COHERENCE OF REDUCIBILITIES WITH RANDOMNESS NOTIONS

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Abstract. Loosely speaking, when A is "more random" than B and B is "random", then A should be random. The theory of algorithmic randomness has some formulations of "random" sets and "more random" sets. In this paper, we study which pairs (R, r) of randomness notions R and reducibilities r have the follwing property: if A is r-reducible to B and A is R-random, then B should be R-random. The answer depends on the notions R and r. The implications hold for most pairs, but not for some. We also give characterizations of n-randomness via complexity.

1. Introduction

1.1. **Degree of randomness.** The theory of algorithmic randomness focused on many randomness notions such as ML-randomness, 2-randomness, Schnorr randomness, Kurtz randomness. Most of the notions are linearly ordered in the sense that one randomness notion implies another randomness notion. For instance, 2-randomness implies ML-random, which in turn implies Schnorr randomness, which also implies Kurtz randomness. Here, we say that 2-randomness is stronger than ML-randomness and so on, and we call this order the *randomness hierarchy*. This fact can be used as a measure of how random a set is.

Another way of measuring randomness is reducibility. The Levin-Schnorr theorem says that a set $A \in 2^\omega$ is ML-random if and only if $K(A \upharpoonright n) > n - O(1)$ where K is the prefix-free Kolmogorov complexity. With this in mind, we say that A is *K-reducible to B*, denoted by $A \leq_K B$, if $K(A \upharpoonright n) < K(B \upharpoonright n) + O(1)$, whose intuitive meaning is that B is more random than A. K-reducibility has been well studied, while similar reducibilities also have been studied.

We expect that, if A is random and B is more random than A, then B should be random. ML-randomness with K-reducibility satisfies this property by the Levin-Schnorr theorem. Furthermore, Miller and Yu [18] showed that, if A is n-random and $A \leq_K B$, then B is n-random. This does not hold for K-reducibility and Schnorr randomness in the sense that, even if $A \leq_K B$ and A is Schnorr random, B may not be Schnorr random (Proposition 2.2). Thus, the measure by K-reducibility and the one induced by the randomness hierarchy are not completely coherent.

We say that a reducibility $\leq_r$ of randomness is *coherent* with a randomness notion R if the following holds: if $A \leq_r B$ and A is R-random, then B is R-random (Definition 2.1). Now, we ask which pairs of reducibilities and randomness notions are coherent. The reducibilities we consider in this paper are K-reducibility, C-reducibility, Schnorr reducibility, and decidable prefix-free machine reducibility and total-machine reducibility, and the randomness notions are ML-randomness, Schnorr randomness, Kurtz randomness and n-randomness.

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The answers for some pairs are immediate from known results. We also have some pairs that can be solved by easy or known arguments: Proposition 2.2, Theorem 2.4, Theorem 2.6 and Proposition 2.8. As a by-product of this project, 2-randomness can be characterized by computable measure machines (Theorem 3.4), whose proof requires an extension of the Counting theorem, which will be developed in Section 3. The main result of this paper is incoherence of Schnorr reducibility with some randomness notions (Theorem 3.6), whose proof uses the method separating Schnorr randomness and computable randomness.

1.2. **n-randomness via complexity.** Characterization of randomness notions via complexity has been a central topic in the theory of algorithmic randomness. The first achievement is the Levin-Schnorr theorem, which characterizes ML-randomness via K. Miller and Yu [18] have given a characterization of ML-randomness via C. Miller [16] also gave characterizations of 2-randomness via C and K. A natural question is whether there exist characterizations of 3-randomness via complexity.

Roughly speaking, a set is ML-random if and only if the complexities of its initial segments of the set are large. A set is 2-random if and only if the complexities are infinitely often maximal up to a constant, in other words, the set of the lengths such that the complexities are maximal is infinite. A theorem we will prove in this paper (Corollary 4.5) says that, a set is n-random if and only if the set of the lengths such that the complexities are maximal is complex according to n. We obtain this result as a by-product in the project of the main question.

1.3. **Overview of paper.** In Section 2 we define the notion of coherence, and then review some results relating to K and C reducibilities and look at some immediate corollaries. We also investigate coherence of decidable prefix-free machine reducibility and total-machine reducibility with randomness notions. In Section 3 we focus on Schnorr reducibility and see incoherence with stronger randomness notions. In Section 4 we show characterizations of n-randomness via complexity. In Section 5 we investigate the notion of dm-triviality.

2. Basic results

We generally follow the notations in Downey and Hirschfeldt [11] and Nies [21]. We also refer a survey [1] by Barmpalias on randomness reducibilities.

The central notion of this paper is *coherence* of a reducibility with a randomness notion defined as follows.

Definition 2.1. We say that a reducibility $\leq_r$ is *coherent* with a randomness notion $\mathcal{R}$ if the following statement holds: For all sets A and B, if $A \leq_r B$ and $A \in \mathcal{R}$, then $B \in \mathcal{R}$.

The intended reducibilities $\leq_r$ are K-reducibility, C-reducibility, Schnorr reducibility, and so on. The relation $\leq_r$ can be any reducibility that is interpreted as a "more random" relation. Then, the coherence of a pair (R, r) means that the following natural reasoning holds: if A is more random than B and B is random, then A should be random. If a reducibility is coherent with many randomness notions, then the reducibility can be seen as a refinement of the randomness hierarchy.

2.1. **Randomness notions.** The randomness notions we consider in this paper are ML-randomness, Schnorr randomness, Kurtz randomness and n-randomness.

We consider Cantor space 2^ω with the product topology equipped with the uniform measure μ. A *ML-test* is a sequence $\{U_n\}_{n\in\omega}$ of uniformly c.e. open sets with $\mu(U_n) \leq 2^{-n}$. A set $A \in 2^\omega$ is *ML-random* if A passes every ML-test $\{U_n\}_{n\in\omega}$, that is, $A \notin \bigcap_n U_n$. A *Schnorr test* is a ML-test such that $\mu(U_n)$ is uniformly computable. A set A is *Schnorr random* if A passes every Schnorr test. A set A

is *Kurtz random* if $A \in U$ for every c.e. open set U with measure 1. A set A is called *n-random* if ML-random relative to $\mathbf{0}^{(n-1)}$. A *martingale* is a non-negative function $M : 2^{<\omega} \to \mathbb{R}$ such that $2M(\sigma) = M(\sigma 0) + M(\sigma 1)$ for every $\sigma \in 2^{<\omega}$. A set A is called *computably random* if $\limsup_n M(A \upharpoonright n) < \infty$ for every computable martingale, where $A \upharpoonright n$ is the initial segment of A with length n. Then, we have the following proper implications:

$$\begin{aligned}(n+1)\text{-random} \Rightarrow & n\text{-random} \Rightarrow \text{ML-random}\\ & \Rightarrow \text{computably random} \Rightarrow \text{Schnorr random} \Rightarrow \text{Kurtz random}\end{aligned}$$

where $n \geq 2$.

2.2. **The reducibilities $\leq_K$ and $\leq_C$.** In many cases, coherence of a reducibility with a randomness notion is derived from a characterization of the randomness notion via complexity.

Let C denote the plain Kolmogorov complexity and K the prefix-free Kolmogorov complexity. The Levin-Schnorr theorem says that a set X is ML-random if and only if $K(X \upharpoonright n) > n - O(1)$. The theorem roughly says that the complexities of initial segments of a random set are large. Then, it is natural to measure randomness of a set by the complexities of initial segments of the set. A set A is K-reducible to B, denoted by $A \leq_K B$, if $K(A \upharpoonright n) \leq K(B \upharpoonright n) + O(1)$. Similarly, A is C-reducible to B, denoted by $A \leq_C B$, if $C(A \upharpoonright n) \leq C(B \upharpoonright n) + O(1)$. Trivially, we have the following observation: The reducibility $\leq_K$ is coherent with ML-randomness.

Our interest is coherence for other pairs. We also have some other characterizations that imply coherence of K and C-reducibilities. The characterization of ML-randomness by C by Miller and Yu [18] immediately implies the coherence of C-reducibility with ML-randomness. The following are equivalent for $X \in 2^\omega$:

(i) X is ML-random.
(ii) $C(X \upharpoonright n) > n - K(n) - O(1)$ for all n.
(iii) $C(X \upharpoonright n) > n - G(n) - O(1)$ for all n, where

$$G(n) = \begin{cases} K_{s+1}(t) & \text{if } n = 2^{\langle s,t \rangle} \text{ and } K_{s+1}(t) \neq K_s(t) \\ n & \text{otherwise.} \end{cases} \tag{1}$$

Miller [16] and Nies, Stephan, and Terwijn [22] showed that a set X is 2-random if and only if $C(X \upharpoonright n) > n - O(1)$ for infinitely many n. Miller [17] also showed that a set X is 2-random if and only if $K(X \upharpoonright n) > n + K(n) - O(1)$ for infinitely many n. Then, the reducibilities $\leq_K$ and $\leq_C$ are coherent with 2-randomness.

So far we have seen that the reducibilities $\leq_K$ and $\leq_C$ are coherent with ML-randomness and 2-randomness. In contrast, the reducibilities $\leq_K$ and $\leq_C$ are not coherent with weaker randomness notions.

Proposition 2.2. *The reducibilities $\leq_K$ and $\leq_C$ are not coherent with computable randomness, Schnorr randomness, or Kurtz randomness.*

Proof. There exists a computably random set X such that, for every computable order h, we have $K(X \upharpoonright n \mid n) \leq h(n) + O(1)$ (see Nies [21, Theorem 7.4.11]). Let A be the witness and B be a ML-random set. Then, $A \leq_K B \oplus \emptyset$ because

$$K(B \oplus \emptyset \upharpoonright n) \geq K(B \upharpoonright \lfloor n/2 \rfloor) - O(1) > \frac{n}{2} - O(1)$$

and

$$K(A \upharpoonright n) \leq K(A \upharpoonright n | n) + K(n) + O(1) \leq \frac{n}{3} + O(1).$$

However, A is computably random, Schnorr random and Kurtz random while $B \oplus \emptyset$ is not computably random, Schnorr random or Kurtz random.

The same holds for $\le_C$ because $C(x) \le K(x) + O(1) \le C(x) + 2\log(|x|) + O(1)$ for all $x \in 2^{<\omega}$ (see [11, Corollary 2.4.2] etc.). □

Thus, although the reducibilities $\le_K$ and $\le_C$ can be seen as measures of randomness, they are not refinements of the randomness hierarchy.

2.3. **Decidable prefix-free machine reducibility.** We will see coherence of other reducibilities with the randomness hierarchy. We ask which reducibilities are coherent with all (or many) randomness notions and can be seen as refinements of the randomness hierarchy. The first reducibility we should look at would be decidable prefix-free machine reducibility because we know that decidable prefix-free machines characterize many randomness notions. We say that a machine M is *decidable* if its domain $\mathrm{dom}(M)$ is decidable.

The following is due to Bienvenu and Merkle [6]. A set X is ML-random if and only if $K_M(X \upharpoonright n) > n - O(1)$ for every decidable prefix-free machine M. Furthermore, there exists a decidable prefix-free machine N such that X is ML-random if and only if $K_N(X \upharpoonright n) > n - O(1)$. A set X is Schnorr random if and only if, for every decidable prefix-free machine M and every computable order h, $K_M(X \upharpoonright n) > n - h(n) - O(1)$. A set X is Kurtz random if and only if, for every decidable prefix-free machine M and every computable order h, $K_M(X \upharpoonright n) > n - h(n) - O(1)$ for infinitely many n.

Definition 2.3. We say that $A \le_{dm} B$ if, for every decidable prefix-free machine M, there exists a decidable prefix-free machine N such that $K_N(A \upharpoonright n) \le K_M(B \upharpoonright n) + O(1)$.

This is a decidable-prefix-free-machine version of Schnorr reducibility of Definition 2.7. A similar notion can be found in [19, Definition 3.1].

An immediate corollary is that the reducibility $\le_{dm}$ is coherent with ML-randomness, Schnorr randomness, Kurtz randomness. Furthermore, we also have the following.

Theorem 2.4. *The reducibility $\le_{dm}$ is coherent with 2-randomness.*

Proof. We use a result from [3] that in fact has studied a similar topic. A *computable upper bound of* K is a total computable function $\hat{K} : 2^{<\omega} \to \mathbb{N} \cup \{+\infty\}$ such that $K(\sigma) \le \hat{K}(\sigma) + O(1)$ for all $\sigma \in 2^{<\omega}$. Let $\mathcal{K}$ be the class of computable upper bounds of K. (This terminology is from [3, Definition 2.3.1].) Then, there exists $K^* \in \mathcal{K}$ such that, $X \in 2^\omega$ is 2-random if and only if $(\forall \hat{K} \in \mathcal{K})\hat{K}(X \upharpoonright n) \ge n + K^*(n) - O(1)$ for infinitely many n ([3, Theorem 2.3.24]).

Although computable upper bounds of K are not exactly the same as complexities with respect to decidable prefix-free machines, they behave similarly for random sets in the following sense. For a decidable prefix-free machine M such that $K_M(\sigma) < \infty$ for every $\sigma \in 2^{<\omega}$ (that is, M is surjective as a function), $K_M(\sigma)$ is a computable upper bound because $K(\sigma) \le K_M(\sigma) + O(1)$ by the minimality of K and $\sigma \mapsto K_M(\sigma)$ is computable. Conversely, let $\tilde{K}$ be a computable upper bound of K and X be a ML-random set (actually the effective Hausdorff dimension $\dim(X) > 1$ is sufficient). Bienvenu [3, Proposition 2.3.15] showed that, for every $\hat{K} \in \mathcal{K}$ and every computable order h, there exists a surjective prefix-free decidable machine M such that $K_M(\sigma) \le \max(h(|\sigma|), \hat{K}(\sigma)) + O(1)$. By considering the function $h(n) = o(n)$, we have

$$K_M(X \upharpoonright n) \le \max(h(n), \tilde{K}(X \upharpoonright n)) + O(1) = \tilde{K}(X \upharpoonright n) + O(1).$$

Suppose that $A \le_{dm} B$ and B is not 2-random. If B is not ML-random, then A is not ML-random because $\le_{dm}$ is coherent with ML-random, and thus A is not 2-random. Then, we can assume that B is ML-random. Then, there exists a computable upper bound $\hat{K}$ such that

$$\lim_n (n + K^*(n) - \hat{K}(B \upharpoonright n)) = \infty.$$

Since B is ML-random, there exists a prefix-free decidable machine M such that

$$K_M(B \upharpoonright n) \leq \hat{K}(B \upharpoonright n) + O(1).$$

By $A \leq_{dm} B$, there exists a decidable prefix-free machine N such that

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + O(1).$$

Combined with these, we have

$$\lim_n (n + K^*(n) - K_N(A \upharpoonright n)) = \infty.$$

Since K_N is a computable upper bound, this implies that A is not 2-random. □

2.4. **Total-machine reducibility.** Next, we see coherence of total-machine reducibility. The total-machine reducibility is defined in Miyabe [20] as a Schnorr-randomness version of C.

Definition 2.5. We say that $A \leq_{tm} B$ if, for every total machine M, there exists a total machine N such that $C_N(A \upharpoonright n) \leq C_M(B \upharpoonright n) + O(1)$.

Bienvenu and Merkle [6] showed that a set X is Kurtz random if and only if, for every total machine M and every computable order h, we have $C_M(X \upharpoonright n) > n - h(n) - O(1)$ for infinitely many n. Then, the reducibility $\leq_{tm}$ is coherent with Kurtz randomness.

Next, we see coherence with 2-randomness. As a slight extension, we have a characterization of 2-randomness by the time-bounded complexity $C^g(\sigma) = \min\{|\tau| : U(\tau) = \sigma \text{ in } g(|\sigma|) \text{ steps}\}$ where U is the fixed universal machine. Notice that $\sigma \mapsto C^g(\sigma)$ is computable if g is computable. There exists a computable order g such that X is 2-random if and only if $C^g(X \upharpoonright n) > n - O(1)$ for infinitely many n (see [21, before Lemma 3.6.14]). Since every time-bounded plain machine is a decidable machine and can be seen as a total machine, we have the following: X is 2-random if and only if, for every total machine M, $C_M(X \upharpoonright n) > n - O(1)$ for infinitely many n. Thus, the reducibility $\leq_{tm}$ is coherent with 2-randomness.

Next we see coherence with Schnorr randomness. Miyabe [20, Theorem 3] showed that the following are equivalent for a set $X \in 2^\omega$:

(i) X is Schnorr random.
(ii) For every total machine N and every computable order g, we have $C_N(X \upharpoonright g(n)) \geq g(n) - n - O(1)$.
(iii) For every computable measure machine M and every total machine N, we have $C_N(X \upharpoonright n) \geq n - K_M(n) - O(1)$.

Then, the reducibility $\leq_{tm}$ is coherent with Schnorr randomness.

Finally, we see coherence with ML-randomness by giving a characterization of ML-randomness via total machines.

Theorem 2.6. *The following are equivalent:*

(i) *A set X is ML-random.*
(ii) *For every total machine M, we have $C_M(X \upharpoonright n) > n - K(n) - O(1)$.*

Furthermore, there exists a total machine L such that $C_L(X \upharpoonright n) > n - G(n) - O(1)$ is equivalent to (i) where G is defined in (1).

Proof. The direction (i)⇒(ii) follows from the characterization of ML-randomness by C. Since $K(n) \le G(n)$ for all $n \in \omega$, it suffices to construct a total machine L such that $C_L(X \restriction n) > n - G(n) - O(1)$ implies (i).

We generally follow the argument of the characterization of ML-randomness by C (Theorem 6.7.2 in [11]), so we refer there for some details.

The goal is to construct a decidable machine L such that, if X is not ML-random, then the complexity $C_L(X \restriction n)$ for some n is largely below $n - G(n)$. Since $G(n)$ can be small only when $n = 2^{\langle s,t\rangle}$, we construct L that compress strings with such length n when $X \restriction t$ is compressible. In Miller-Yu's proof, the condition of compressibility is $K(\sigma) \le t - k$, which is not a decidable relation. In our case we use the decidable relation of $K_s(\sigma) \le t - k$.

At first we pick up compressible strings. By the Counting Theorem, there exists a constant $c \in \omega$ such that for all t and k we have

$$|\{\sigma \in 2^t \ : \ K(\sigma) \le t - k\}| \le 2^{t-K(t)-k+c}.$$

A rough idea of the construction of L is as follows. The machine L produces the strings with length $n = 2^{\langle s,t\rangle}$ that are extensions of compressible strings with length t. The length of its input string is between $\frac{n}{2}+c+1$ and $n+c$ both included. Furthermore, if we want to compress $\rho \in 2^n$ with length k, the output string ρ will be produced by an input string with length $n - K_{s+1}(t) - k + c$.

We give the construction of L in detail. We do the following construction for every $s, t, k \in \omega$. Let $n = 2^{\langle s,t\rangle}$ and $m = n - K_{s+1}(t) - k + c$. The value n will be the length of the output strings σ and m will be the length of the input strings. If $m \ge \frac{n}{2} + c + 1$, then for each $\sigma \in 2^n$ such that $K_{s+1}(\sigma \restriction t) \le t - k$, we try to pick a string $\tau \in 2^m$ and define $L(\tau) = \sigma$. Notice that L can be decidable.

We claim that if $K_{s+1}(t) = K(t)$, then there are enough strings of length m for all such strings σ. This is because the number of $\sigma \in 2^n$ for which $K_{s+1}(\sigma \restriction t) \le t-k$ is at most

$$2^{n-t}|\{\rho \in 2^t \ : \ K(\rho) \le t - k\}| \le 2^{n-t}2^{t-K(t)-k+c} = 2^{n-K(t)-k+c} = 2^m.$$

Suppose that X is not ML-random. Then, for every k, there exists t such that $K(X \restriction t) \le t - k$ and $K(t) \le 2^{t-1} - k - 1$. Let s be the least such that $K_{s+1}(t) = K(t)$ and $K_{s+1}(X \restriction t) \le t - k$. Let $n = 2^{\langle s,t\rangle}$ and $m = n - K_{s+1}(t) - k + c$. Then,

$$m \ge n - (2^{t-1} - k - 1) - k + c \ge n - 2^{\langle s,t\rangle -1} + c + 1 \ge \frac{n}{2} + c + 1.$$

Thus, $C_L(X \restriction n) \le m$. Since $G(n) = K_{s+1}(t) = K(t)$, we have

$$C_L(X \restriction n) \le n - K(t) - k + c = n - G(n) - k + c.$$

Here, k is arbitrary, (iv) does not hold. □

As a corollary, the reducibility $\le_{tm}$ is coherent with ML-randomness.

2.5. **Schnorr reducibility.** Schnorr reducibility is induced by computable measure machines, which characterize Schnorr randomness.

A *computable measure machine* is a prefix-free machine M such that its measure $\mu([\![\mathrm{dom}(M)]\!])$ is computable. Downey and Griffiths [10] showed that a set X is Schnorr random if and only if $K_M(X \restriction n) > n - O(1)$ for every computable measure machine. Note that every computable measure machine is a decidable prefix-free machine and compare the following with the characterization by decidable machines.

Definition 2.7 (Downey and Griffiths [10])**.** We say that A is *Schnorr reducible* to B (denoted by $A \le_{Sch} B$) if, for every computable measure machine M, there exists a computable measure machine N such that $K_N(A \restriction n) \le K_M(B \restriction n) + O(1)$.

Schnorr reducibility is coherent with Schnorr randomness.

We also have a characterization of Kurtz randomness via computable measure machines by Downey, Griffiths, and Reid [12], from which coherence with Kurtz randomness follows. A set A is not Kurtz random if and only if, there is a computable measure machine M and a computable order f such that, for all n, we have $K_M(A \restriction f(n)) < f(n) - n$.

Proposition 2.8. *Schnorr reducibility is coherent with Kurtz randomness.*

Proof. Let $A, B \in 2^\omega$ be sets such that $A \leq_{Sch} B$ and B is not Kurtz random. Then, there is a computable measure machine M and a computable order f such that, for all n, we have $K_M(B \restriction f(n)) < f(n) - n$. By $A \leq_{Sch} B$, for this machine M, there exists a computable measure machine N such that $K_N(A \restriction m) \leq K_M(B \restriction m) + c$. Hence, $K_N(A \restriction f(n)) < f(n) - n + c$ for every n. Finally, let $g(n) = f(n + c)$ for every n. Then,

$$K_N(A \restriction g(n)) = K_N(A \restriction f(n + c)) < f(n + c) - (n + c) + c = g(n) - n$$

for every n. Hence, A is not Kurtz random. □

3. Incoherence of Schnorr reducibility with randomness notions

First we give a characterization of 2-randomness via computable measure machines, wishing that it might imply coherence of Schnorr reducibility with 2-randomness. The key in the proof of the characterization is an extension of the Counting theorem. Thereafter, we show the incoherence of Schnorr reducibility with stronger randomness notions.

3.1. **Extended Counting Theorem.** The Counting Theorem (see [11, Theorem 3.7.6]) says that

$$|\{\sigma \ : \ |\sigma| = n \wedge K(\sigma) \leq n + K(n) - r\}| \leq 2^{n-r+O(1)}$$

where the constant does not depend on n and r. This is a basic tool in the theory of algorithmic randomness, and its applications are so widespread that people often do not refer the Counting theorem in a logical argument. To extend the theorem for machines that may not be universal, we start from the observation of the connection to the Coding theorem.

For a prefix-free machine M, let $Q_M(\sigma) = \mu([\![\{\tau \ : \ M(\tau) \downarrow = \sigma\}]\!])$. We write $Q(\sigma)$ for $Q_U(\sigma)$ where U is the fixed universal prefix-free machine. For details see [11, Definition 3.9.3]. The Coding theorem (see [11, Theorem 3.9.4]) says that

$$K(\sigma) = -\log Q(\sigma) \pm O(1).$$

According to [11, Section 9.4], "One informal interpretation of the Coding Theorem is that if a string has many long descriptions then it also has a short description." The value $Q(\sigma)$ is, roughly speaking, the probability that M produces the string σ. We consider the probability that M produces the strings with length n.

Definition 3.1. Let

$$R_M(n) = -\log \mu([\![\{\tau \ : \ |M(\tau) \downarrow | = n\}]\!]).$$

For convenience, $R_M(n) = \infty$ if $\mu([\![\{\tau \ : \ |M(\tau) \downarrow | = n\}]\!])$=0.

Precisely speaking, $R_M(n)$ is a real number, but for convenience we sometimes see $R_M(n)$ as a natural number.

Proposition 3.2. *For a universal prefix-free machine U, we have*

$$K(n) = R_U(n) \pm O(1).$$

Proof. Notice that $R_M(n)$ is an information content measure ([11, Definition 3.7.7]) for every machine M. This is because

$$\sum_n 2^{-R_M(n)} = \sum_n \mu([\![\{\tau \ : \ |M(\tau)\downarrow| = n\}]\!]) = [\![\mathrm{dom}(M)]\!] \leq 1.$$

Furthermore, the relation $|M(\tau)\downarrow| = n$ is c.e., and so is $R_M(n) \leq k$. By the minimality of K, we have $K(n) \leq R_U(n) + O(1)$ ([11, Theorem 3.7.8]).

The converse means, loosely speaking, if $K(0^n)$ is small, then $R_U(n)$ is small, which means that there are many or short strings σ producing strings with length n. Let σ_n be one of the shortest strings such that $U(\sigma_n) = 0^n$. Then,

$$\sigma_n \in \{\tau \ : \ |U(\tau)\downarrow| = n\},$$

and

$$\mu([\![\{\tau \ : \ |U(\tau)\downarrow| = n\}) \geq 2^{-K(0^n)}.$$

Hence,

$$R_U(n) \leq K(0^n) \leq K(n) + O(1).$$

□

With this observation, we rethink the Counting theorem.

Theorem 3.3 (Extended Counting theorem)**.**

$$|\{\sigma \ : \ |\sigma| = n \wedge K_M(\sigma) \leq n + R_M(n) - r\}| \leq 2^{n-r}.$$

Notice that $K(n) = K_U(n) = R_U(n) \pm O(1)$ and this is an extension.

Proof. Let

$$S_{n,r} = \{\sigma \ : \ |\sigma| = n \wedge K_M(\sigma) \leq n + R_M(n) - r\}.$$

For each $\sigma \in S_{n,r}$, let σ^* be one of the shortest strings such that M produces σ. Then, σ^* for $\sigma \in S_{n,r}$ contributes in the enumeration of $\mu([\![\{\tau \ : \ |M(\tau)\downarrow| = n\}]\!])$. Thus,

$$2^{-R_M(n)} = \mu([\![\{\tau \ : \ |M(\tau)\downarrow| = n\}]\!]) \geq 2^{-(n+R_M(n)-r)}|S_{n,r}|,$$

and $|S_{n,r}| \leq 2^{n-r}$. □

3.2. **2-randomness via c.m.m.** Using the extended Counting Theorem we provide a characterization of 2-randomness via computable measure machines.

Theorem 3.4. *A sequence $X \in 2^\omega$ is 2-random iff, for every computable measure machine M,*

$$K_M(X \upharpoonright n) \geq n + R_M(n) - O(1)$$

for infinitely many n.

In the following, we use the following fact repeatedly: For a sequence $\{a_n\}_{n\in\omega}$ of uniformly computable positive reals such that $a_n = O(2^{-n})$, the sum $\sum_n a_n$ is a computable real. For each $k \in \omega$, $\sum_{n\leq k} a_n$ is computable because this is a finite sum of computable reals, and the error $\sum_{n>k} a_n$ is bounded by $O(2^{-k})$.

Proof of "if" direction. The key idea in the proof is that $R_M(n)$ need not be small. In fact we construct a computable measure machine M such that $R_M(n) \geq n - O(1)$ for every n. This is quite large considering Proposition 3.2.

We start from an oracle universal ML-test $\{U^X_k\}$. First we construct an approximation of U^X_k. For each k, X, we consider a computable sequence $V^X_{k,s}$ of finite prefix-free sets of strings with length s that generates an approximation of U^X_k, that is,

$$\bigcup_{s\in\omega}\{[\sigma] \ : \ \sigma \in V^X_{k,s}\} = U^X_k.$$

We can assume that, if $\sigma \in V_{k,s}$, then $\sigma 0, \sigma 1 \in V_{k,s+1}$, that is, the left-hand side is increasing. Notice that the relation $\sigma \in V_{k,s}^X$ is a decidable relation if X is computable.

We fix $\{Z_s\}$ of an approximation of $\emptyset'$, that is, $\{Z_s\}$ is a computable sequence of finite subsets of ω such that $Z_s \uparrow \emptyset'$. Then, if X is not 2-random, then for every k some initial segment of X is in $V_{k,s}^{Z_s}$ for every sufficiently large s.

We construct a computable measure machine M_s for each $s \geq 1$ from $\{V_{2k,s}^{Z_s}\}_{k\in\omega}$ by the KC-theorem. We request every $\sigma \in V_{2k,s}^{Z_s}$ for every $k \geq 1$ with cost $2^{-(2|\sigma|-k)}$. The cost for each k is

$$\sum_{\sigma \in V_{2k,s}^{Z_s}} 2^{-(2|\sigma|-k)} \leq \sum_{\sigma \in V_{2k,s}^{Z_s}} 2^{-|\sigma|-s+k} \leq 2^{-s-k},$$

and is a computable real because $V_{2k,s}^{Z_s}$ is a computable finite set. Thus, the total cost is

$$\sum_{k\geq 1} \sum_{\sigma \in V_{2k,s}^{Z_s}} 2^{-(2|\sigma|-k)} \leq \sum_{k\geq 1}\sum_{n\geq 1} 2^{-s-k} = 2^{-s},$$

and is also a computable real. Since $|\sigma| = s$ for every $\sigma \in V_{2k,s}^{Z_s}$ and every k, we have $|M_s(\tau)| = s$ for every $\tau \in \mathrm{dom}(M_s)$. Thus, the measure of M_s is equal to $2^{-R_{M_s}(s)}$ and $R_{M_s}(t) = \infty$ for every $t \neq s$.

Next we define another computable measure machine M. For $\sigma \in 2^{<\omega}$ let $\widehat{\sigma}$ denote the piece-wise iterated string of σ. For instance $\widehat{101} = 110011$ and $\widehat{011} = 001111$. We define M by

$$M(\widehat{\sigma}01\tau) = M_{|\sigma|}(\tau)\sigma.$$

The machine M is clearly prefix-free.

Before we prove that the measure of M is computable, we evaluate $R_M(n)$. First notice that $R_M(n) = \infty$ if n is odd because $|M_{|\sigma|}(\tau)| = |\sigma|$. For even n, we have

$$2^{-R_M(n)} = \sum_{\sigma\in 2^{n/2}} \sum_{|M_{n/2}(\tau)|=n/2} 2^{-(n+|\tau|+2)} \leq \sum_{\sigma \in 2^{n/2}} 2^{-n-2}\cdot 2^{-n/2} = 2^{-n-2}.$$

Hence, $R_M(n) \geq n+2$. Since $2^{-R_M(n)}$ is a computable real, M is a computable measure machine.

Finally, suppose that X is not 2-random. Then, for every k, we have $X \in U_{2k}^{\emptyset'}$. Thus, $X \upharpoonright s \in V_{2k,s}^{\emptyset'}$ for all sufficiently large s. Since the use of the oracle is bounded, $X \upharpoonright s \in V_{2k,s}^{Z_s}$ for all sufficiently large s. For such s, we have

$$K_M(X \upharpoonright 2s) \leq 2s + 2 + 2s - k = 2s + R_M(2s) - k.$$

We also have the inequality for odd n because $R_M(n) = \infty$ for odd n. □

For the other direction, we follow the argument of Theorem 8 in [8] (Theorem 11 in [7]) or Theorem 1 in [2], and combine with the extended counting theorem.

Lemma 3.5 (Conidis [9])**.** *Let $\epsilon > 0$ be a rational number and let $U_0, U_1, \cdots$ be a sequence of uniformly c.e. open sets of measure at most ϵ each. Then for every rational $\epsilon' > \epsilon$ there exists a $\mathbf{0}'$-c.e. open set V of measure at most ϵ' that contains $\liminf_{n\to\infty} U_n$. Furthermore, the $\mathbf{0}'$-enumeration algorithm for V can be effectively given in ϵ, ϵ' and U_i.*

Proof of "only if" direction. Suppose that, for all c, we have $K_M(X \upharpoonright n) < n + R_M(n) - c$ for all sufficiently large n. Let

$$U_n = \{\sigma \in 2^n \ : \ K_M(\sigma) < n + R_M(n) - (c + d + 1)\}$$

where d is a witness constant for the extended counting theorem for M. Since $R_M(n)$ is computable, so is U_n. By the extended counting theorem, The measure of $[\![U_n]\!]$ is at most

$$2^{-n} \cdot 2^{n-(c+d+1)+d} = 2^{-c-1}$$

for all n. By Lemma 3.5, there exists $\mathbf{0}'$-c.e. open set V_c of measure at most 2^{-c} that contains $\liminf_{n\to\infty} U_n$, which contains X. Since the construction is uniform and c is arbitrary, the sequence $\{V_c\}$ is a $\mathbf{0}'$-ML-test that covers X. Thus, X is not 2-random. □

3.3. **Incoherence of Schnorr reducibility.** We have seen a characterization of 2-randomness by computable measure machines. However, this does not imply coherence of Schnorr reducibility with 2-randomness, mainly because the right hand side $n + R_M(n)$ in the characterization depends on M. In fact, Schnorr reducibility is not coherent with computable randomness, nor with any randomness notions stronger than computable randomness.

Recall that a set $X \in 2^\omega$ is called computably random if $\sup_n M(X \upharpoonright n) < \infty$ for every computable martingale M, and a set $X \in 2^\omega$ is Schnorr random if and only if, for every computable martingale M and every computable order f, we have $M(X \upharpoonright f(n)) < n$ for almost all n ([14]). Thus, the difference between Schnorr randomness and computable randomness is the rate of divergence of capitals for a computable martingale.

Theorem 3.6. *For every set $A \in 2^\omega$, there exists $B \in 2^\omega$ such that $A \leq_{Sch} B$ and B is not computably random.*

The goal is, for a given set $A \in 2^\omega$, to construct a set $B \in 2^\omega$ such that $A \leq_{Sch} B$ but B is not computably random. The set A may be Schnorr random and B may be Schnorr random. In fact we use the method of constructing a Schnorr random set that is not computably random in each high degree in [22].

Loosely speaking, the set B is almost the same as A but we force $B(n) = 0$ in some positions. The forced positions are too sparse for Schnorr reducibility to distinguish A and B. The number of candidates of the forced positions is so limited that some computable martingale succeeds on B.

Proof. We define functions ψ, p and h. The image of h will be the positions where $B(n) = 0$ are forced. The value $\log p(x) - 1$ is the number of candidates of the forced positions.

We define ψ by

$$\psi(e,x) = \begin{cases} \langle e,x,s\rangle + 1 & \text{where } s \text{ is the least such that } \Phi_e(x)[s]\downarrow, \\ \uparrow & \text{if } \Phi_e(x)\uparrow. \end{cases}$$

Notice that ψ is one-to-one where defined, $\psi(e,x) > x$ for every $x \in \omega$, and the relation $n \in \mathrm{rng}\psi$ is decidable. Furthermore, the numbers e, x such that $\psi(e,x) = n$ are computable from n.

We define a computable function $p : \omega \to \omega$ by

$$p(n) = \begin{cases} p(x) + 1 & \text{if } \exists x < n, \exists e < \log p(x) - 1,\ \psi(e,x) = n, \\ n + 4 & \text{otherwise.} \end{cases}$$

Notice that $p(n)$ is well-defined. The value $p(n)$ will be defined if $p(x)$ is defined. From given n, the numbers e, x are fixed and computable from n if n is in the range of ψ. Since $x < \psi(e,x) = n$, the function p is defined inductively. We also note that $\lim_n p(n) = \infty$.

We define a set H_x for each $x \in \omega$ by

$$H_x = \{\psi(e,x) \ :\ \psi(e,x)\downarrow,\ e < \log p(x) - 1\}.$$

The set H_x is the candidates of forcing positions at stage x. Notice that $|H_x| \leq p(x) - 1$ and the number of the candidates is bounded by $p(x) - 1$. We assume Φ_0 is total and thus H_x is not empty for each $x \in \omega$. We also note that the H_x are pairwise disjoint.

Let $h(x) = \max(H_x)$. We claim that h dominates all computable functions, that is, for every computable function f, we have $f(x) \leq h(x)$ for all sufficiently large x. (We do not claim that h is monotone.) Let f be a computable function with an index e. Then,

$$f(x) = \Phi_e(x) < \psi(e, x).$$

Let x_0 be such that $e < \log p(x) - 1$ for all $x > x_0$. Such x_0 exists because $\lim_{n\to\infty} p(n) = \infty$. For all $x > x_0$, we have

$$\psi(e, x) \leq \max(H_x) = h(x).$$

Since f is arbitrary, h dominates all computable functions.

Given a set $A \in 2^\omega$, we define $B \in 2^\omega$ by

$$B(n) = \begin{cases} 0 & \text{if there exists } x < n \text{ such that } h(x) = n, \\ A(n) & \text{otherwise.} \end{cases}$$

We claim that B is not computably random. We construct a computable martingale M that succeeds on B. The martingale M uses the so-called martingale strategy S at each stage. The martingale strategy S with initial capital c bets the capital $2^k c$ that the next bit is 0 until S wins for the first time where k is the number of loss.

At stage n, M uses the strategy S at H_n with the initial capital $\frac{1}{p(n)}$. Since $|H_n| \leq \log p(n) - 1$, the number of failure is at most $\log p(n) - 1$ and the amount of the decreased capital is at most

$$\sum_{i=1}^{\log p(n)-1} \frac{2^{i-1}}{p(n)} = \frac{2^{\log p(n)} - 1}{p(n)} = \frac{p(n) - 1}{p(n)} < 1.$$

Hence, the capital is never negative. Since $\Phi(X)(h(n)) = 0$ for every n, the strategy S increases his capital with

$$\frac{2^k}{p(n)} - \sum_{i=1}^{k-1} \frac{2^{i-1}}{p(n)} = \frac{2^k - (2^k - 1)}{p(n)} = \frac{1}{p(n)}$$

where S wins at the k-th time.

The strategy starts at stage 0 with the initial capital $\frac{1}{p(0)}$. At stage n, if S increases his capital at $\psi(e, n) \in H_n$, then the stage n is over. The next stage number is $\psi(e, n)$. Notice that $\psi(e, n) > n$ for every $\psi(e, n) \in H_n$. Then, the initial capital at the next stage $\psi(e, n)$ is $\frac{1}{p(\psi(e,n))} = \frac{1}{p(n)+1}$. By $p(0) = 4$, the sup of M is

$$1 + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots = \infty.$$

Finally, we claim that $A \leq_{Sch} B$. We denote by $X \leq_{wdm} Y$ that, for each decidable prefix-free machine M and a computable order g there exists a decidable prefix-free machine N such that $K_N(X \upharpoonright n) \leq K_M(Y \upharpoonright n) + g(n) + O(1)$. Miyabe [19, Theorem 3.5] showed that, $X \leq_{Sch} Y \iff X \leq_{wdm} Y$ for all X, Y. Then, it suffices to show $A \leq_{wdm} B$.

Now fix a prefix-free decidable machine M and a computable order g. The goal is to construct a decidable prefix-free machine N such that $K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + g(n) + O(1)$. The sets $A \upharpoonright n$ and $B \upharpoonright n$ are the same except the positions

$$G(n) = \{h(x) \ : \ \exists x < n, \ h(x) \leq n\}.$$

Thus, it suffices to encode $G(n)$ into strings less than $g(n)+O(1)$ in length. Since h grows fast, $|G(n)|$ grows slower than any computable order. It seems possible but the problem is that $G(n)$ is not computable. Thus, we construct a computable superset $\overline{G}(n) \supseteq G(n)$ such that $|\overline{G}(n)|$ is sufficiently small, and encode all subsets of $G(n)$.

First we construct an approximation $G(n)[t]$ of $G(n)$. Let

$$H_x[t] = \{\psi(e,x) \ : \ \Phi_e(x)[t]\downarrow, \ e < \log p(x) - 1\}.$$

Then, for each x, we have $\bigcup_t H_x[t] =$. Let

$$h(x)[t] = \max(H_x[t]).$$

Then, for each x, $\{h(x)[t]\}_{t\in\omega}$ is non-decreasing. Let

$$G(n)[t] = \{h(x)[t+n] \ : \ \exists x < n, \ h(x)[t+n] \le n\}.$$

We claim that $G(n)[t] \supseteq G(n)$ for every $t \in \omega$. Suppose that $h(x) \in G(n)$ for some $x < n$. Then, there exists $e < \log p(x) - 1$ such that $h(x) = \psi(e,x) = \langle e,x,s\rangle + 1 \le n$ where s is the least such that $\Phi_e(x)[s]\downarrow$. In particular we have $s < n$. Thus, $h(x) \in H_x[t+n]$. Since $\max(H_x[t+n]) \le \max(H_x) = h(x)$, we have $h(x) = h(x)[t+n] \in G(n)[t]$.

Furthermore, we have $G(n)[t] = G(n)$ for sufficiently large t. This is because, for sufficiently large t, we have $H_x[t] = H_x$, $h(x)[t] = h(x)$ and $G(n)[t] = G(n)$.

Now we define a finite set $\overline{G}(n)$ of ω as a sufficiently good approximation of $G(n)$. Let $c \in \omega$ be a constant such that

$$|G(n)| \le g(n) + c$$

for every n. Then, there exists a stage $s \in \omega$ such that

$$|G(n)[s]| \le g(n) + c.$$

Pick such a stage t and define $\overline{G}(n)$ by

$$\overline{G}(n) = G(n)[t]$$

for this t. Notice that $\overline{G}(n)$ is computable uniformly in n.

Next we construct a prefix-free decidable machine N. We define $E(\sigma,\tau) \in 2^{|\sigma|}$ for $|\tau| = |\overline{G}(|\sigma|)$ by

$$E(\sigma,\tau)(n) = \begin{cases} \tau(k) & \text{if } n \text{ is the } k\text{-th minimal element in } \overline{G}(|\sigma|) \\ \sigma(n) & \text{otherwise.} \end{cases}$$

Thus, $E(\sigma,\tau)$ is almost the same as σ except the positions of $\overline{G}(|\sigma|)$. We define N by

$$N(\sigma\tau) = E(M(\sigma),\tau) \text{ if } |\tau| = |\overline{G}(|M(\sigma)|)|.$$

Since M is a prefix-free decidable machine, so is N.

Finally, we claim that $K_N(A \upharpoonright n) \le K_M(B \upharpoonright n) + g(n) + O(1)$. For every n, let σ be a shortest string such that $M(\sigma) = B \upharpoonright n$. Since A and B are different only at $G(n) \subseteq \overline{G}(n)$, there exists a string τ such that $|\tau| = |\overline{G(n)}|$ and $E(B \upharpoonright n, \tau) = A \upharpoonright n$. Then, $N(\sigma\tau) = A \upharpoonright n$. Hence,

$$K_N(A \upharpoonright n) \le |\sigma| + |\tau| = K_N(B \upharpoonright n) + |\overline{G(n)}| \le K_N(B \upharpoonright n) + g(n) + c.$$

Hence, we have $A \le_{wdm} B$. □

Thus, Schnorr reducibility is not coherent with ML-randomness or 2-randomness.

4. Coherence with n-randomness

In this section we study coherence with n-randomness. Note that, by Theorem 3.6, Schnorr reducibility is not coherent with n-randomness for every $n \geq 1$. We will see that, except Schnorr reducibility, most reducibilities are coherent with n-randomness.

4.1. **Implication of vL-reducibility.** We show that some reducibilities are coherent with n-randomness by showing that each reducibility implies vL-reducibility.

First, we see immediate corollaries from known results. We say that X is vL-reducible to Y, denoted by $X \leq_{vL} Y$, if for all Z, if $X \oplus Z$ is ML-random, then $Y \oplus Z$ is ML-random. One interesting basic property of vL-reducibility is that, as is shown in [18], if $X \leq_{vL} Y$ and X is n-random, then Y is n-random. This can be checked easily by letting Z be a ML-random set Turing equivalent to $0^{(n-1)}$. If X is n-random, then Z is ML-random and X is ML-random relative to Z. Thus $X \oplus Z$ is ML-random by van Lambalgen's theorem. Hence $Y \oplus Z$ is ML-random and Y is n-random by van Lambalgen's theorem again.

In what follows, a string σ is identified with the natural number k such that σ is the k-th string in lexicographical order.

Theorem 4.1 ([18]). *$X \oplus Z$ is ML-random if and only if*

$$K(X \upharpoonright (Z \upharpoonright n)) \geq (Z \upharpoonright n) + n - O(1).$$

Thus, K-reducibility implies vL-reducibility, and $\leq_K$ is coherent with n-randomness for every $n \geq 2$.

Miller and Yu [18] also showed that, for a ML-random set Z, X is Z-ML-random if and only if

$$C(X \upharpoonright n) + K(Z \upharpoonright n) \geq 2n - O(1).$$

Thus, C-reducibility implies vL-reducibility, and $\leq_C$ is coherent with n-randomness for every $n \geq 2$.

To see that $\leq_{dm}$ is coherent with n-randomness, we give a characterization of ML-randomness by prefix-free decidable machines.

Theorem 4.2. *A set $X \oplus Z$ is ML-random if and only if, for every prefix-free decidable machine, we have $K_M(X \upharpoonright (Z \upharpoonright n)) > (Z \upharpoonright n) + n - O(1)$.*

Proof. The "only if" direction follows from Theorem 4.1.

For the other direction, the proof generally goes along with the one of Theorem 10 in [20]. The only difference is that we need to use a prefix-free decidable machine whose existence was claimed in the characterization of ML-randomness by prefix-free decidable machines. □

As a corollary, we have $\leq_{dm} \Rightarrow \leq_{vL}$. In particular, for every $n \geq 2$, $\leq_{dm}$ is coherent with n-randomness.

The following is the tm-reducibility version.

Theorem 4.3. *Let Z be a ML-random set. Then, the following are equivalent.*

(i) *X is ML-random relative to Z.*
(ii) *$C_N(X \upharpoonright n) + K(Z \upharpoonright n) \geq 2n - O(1)$ for every total machine N.*

Proof. The direction (i)⇒(ii) is immediate from the case of C.

For the other direction, first note that there exists a total machine L such that

$$C_L(X \upharpoonright (X \upharpoonright n)) \leq (X \upharpoonright n) - n + 1$$

for every $X \in 2^\omega$ (Lemma 9 in [20]). Then,

$$\begin{aligned} K(Z \restriction (X \restriction n)) \geq& 2(X \restriction n) - C_L(X \restriction n) - O(1) \\ \geq& 2(X \restriction n) - (X \restriction n) + n - O(1) = (X \restriction n) + n - O(1). \end{aligned}$$

Thus, $X \oplus Z$ is ML-random. □

Hence, we have $\leq_{tm} \Rightarrow \leq_{vL}$. In particular, for every $n \geq 2$, $\leq_{tm}$ is coherent with n-randomness.

4.2. **Characterizations of 2-randomness.** We have already seen that Schnorr reducibility is not coherent with 2-randomness. The following results were found when the author was trying to show coherence of Schnorr reducibility with 2-randomness.

Theorem 4.4. *The following are equivalent:*

(i) *X is 2-Z-random.*
(ii) *$C(X \restriction (Z \restriction n)) > Z \restriction n - O(1)$ for infinitely many n.*
(iii) *$C(X \restriction (Z \restriction n) \mid (Z \restriction n)) > Z \restriction n - O(1)$ for infinitely many n.*

Proof. (i)⇒(ii). Suppose that $\liminf_{n\to\infty}(C(X \restriction (Z \restriction n)) - Z \restriction n) = -\infty$. Fix k. Let $U_n = \bigcup\{[\sigma] \ : \ |\sigma| = Z \restriction n \text{ and } C(\sigma) < |\sigma| - k\}$. Then, U_n is uniformly Z-c.e. open. Furthermore, the number of σ such that $C(\sigma) < |\sigma| - k$ is at most

$$1 + 2 + 2^2 + \cdots + 2^{|\sigma|-k-1} = 2^{|\sigma|-k} - 1,$$

and thus $\mu(U_n) \leq 2^{-k}$. Thus, by a relativized version of Conidis' result (Lemma 3.5), we can construct a Z'-c.e. open set V such that $\liminf_{n\to\infty} U_n \subseteq V$ and $\mu(V) \leq 2^{-k+1}$. Since the construction is uniform in k and X is covered by $\liminf_{n\to\infty} U_n$, X is not 2-Z-random.

(ii)⇒(iii). The argument is the same as Claim 2.5.1 in Li and Vitányi [15]. See also Theorem 6.11.2 in Downey and Hirschfeldt [11].

(iii)⇒(i). Suppose that X is not 2-Z-random. Then, there exists a Z'-ML-test $\{U_d\}$ that covers X. Let Φ be a Turing functional such that $\{\Phi^{Z'}(d,k)\}_k$ is a set of strings that generates U_d.

We define a conditional machine M as follows. Let σ be the conditional string and $\alpha = (\sigma 0^\omega)'[\sigma]$. At stage $s = \langle d, k \rangle$, if the length of $\Phi^\alpha(d,k)$ is less than σ, then declare $M(\tau|\sigma) = \Phi^\sigma(d,k) \restriction \sigma$ where τ is the lexicographically minimal undeclared string with length $\sigma - d$, if such τ exists.

We claim that, for each $d \in \omega$, there exists n such that $C_M(X \restriction (Z \restriction n)|(Z \restriction n)) \leq (Z \restriction n) - d$. Fix d and let k_0 be such that $\Phi^{Z'}(d,k_0) \prec X$ where $\prec$ is the prefix relation. When the conditional string σ is $Z \restriction n$ and n is large enough, we have $\Phi^\alpha(d,k) = \Phi^{Z'}(d,k)$ for every $k \leq k_0$ and the length $\Phi^\alpha(d,k_0)$ is less than σ. Furthermore, since $\mu(U_d) \leq 2^{-d}$, the number of $\Phi^{Z'}(d,k)$ whose length is less than σ is at most $2^{\sigma-d}$. At stage $s_0 = \langle d, k_0 \rangle$, we have $\Phi^\alpha(d,k) = \Phi^{Z'}(d,k)$ for every $k \leq k_0$ and the number of declared strings is at most $2^{\sigma-d}$. Thus, there exists $\tau \in 2^{\sigma-d}$ such that

$$M(\tau|\sigma) = \Phi^\alpha(d,k_0) \restriction \sigma = \Phi^{Z'}(d,k_0) \restriction \sigma = X \restriction \sigma.$$

Hence, $C_M(X \restriction \sigma) \leq \sigma - d$. □

An interesting corollary of this theorem is the following.

Corollary 4.5. *A set X is 3-random if and only if $C(X \upharpoonright (\Omega \upharpoonright n)) > \Omega \upharpoonright n - O(1)$ for infinitely many n. A set X is n-random if and only if $C(X \upharpoonright (Z \upharpoonright n)) > Z \upharpoonright n - O(1)$ for infinitely many n, where Z is a set Turing equivalent to $0^{(n-2)}$ and $n \geq 2$.*

In the past we did not have any characterization of 3-randomness by complexity except obvious relativized versions of characterizations of ML-randomness and 2-randomness. Here, we have one characterization of 3-randomness. Although it uses Ω, it does not use any oracles.

Theorem 4.4 roughly means that we can measure how random a set X is by looking at the set of lengths such that the complexities of the strings with the lengths are maximal up to a constant.

Theorem 4.6. *The following are equivalent:*

(i) *$X \oplus Z$ is 2-random.*
(ii) *$K(X \upharpoonright (Z \upharpoonright n)) \geq (Z \upharpoonright n) + n + K(n) - O(1)$ for infinitely many n.*

For the proof, we recall that $X \widehat{\oplus} Z$ is defined to be

$$z_0 x_0 x_1 z_1 x_2 x_3 x_4 x_5 z_2 \cdots z_{n-1} x_{2^n-2} \cdots x_{2^{n+1}-3} z_n \cdots$$

for $X = x_0 x_1 \cdots$ and $Z = z_0 z_1 \cdots$.

Proof. (i)⇒(ii). Let $\sigma = Z \upharpoonright n$. Then

$$K(X \upharpoonright (Z \upharpoonright n)) = K(X \widehat{\oplus} Z \upharpoonright (\sigma + n + 1)) \geq \sigma + n + 1 + K^{X \oplus Z}(\sigma + n + 1)$$

where the second inequality follows from a corollary of Ample Excess Lemma [18]. Notice that

$$K^{X \oplus Z}(\sigma + n) = K^{X \oplus Z}(n) \pm O(1).$$

Since $X \oplus Z$ is 2-random and thus weakly low for K ([17]), we have

$$K^{X \oplus Z}(n) = K(n)$$

for infinitely many n.

(ii)⇒(i). Notice that

$$K(X \upharpoonright (Z \upharpoonright n)) \leq (Z \upharpoonright n) + K(Z \upharpoonright n) + O(1).$$

Then, we have

$$K(Z \upharpoonright n) \geq n + K(n) - O(1)$$

for infinitely many n. Hence, Z is 2-random.

Let $m_C(\sigma) = |\sigma| - C(\sigma)$ and $m_K(\sigma) = |\sigma| + K(|\sigma|) - K(\sigma)$. The the statement (ii) implies that $m_K(X \upharpoonright (Z \upharpoonright n)) = O(1)$ for infinitely many n because $K(Z \upharpoonright n) \leq n + K(n) + O(1)$. Since $m_K(\sigma) \geq m_C(\sigma) - O(\log m_C(\sigma))$ ([11, Theorem 4.3.2]), we have $m_C(X \upharpoonright (Z \upharpoonright n)) = O(1)$ for infinitely many n. Hence, $C(X \upharpoonright (Z \upharpoonright n)) > Z \upharpoonright n - O(1)$ for infinitely many n. Thus, X is 2-Z-random by Theorem 4.4. Finally, by van Lambalgen's theorem ([23]), $X \oplus Z$ is 2-random. □

5. DM-TRIVIALITY

In this section, with independent interest, we ask the following question.

Question 5.1. If $A \leq_{dm} \emptyset$, then should A be computable?

Since $\leq_{dm}$ implies $\leq_{wdm}$ which is equivalent to $\leq_{Sch}$, dm-triviality implies Schnorr triviality.

We can also show that dm-triviality implies K-triviality. To see this, we use Solovay functions.

Definition 5.2 (Bienvenu and Downey [4])**.** A function $g : \omega \to \omega$ is a *Solovay function* if g is computable and it holds that

(i) $\sum_n 2^{-g(n)} < \infty$, and
(ii) $g(n) \le K(n) + O(1)$ for infinitely many n.

Theorem 5.3 (Bienvenu, Downey, Nies, and Merkle [5])**.** *Let g be a Solovay function. If $K(A \upharpoonright n) \le g(n) + O(1)$, then A is K-trivial.*

Theorem 5.4 (Bienvenu, Downey, Nies and Merkle [5])**.** *There exists a Solovay function that is an order.*

Theorem 5.5. *If $A \le_{dm} \emptyset$, then A is K-trivial.*

Proof. Let g be a Solovay function that is an order. Since g is a Solovay function, we have $\sum_n 2^{-g(n)} < \infty$. By the KC-theorem, there exists a machine M such that $K_M(0^n) = g(n) + c$ for some constant c. Since g is an order, M is decidable. By $A \le_{dm} \emptyset$, there exists a decidable prefix-free machine N such that

$$K(A \upharpoonright n) \le K_N(A \upharpoonright n) + O(1) \le K_M(\emptyset \upharpoonright n) + O(1) = g(n) + O(1).$$

Since g is a Solovay function, A is K-trivial. □

Thus, the class of dm-trivial reals is a subset of the class of K-trivial reals and the class of Schnorr trivial reals. Since K-trivial reals and Schnorr trivial reals are incomparable [13, Corollary 3.14], dm-trivial reals do not coincide with either of them.

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