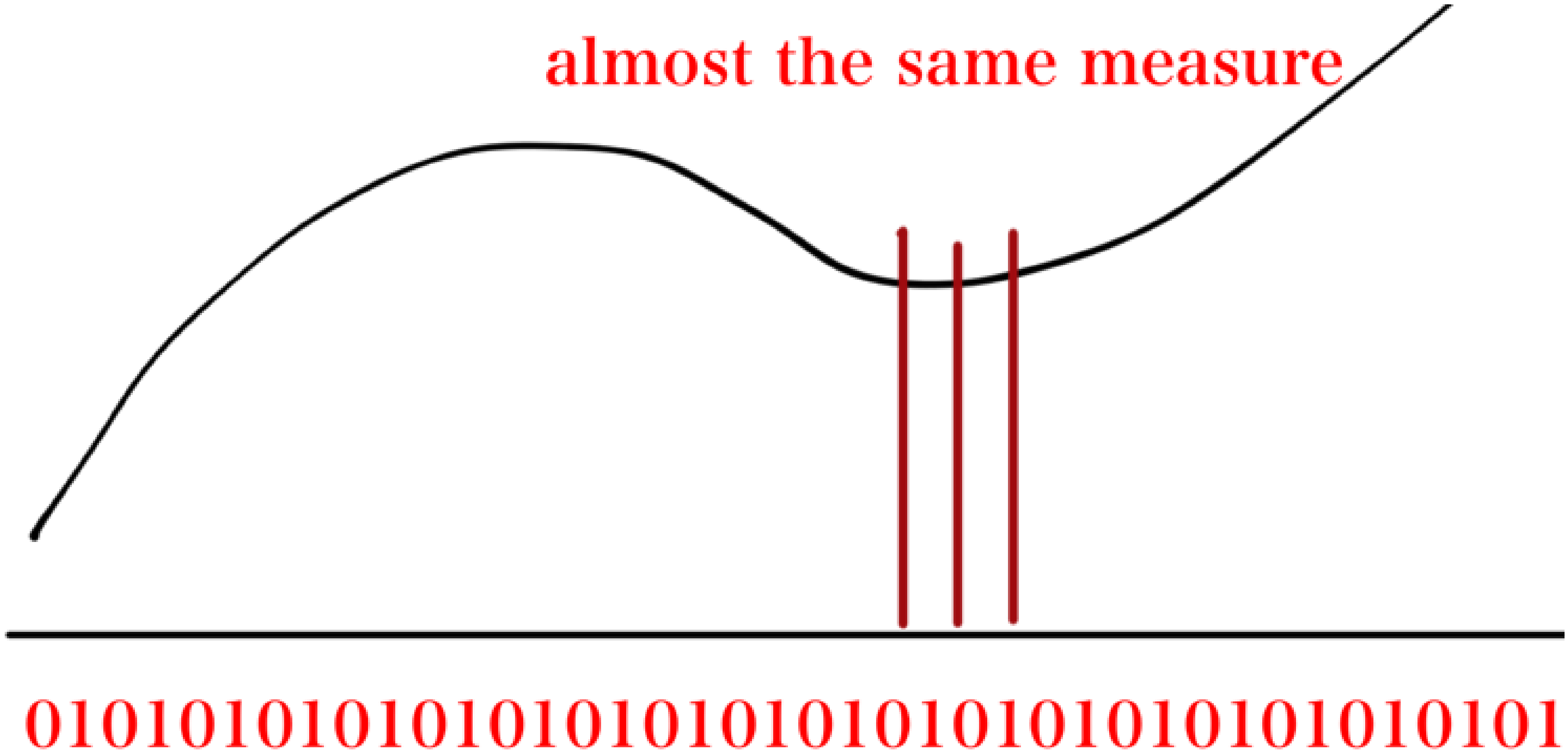
$\ll$ *means absolute continuity.*

Lemma 4. *Let Φ be an a.e. computable function. Let μ be the induced measure from Φ and λ. Assume $\lambda \ll \mu$. For each $\sigma \in 2^{<\omega}$,*

$$\lim_{n\to\infty} \lambda\{X \in [\sigma] \;:\; \Phi(X)(n) = 0\} = \frac{1}{2}\lambda(\sigma).$$

Proof. By the Radon-Nikodym theorem and Lévy's zero-one law. □

$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$



Summary

Question

Question 5. Does there exist $X \in \mathrm{SR}$ such that, if $Y \leq_{tt} X$ then $Y \notin \mathrm{CR}$.
How about wtt?

I conjecture that we can not tt-compute (or wtt-compute) a computably random from a Schnorr random even nonuniformly.

Summary

- We found two problems that is possible non-uniformly but not possible uniformly.
- Analytical tools are useful to show results in computability. In particular, a.e. computable functions can be studied more from the measure-theoretic perspective.